Affine algebraic curves with zero Euler characteristics

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- $C \simeq \mathbb{C}^*$ and C has no finite self–intersections.
- C has one place at infinity and one finite self–intersection.

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Koras, Russell case $C \simeq \mathbb{C}^*$ and C smooth.

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$$\begin{cases} x(t) & = t^a + \alpha_1 t^{a-1} + \cdots + \alpha_{a+b} t^{-b} \\ y(t) & = t^c + \beta_1 t^{c-1} + \cdots + \beta_{c+d} t^{-d}. \end{cases}$$

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If (x_0, y_0) is a singular point of $(x(t), y(t))$, 2δ is the number of solutions to

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For an ordinary double point we have $2\delta = 2$.

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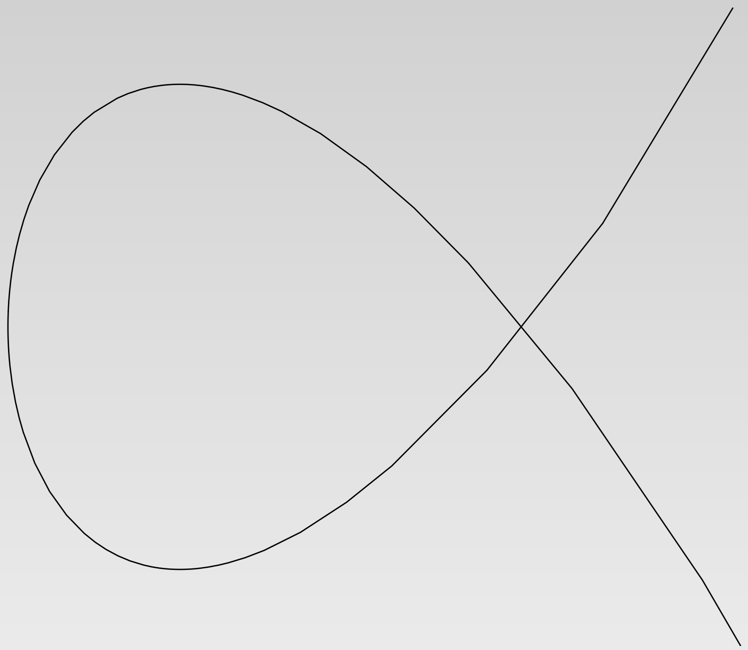

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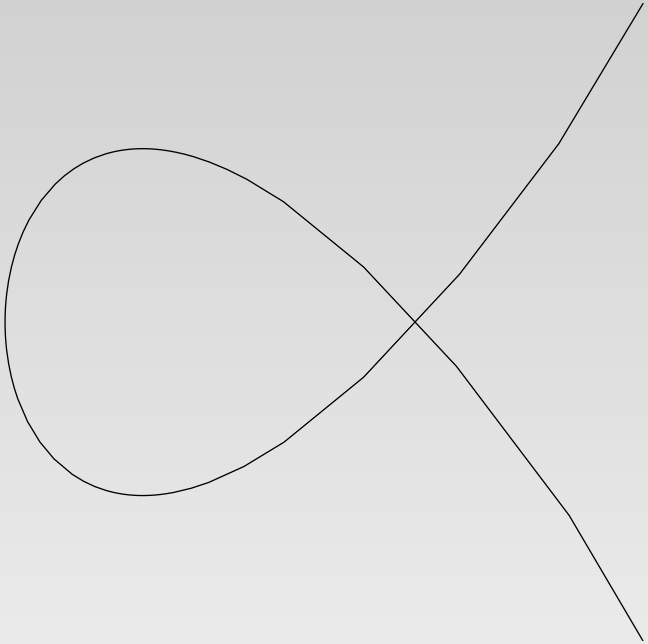

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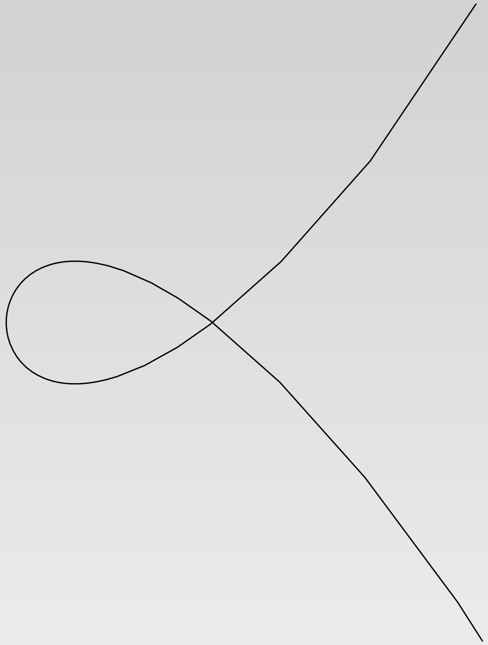

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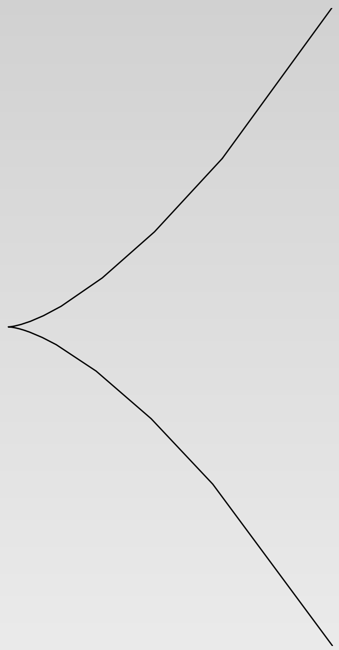

