Differential geometry I, problem set

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Vector fields and differential forms

Problem 1. Let v be a vector field on M which is a gradient–like vector field for some Morse function f. Prove that there exists a Riemannian metric on M such that v is actually a gradient vector field for M.

Problem 2. Let M be a compact manifold presented as a union of $M_+ \cup M_-$ along their common boundary M_0. Suppose that there are two smooth functions $f_\pm \colon M_\pm \to \mathbb{R}_\pm$ such that $f_\pm^{-1}(0) = M_0$ and have non-vanishing gradient on M_0. Prove that there exists a smooth function $f \colon M \to \mathbb{R}$, which is equal to $f_\pm$ away from an arbitrary small neighbourhood of M_0 and such that the set of critical points of f is a union of the set of critical points of f_+ and f_-.

Hint: construct f as an integral of a suitably defined vector field.

Problem 3. Let M be a compact manifold with boundary $-M_0 \sqcup M_1$. Suppose v is a vector field with finitely many critical points, all of them in the interior of M. Assume that near each critical point the vector field can be written in local coordinates as $(\pm x_1, \dots, \pm x_n)$. Suppose that for any $x \in M$ a trajectory of v starts either on M_0 or at a critical point of v and ends either on M_1 or at a critical point of v.

Prove that if there are no 'broken circular trajectories' (unions of trajectories that start at one point and end at the same), then the vector field is a gradient–like vector field for some function f.

Problem 4. Prove that S^n has a nowhere vanishing vector field if and only if n is odd.

Problem 5. Prove that $*$ is an idempotent. The eigenvalues of $*$ are ± 1 and the eigenspaces have the same dimension. We denote them by V_+^{tot} and V_-^{tot}.

Problem 6. Suppose $\dim V = 4$. Find explicitly the basis of V_+^2 and V_-^2.

* Using this prove that $SO(4) = SO(3) \times SO(3)$.

Problem 7. Compute the star operator for the induced metric on a torus in $\mathbb{R}^2$.

Problem 8. Use the star operator to compute Laplace operator in polar coordinates in $\mathbb{R}^4$.

Problem 9. Prove that for a vector field v on a manifold M the divergence is given by $di_v\omega$, where ω is the volume form, i_v is a contraction and d the exterior derivative.

Problem 10. Argue that the Liouville theorem $L_X\omega = i_X d\omega + di_X\omega$ is morally the same statement as *trace is the derivative of the determinant*.

Vector bundles

Problem 11. Prove that over a compact manifold M every vector bundle is a subbundle of a trivial bundle.

Problem 12. Suppose E is a rank k vector subbundle of trivial bundle $F \cong \mathbb{F}^m$ (here $\mathbb{F}$ is the base field) over a compact space X. There is an induced map $\rho_{E,m} \colon M \to G_{\mathbb{F}}(k,m)$.

(a) Prove that there is a $1-1$ correspondence between homotopy classes of maps from X to $G_{\mathbb{F}}(k,m)$ and homotopy classes of rank k vector subbundles of $\mathbb{F}^m$ (the notion of a homotopy class of a bundle is simple: it is the homotopy class of transition functions).
(b) There is a map $p_m \colon G_{\mathbb{F}}(k,m) \to G_{\mathbb{F}}(k,m+1)$ given by the inclusion. Prove that $\rho_{E,m+1}$ is homotopy equivalent to $p_m \circ \rho_{E,m}$.
(c) Deduce that all homotopy types of rank k vector bundles are in a 1—1 correspondence with homotopy types of maps from M to $G_{\mathbb{F}}(k,\infty) = \lim G_{\mathbb{F}}(k,m)$.

Problem 13. Determine the tangent bundle to the projective plane and to the grassmanian in terms of the tautological bundle.

Problem 14. Let $\pi \colon E \to F$ be a linear map between vector bundles over the same manifold M. Suppose the rank of the image $\pi_x \colon E_x \to F_x$ (the subscript x means that we take the fibre over a point $x \in M$) is independent of x. Does this mean that $\ker \pi$ is a vector bundle?