One double point „hides” in a singular point. $2\delta = 2.$

Another example

Curves depend on λ.

$$\begin{cases} x_\lambda(t) & = t^3 - 15\lambda^2 t \\ y_\lambda(t) & = t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t, \end{cases}$$

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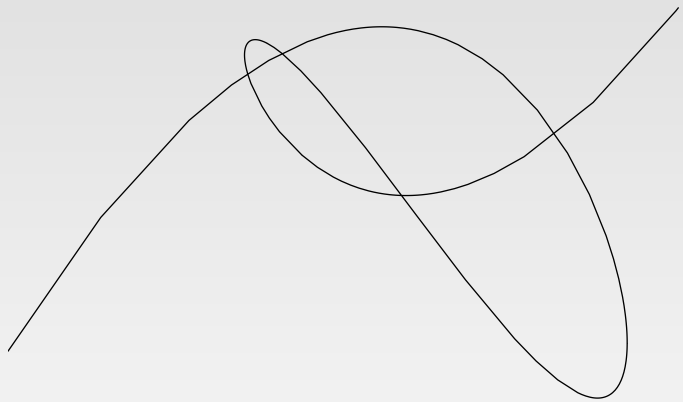
$$\lambda = 1$$



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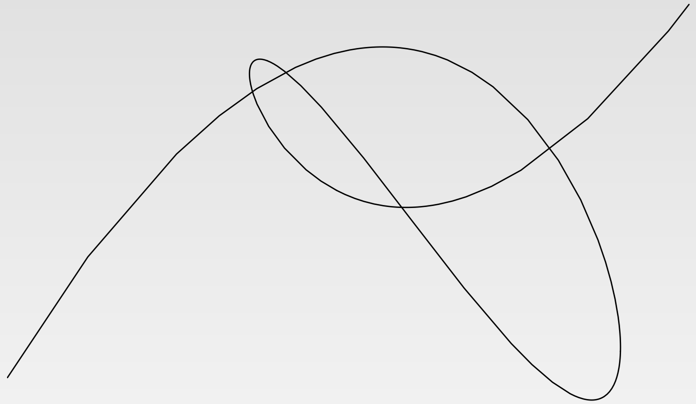

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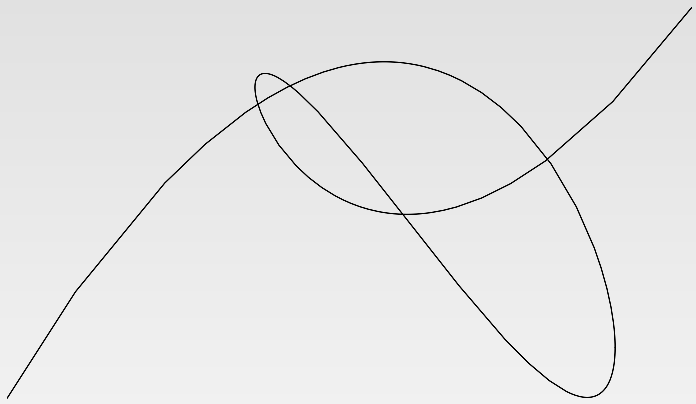

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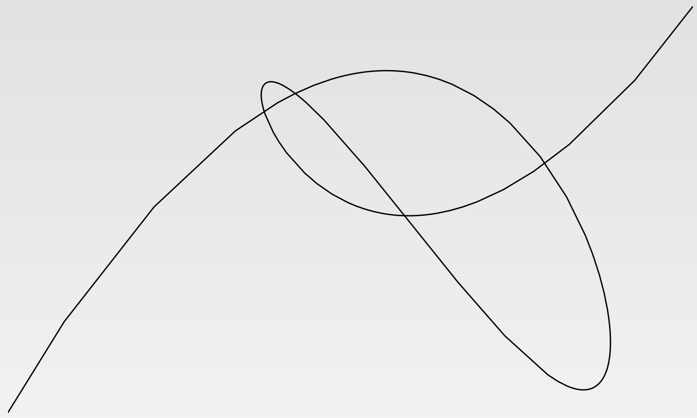

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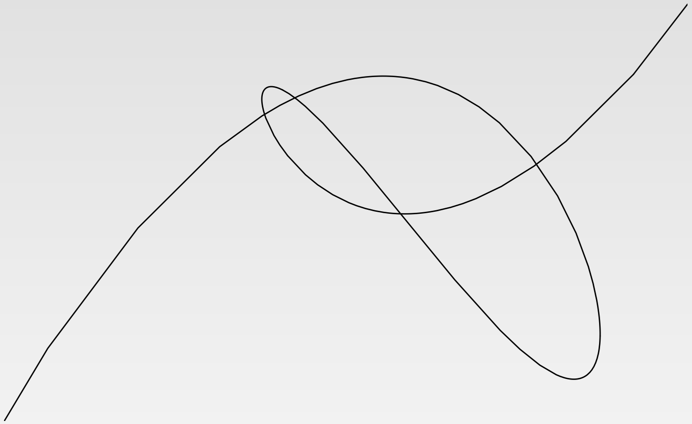

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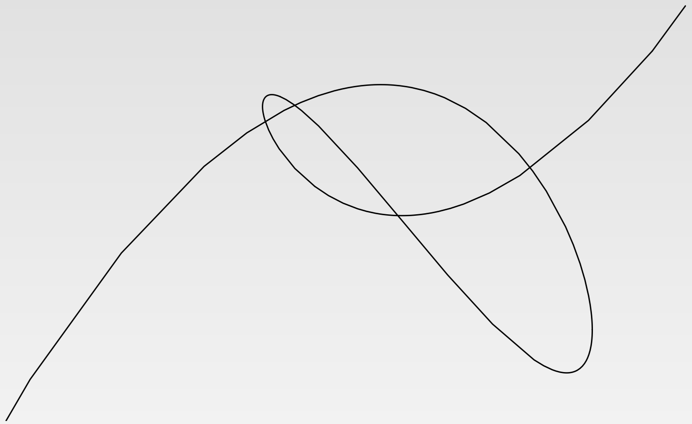

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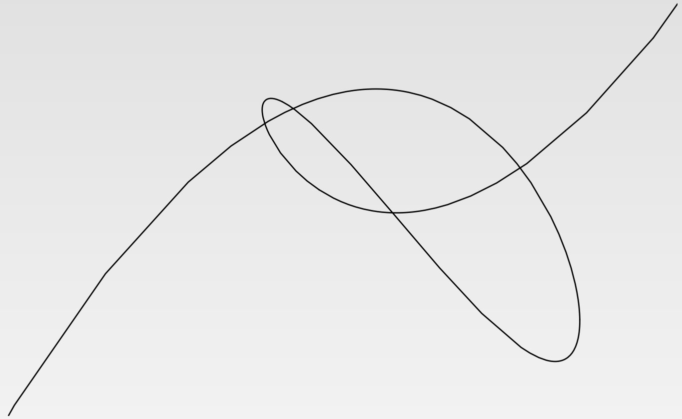

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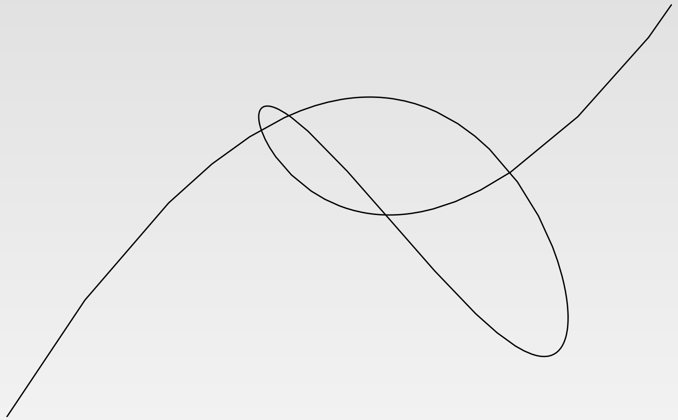
$$\lambda = 0.93$$



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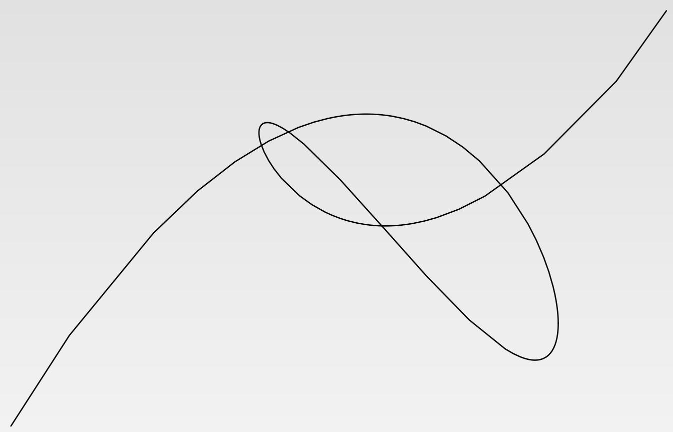

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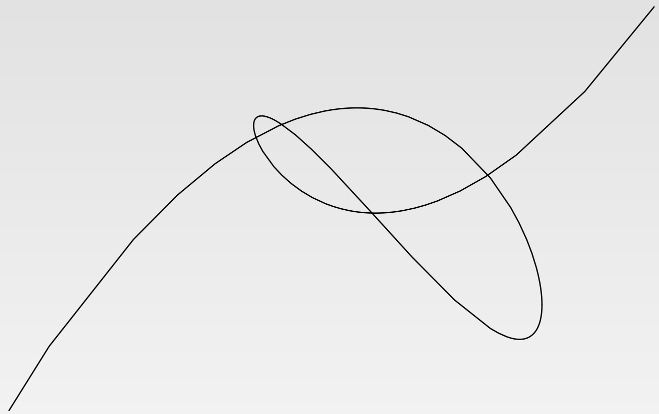

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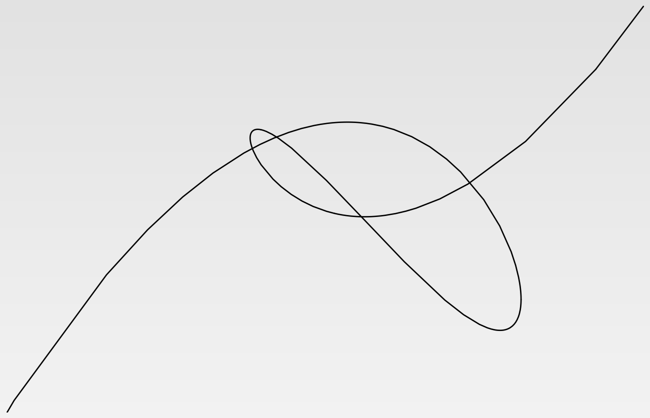