Problem 15. A *Riemannian metric* on the vector bundle E is a choice of a scalar product in each fibre. Prove that a bundle over a paracompact manifold admits a Riemannian metric. Show that if E has a Riemannian metric, then the transition functions can be choosen to sit in the orthogonal group.

Problem 16. Let E be a real vector bundle over M of rank $2k$. Suppose that the transition functions of E sit in $U(k)$. Prove that E can be given a structure of a complex bundle of rank k.

Problem 17. Let E be a real vector bundle over M of rank $2k$. Explain that it has a complex structure if and only if a corresponding map $\rho_E \colon M \to G_{\mathbb{R}}(2k,\infty)$ lifts to a map $G_{\mathbb{C}}(k,\infty)$.

Problem 18. Prove or find in the literature, that the tangent bundle to S^{2k} for $k \neq 1,3$ does not admit a complex structure (the default proof is completely different that the maps into grassmanian). The case $k=2$ is 'easiest', the case k is big is also not too hard.

Morse homology.

For some problems an elementary knowledge of homological algebra might be helpful. You can consult Weibel's book for instance.

Problem 19. Let M be a closed smooth manifold of dimension n admitting a Morse function with exactly two critical points. Prove that M is homeomorphic to S^n.

Problem 20. Let D be a two–dimensional disk and $f \colon D \to \mathbb{R}$ be a smooth function with $f|_{\partial D} \equiv 0$ and f takes both positive and negative values in the interior. Prove that f must have at least three critital points (note that f doesn't have to be Morse).

Problem 21. Prove, using Morse homology, that if M and N are two closed manifolds, then $H_*(M \times N; \mathbb{F}) \cong H_*(M; \mathbb{F}) \times H_*(N\mathbb{F})$ for any field $\mathbb{F}$. Hint: show that there is a quasiisomorphism $C_*(M \times N) \cong C_*(M) \otimes C_*(N)$. This is easy.

Problem 22. Using Morse homology (over a field) prove the Poincaré duality for a closed manifold M of dimension n: $H_k(M) \cong H_{n-k}(M)$.

Problem 23. Suppose M is a compact manifold with boundary. Consider Morse functions $f\colon M \to [0,1]$ that are identically equal to zero (respectively one) on ∂M. For such a function consider the Morse complex $C_*(M)$. Prove that the homology of this complex is either $H_*(M)$ or $H_*(M; \partial M)$ (singular homology of a space or singular homology of the space). Which is which?

Problem 24. Establish a variant of the Mayer-Vietoris exact sequence in Morse homology. That is, we assume that a closed manifold M is presented as a union of M_1 and M_2 along their common boundary N. The Mayer-Vietoris sequence computes $H_*(M)$ in terms of $H_*(M_i)$ and $H_*(N)$.

Problem 25. Show that the Euler characteristic of a closed manifold is equal to the alternating sum of numbers of critical points of a Morse function on it. Notice that the Morse–Smale condition is not necessary.

Transversality theorems.

Problem 26. Let A, B be two closed submanifolds of a compact manifold M intersecting transversally. Prove that $A \cap B$ is a smooth manifold of dimension $\dim A + \dim B - \dim M$.

Problem 27. Let A, B be as above. Suppose x_n is a sequence of points in $A \cap B$ converging to x_0, such that $x_n \neq x_0$. Prove that $T_{x_0}A \cap T_{x_0}B$ has positive dimension. Do not use the fact that $A \cap B$ is a manifold.

Problem 28. Show that given a generic closed surface in $\mathbb{R}^3$ there is no line tangent to it with tangency order 5.

Problem 29. Find out all possible singularities that can occur in a generic one-parameter family of smooth functions on a closed manifold M. What are singularities in two-parameter families?

Hint: see Arnold–Varchenko–Gussein-Zade's book or Cerf's paper. At best both.

Riemannian geometry