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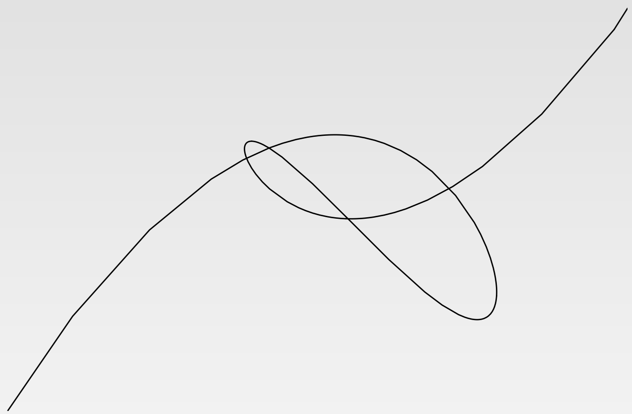

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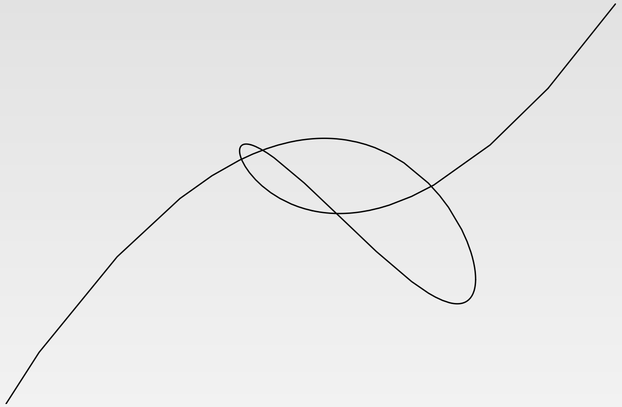

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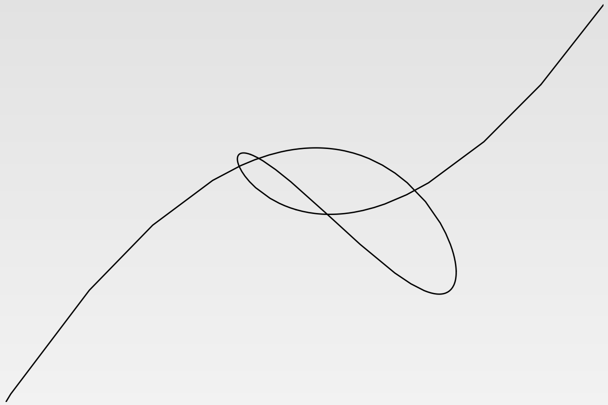

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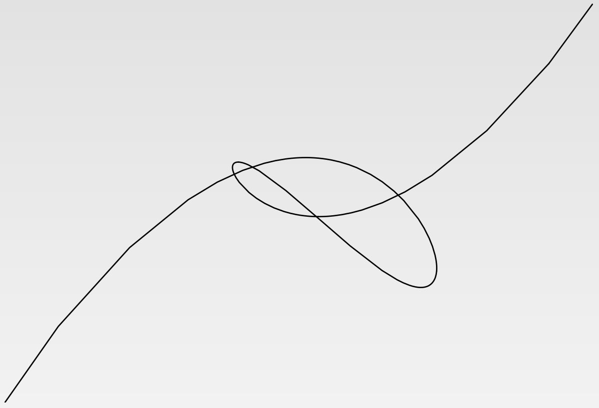

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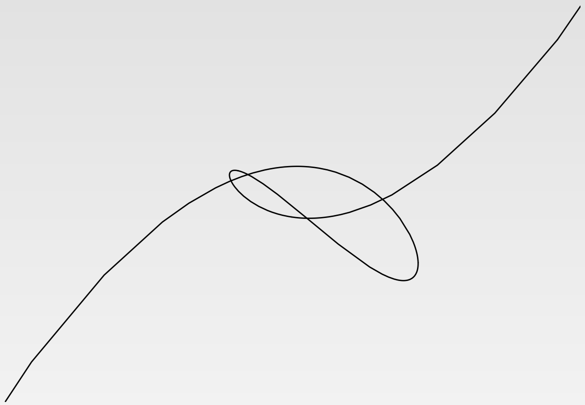

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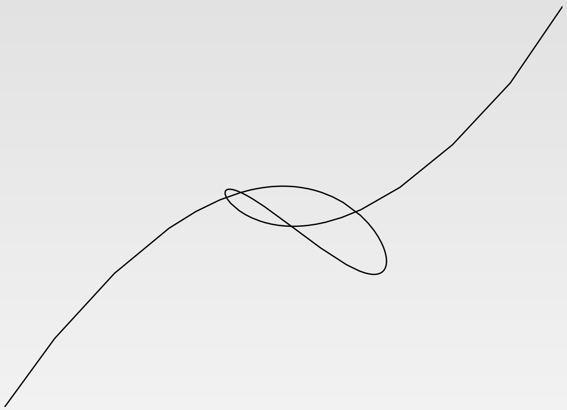

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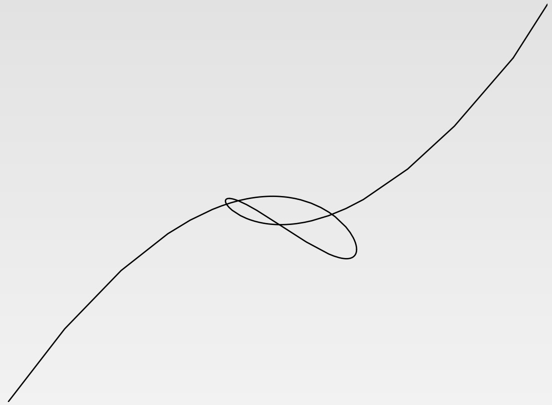

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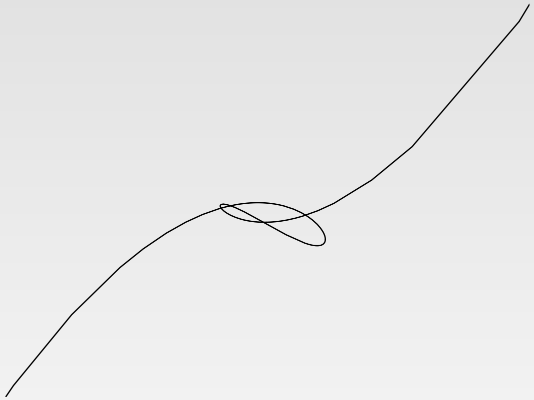

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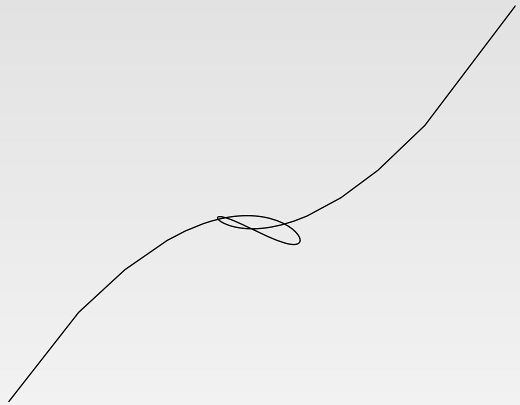

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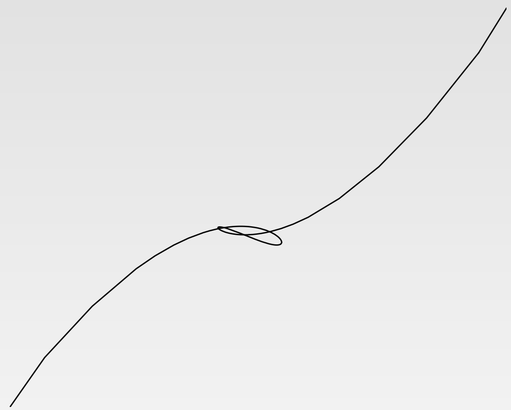

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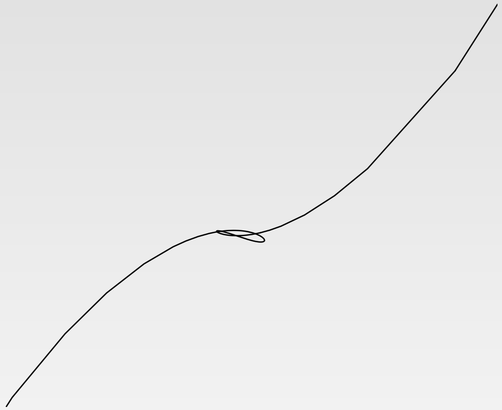

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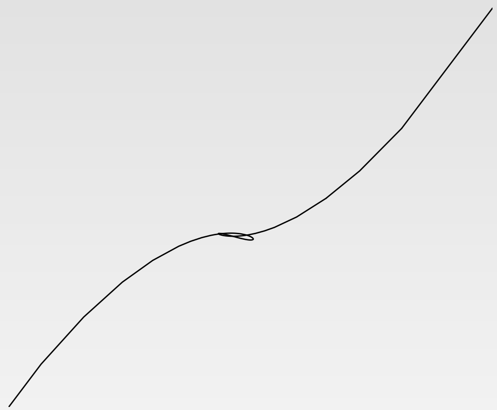
$$\lambda = 0.45$$



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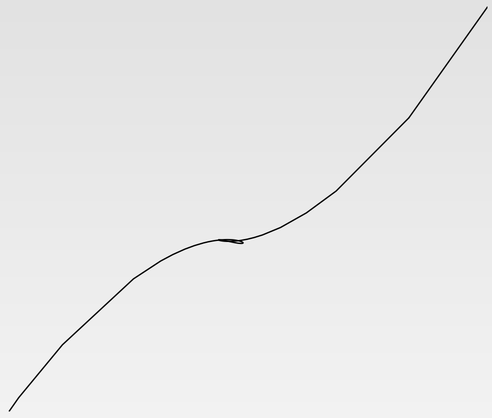
$$\begin{cases} x_\lambda(t) & = t^3 - 15\lambda^2 t \\ y_\lambda(t) & = t^5 - 30\lambda^2 t^3 + 10\lambda^3 t^2 + 201\lambda^4 t, \end{cases}$$

$\lambda = 0.4$



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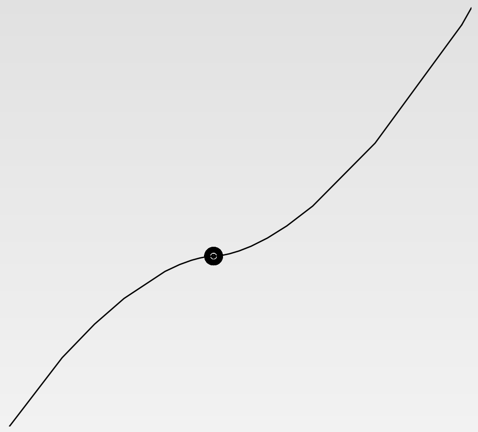

Four double points hide in a singular point (t^3, t^5).
Thus $2\delta = 8$.

Serre formula

For a curve C of degree d we have

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$\sum$ — sum over all singular points and double points.

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We require $g = 0$. Thus

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- For a typical curve δ_i correspond to ordinary double points.
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- Maybe at infinity.

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To control the deformations of a parametric curves we introduce

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Strongly resembles $\bar{M}$ number of Orevkov.

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Parametrise locally $x(t) \sim t^p$, $y(t) \sim t^q + \dots$. Write

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- ν is determined by the characteristic sequence and the order p.

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- One can find all cases with an equality.
- Direct calculations.
- Resembles Zajdenberg–Orevkov inequality.

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$$\begin{cases} x &= t^4, \\ y &= 2t^4 + t^6 + 2t^8 + t^9 \end{cases}$$

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$$\mu = 15 + 3 = 18 \leq 4 \cdot 5$$

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Change 9 to 13.

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The more complicated singularity, the less sharp is the inequality.

Tangent codimension

- Two branches at a singular point

$$y = c_1 x^{1/p_1} + c_2 x^{2/p_1} + \cdots + c_k x^{k/p_1} + \ldots$$

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— and possibly some equality relations between non–vanishing c's and d's.

- number of these relation ν_{tan}: the tangent codimension.

Example

$$\text{Branch I} \begin{cases} x &= t^4 \\ y &= t^8 + 3t^{10} + 2t^{14} + 5t^{15} \end{cases}$$

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Consider Puiseux expansion

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• The sign change results from chosing different root of unity of order 6.

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Here terms at x, x^2, $x^{5/2}$, x^3 agree.

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In other words, $c_4 = d_6$, $c_8 = d_{12}$, $c_{10} = d_{15}$ and $c_{12} = d_{18}$.

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This $+1$ is very inconvenient. We can get rid of it almost all cases.

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  it equals $\min(q_1 p_2, q_2 p_1)$ — leads to better estimate.

External codimension

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The subspace of curves with such singularity in the space curves $x = t^p + \cdots + a_0$, $y = t^q + b_1 t^{q-1} + \ldots$ *for* p, q *sufficiently large has codimension* $\operatorname{ext} \nu$.

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  If we swap x with y, the codimension may change.

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- We have $p-1$ condition on x, ν on y and can move parameter t.

  For $x = t^4$, $y = t^8 + t^9$, *we have* $\text{ext } \nu = 8$. *For* $x = t^8 + t^9$, $y = t^4$ *we have* $\text{ext } \nu = 9$.

External codimension

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  The codimension is minimal if $\operatorname{ord} x =$*multiplicity.*

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- If we have 2 branches with $\text{ext } \nu_1$, $\text{ext } \nu_2$, $\text{ext } \nu = \text{ext } \nu_1 + \text{ext } \nu_2 + \nu_{tan}^{(12)} + 2.$

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  Additional 2 comes from the condition $x(t_0) = x(t_1)$, $y(t_0) = y(t_1)$.

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- Definition for more branches is similar.

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  E the reduced exceptional divisor.

External codimension II

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  K is the projection of the canonical divisor onto the subgroup of $Pic(\tilde{X}) \otimes \mathbb{Q}$ spanned by components of E.

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  In fact $\bar{M} = K(K + D) + \#$ *branches* $- 1$.

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Proposition. For a given singular curve $C \subset \mathbb{C}^2$, if orders of x at C all branches are multiplicities, then

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The proof follows from calculating both quantities in terms of Eisenbud–Neumann diagrams.

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a, b, c and d need not be positive. We will discuss it later.

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  - $t \longrightarrow t + a$.

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In case of two branches the actual structure of the group depends heavily on a, b, c *and* d.

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  For example if $b < 0$, the change $x \to x + const$ is not allowed.

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- The *regularity* means, that the space of curves with singularities of codimension $\mathrm{ext}\ \nu_1, \ldots, \mathrm{ext}\ \nu_k$ forms a subspace of the codimension at most $\mathrm{ext}\ \nu_1 + \cdots + \mathrm{ext}\ \nu_k$. Thus

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Sum all singular points together with infinity.

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$\text{ext } \nu_i$ external codimensions.

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Codimension is really a codimension.

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- All our examples calculated by hand are regular.
- Slightly more general regularity conjecture fail.

Genus formula revisited

Take curve C

$$\begin{cases} x(t) & = t^a + \alpha_1 t^{a-1} + \cdots + \alpha_a \\ y(t) & = t^c + \beta_1 t^{c-1} + \cdots + \beta_c. \end{cases}$$

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Then $\deg C = c$ and $\sum 2\delta_i + 2\delta_\infty = c(c-1)$.

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Now we plug the $2\delta'_\infty$.

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From this sum we exclude the only finite double point.

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Use the inequality $2\delta_i \leq p_i\nu_i$ for singular point with one branch.

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$$\sum p_i \nu_i + 2\delta_{dbl} + 2\delta'_\infty \leq (a-1)(c-1).$$

The inequality for $2\delta'_\infty$ is similar.
$2\delta'_\infty \leq a'\nu'_\infty + a' - 1$, where $a' = gcd(a, c)$.

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Now we estimate $2\delta_{dbl}$.

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$$\sum p_i\nu_i + (p_{01} + p_{02})(\nu_{01} + \nu_{02} + \nu_{tan} + 1) + \\ + a' - 1 \leq (a-1)(c-1).$$

p_{01} and p_{02} are orders of x at two branches of the double locus.

Genus formula for annuli.

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$$\sum p_i \nu_i \leq (a+b-1)(c+d-1)+$$

$$+ |ad - bc| - a' - b' + 1 - a'\nu'_\infty - b'\nu'_0,$$

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if $ad - bc \neq 0$ and $a + b \leq c + d$ we get

$$\sum p_i \nu_i \leq (a + b - 1)(c + d - 1) + \\ + |ad - bc| - a' - b' + 1 - a'\nu'_\infty - b'\nu'_0,$$

where $a' = gcd(a, c)$, $b' = gcd(b, d)$, ν'_0, ν'_∞ are codimensions at zero and infinity.

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If $ad = bc$ we need to take into account the tangency of branches at infinity.

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if $ad - bc \neq 0$ and $a + b \leq c + d$ we get

$$\sum p_i \nu_i \leq (a+b-1)(c+d-1) + \\ - a' - b' + 1 - (a' + b')(\nu'_{inf} + 1),$$

If $ad = bc$ we need to take into account the tangency of branches at infinity. The formula is suitably changed.

Estimates for annuli.

C annulus with $ad \neq bc$.

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- $$\sum p_i \nu_i \leq (a+b-1)(c+d-1) - a'\nu'_\infty - b'\nu'_0 + \det{}'.$$
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- $$\sum (\nu_i + p_i - 2) + \nu'_0 + \nu'_\infty \leq a + b + c + d - 1 - K - D$$
- $$\sum (p_i - 1) \leq \begin{cases} a+b-1, & b \leq 0 \\ a+b, & b > 0 \end{cases}$$

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- The genus formula.

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- $$\sum (\nu_i + p_i - 2) + \nu'_0 + \nu'_\infty \leq \\ a+b+c+d-1-K-D$$
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- The regularity condition.

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- Counting zeros of $\frac{d}{dt}x(t)$.

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- $$\sum (p_i - 1) \leq \begin{cases} a+b-1, & b \leq 0 \\ a+b, & b > 0 \end{cases}$$
- K is maximal non–negative integer such that $Ka \leq c$ and $Kb \leq d$.

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- $$\sum (p_i - 1) \leq \begin{cases} a+b-1, & b \leq 0 \\ a+b, & b > 0 \end{cases}$$
- $D \in \{0,1,2\}$ is the number of constants: if we can add a constant to x or y.

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- $\sum (\nu_i + p_i - 2) + \nu'_0 + \nu'_\infty \leq a + b + c + d - 1 - K - D$
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Different types.

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- Type $\binom{+}{+}$: $0 < a < c$, $0 < b < d$. Then $D = 2$, $K \geq 1$. K is here very important.

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In type $\binom{+}{+}$ distinguish $ad \neq bc$ and $ad = bc$.

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  If $ad \neq bc$, many singular cases.

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  $ad = bc$: strong condition on a, b, c and d.

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- Annoying type. Many subcases, i.e. $a > c$, $a < c$ etc.

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  Here we do not need regularity neither.

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- Type $\binom{-}{-}$: $a, d > 0$, $b, c < 0$. $D = K = 0$.
- Most important. Contains all smooth cases.

Choice of presentation.

Recall

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$$\sum p_i \nu_i \leq (a+b-1)(c+d-1)+ \\ + |ad-bc| - a' - b' + 1 - a'\nu'_\infty - b'\nu'_0.$$

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- The term $a'\nu'_\infty$ may dominate. *Especially if $a+b$ is small.*

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- The term $a'\nu'_\infty$ may dominate. *It cannot happen, that $a|c$ and $b|d$ at the same time. We can reduce the case by the suitable change* $y \to y - const \cdot x^{\min(c/a, d/b)}$.

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— we can make each curve satisfy this condition.

— we call curve satisfying it handsome.

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— each curve is isomorphic to a handsome curve.

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The most difficult is then case $\binom{-}{-}$.

Example

Instead of the definition of handsomeness.

Example

Consider a curve

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$$y \longrightarrow y^{(1)} = -y/3 + x^3$$

Example

Consider a curve

$$\begin{cases} x = t^4 + t^{-2} \\ y^{(1)} = t^{12} + 3t^6 - \frac{1}{3}t + 3 + \frac{2}{3}t^{-2} - \frac{1}{3}t^{-4}. \end{cases}$$

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The same is with x. Return to previous y.

Example

Consider a curve

$$\begin{cases} x = t^4 + t^{-2} \\ y = t + 2t^{-2} - t^{-4} + 3t^{-6}. \end{cases}$$

We can apply different changes of type $y \longrightarrow y - x^k$, $x \longrightarrow x - y^l$ to that curve.

$$x \longrightarrow x^{(1)} = \tfrac{1}{8}(y^4 - x).$$

Example

Consider a curve

$$\begin{cases} x^{(1)} = t + t^{-1} - \frac{1}{8}t^{-2} + \cdots + \frac{81}{8}t^{-24} \\ y = t + 2t^{-2} - t^{-4} + 3t^{-6}. \end{cases}$$

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Which parametrisation is the best?

Example

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Handsomeness. This one!

Dealing with inequalities

Essentially three methods

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calculations,

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  N is the number of finite singular points.

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  We are left with case $N = 1$.

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  Reject cases with $\nu'_0 + \nu'_\infty \geq 2$ (if $ad \neq bc$).

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  Left with something like
  $p_1(a+b+c+d-K-D-p_1+2) \leq (a+b-1)(c+d-1)+\det'$.

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  Left with something like
  $p_1(a + b + c + d - K - D - p_1 + 2) \leq (a + b - 1)(c + d - 1) + \det'$.
- Now reject $p_1 \leq a + b - 2$ and consider other cases.

Result

In case of polynomial curves with one double locus there are

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namely

$$\begin{cases} x = t^a \\ y = \lambda_1 t^{-a} + \lambda_2 t^{-2a} + \cdots + \lambda_k t^{-ka} + t^{-c}, \end{cases}$$

with $a \nmid c$.

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Moreover these 23 cases contain

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Moreover these 23 cases contain 7 series and 2 special cases

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For annuli we find 19 series and 4 special cases, including one series with continuous parameters.

Moreover these 23 cases contain 7 series and 2 special cases of smooth embeddings $\mathbb{C}^* \longrightarrow \mathbb{C}^2$.

Maps $\mathbb{C} \to \mathbb{C}^2$

(a) $x = t^2$, $y = (t^2 - 1)^k t^{2l+1}$, $k = 1, 2, \ldots,$
$l = 0, 1, \ldots;$

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(g) $x = t^a(t-1)^{kb}, y = t^c(t-1)^{kd},$
$\kappa = |ad - bc| = 1, k = 1, 2, \ldots,$
$2 < a + kb < c + kd;$

Maps $\mathbb{C} \longrightarrow \mathbb{C}^2$

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(i) $x = t^{ka-b}(t-1)^{b}$, $y = t^{kc-d}(t-1)^{d}$, $\kappa = 1$, $k = 1, 2, \ldots$;

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Maps $\mathbb{C} \longrightarrow \mathbb{C}^2$

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(m) $x = [t(t-1)]^{2k+1}$, $y = x^l[t(t-1)]^k(t - \frac{1}{2})$,
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(p) $x = t^3(t-1)^3$, $y = t(t-1)(t-\frac{1}{2}-\frac{1}{6}i\sqrt{3})x^k$,
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(u) $x = t^3 - 3t,\ y = t^5 - \frac{7}{2}t^4 - t^2 + 11t.$

Maps $\mathbb{C}^* \longrightarrow \mathbb{C}^2$

(a) $x = t^m$, $y = t^n + \gamma_1 t^{-m} + \gamma_2 t^{-2m} + \cdots + \gamma_k t^{-mk}$, where $m > 0$, $gcd(m, |n|) = 1$, $k = 0, 1, \ldots,$ $\gamma_j \in \mathbb{C}$, $\gamma_k = 1$ (if $k > 0$) and $k > 0$ if $n > 0$. and at least one $\gamma_i \neq 0$ if $m > 0$.

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(b) $x = t(t-1)$, $y = (x + \frac{1}{4})^m x^n R_l(1/t)$, where $m, n = 0, 1, \ldots$ and R_l is a polynomial satisfying $R_l(1/t) - R_l(1/(1-t)) = (2t-1)t^{-l}(1-t)^{-l}$, $l = 1, 2, \ldots$.

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(c) $x = t^{mn}(t-1)$, $y = S_k^+(1/t)$, where $mn \geq 2$, $k = 1, 2, \ldots$. S_k are polynomials defined recursively by $S_0^+(u) = u^n$, $S_{k+1}^+(u) = [S_k^+(u) - S_k^+(1)]u^{mn+1}/(u-1)$.

Maps $\mathbb{C}^* \to \mathbb{C}^2$

(d) $x = t^{mn-1}(t-1)$, $y = T_k^+(1/t)$, where $mn \geq 2$, $k = 1, 2, \ldots$. T_k^+ are polynomials satisfying
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(e) $x = t^{mn}(t-1)$, $y = S_k^-(1/t)$, where $mn \geq 2$, $k = 1, \ldots,$ and S_m^- is a polynomial such that
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Maps $\mathbb{C}^* \to \mathbb{C}^2$

(g) $x = t^2(t-1)$, $y = U_k(1/t)$, $k = 1, 2, \ldots,$
$U_1(u) = 3u + u^2$,
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(j) $x = t + t^{-1}$, $y = Z(t)$ is a polynomial satisfying
$y(t) + y(1/t) = (t-1)^{2m+1}(t+1)^{n+1}/t^{m+n+1},$
where $0 \leq m \leq n$ i $m + n > 0$.

Maps $\mathbb{C}^* \to \mathbb{C}^2$

(k) $x = (t-1)^3 t^{-2}, y = x^k (t-1)(t-4)t^{-1},$
$k = 1, 2, \ldots.$

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(w) $x = (t-1)^2(t+2)t^{-1}, y = (t-1)^2(t+\frac{1}{2})t^{-2}.$

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- Applications to XVI Hilbert problem (Liénard vector fields).

THANK YOU