Khovanov invariants for knots

Maciej Borodzik

Institute of Mathematics, University of Warsaw

Warsaw, 2018

Knots

A knot is a possibly tangled circle in $\mathbb{R}^3$:

Knots

A knot is a possibly tangled circle in $\mathbb{R}^3$:

Definition

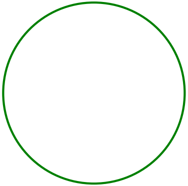
A *knot* in $\mathbb{R}^3$ is an image of a smooth embedding $\phi\colon S^1 \to \mathbb{R}^3$.
A *link* is "a knot with more than one component".

Knots

A knot is a possibly tangled circle in $\mathbb{R}^3$:

Definition

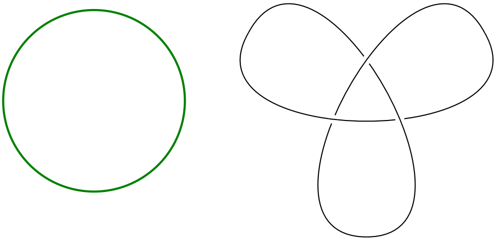
A *knot* in $\mathbb{R}^3$ is an image of a smooth embedding $\phi\colon S^1 \to \mathbb{R}^3$.
A *link* is “a knot with more than one component”.



A knot is a possibly tangled circle in $\mathbb{R}^3$:

Definition

A *knot* in $\mathbb{R}^3$ is an image of a smooth embedding $\phi\colon S^1 \to \mathbb{R}^3$.
A *link* is "a knot with more than one component".

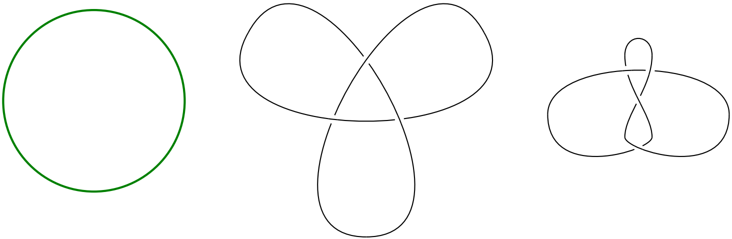


Knots

A knot is a possibly tangled circle in $\mathbb{R}^3$:

Definition

A *knot* in $\mathbb{R}^3$ is an image of a smooth embedding $\phi\colon S^1 \to \mathbb{R}^3$.
A *link* is “a knot with more than one component”.



Distinguishing knots

- A *knot invariant* assigns a simpler (more tractable) object to a knot;

Distinguishing knots

- A *knot invariant* assigns a simpler (more tractable) object to a knot;
- Should be the same no matter how the knot is drawn;

Distinguishing knots

- A *knot invariant* assigns a simpler (more tractable) object to a knot;
- Should be the same no matter how the knot is drawn;
- Should be computable;

Distinguishing knots

- A *knot invariant* assigns a simpler (more tractable) object to a knot;
- Should be the same no matter how the knot is drawn;
- Should be computable;
- Should have a meaning;

Distinguishing knots

- A *knot invariant* assigns a simpler (more tractable) object to a knot;
- Should be the same no matter how the knot is drawn;
- Should be computable;
- Should have a meaning;
- Should really distinguish knots.

Polynomial invariants

- Assign a polynomial to a knot.

Polynomial invariants

- Assign a polynomial to a knot.
- Alexander polynomial defined in 1928.

Polynomial invariants

- Assign a polynomial to a knot.
- Alexander polynomial defined in 1928.
- Jones polynomial discovered in 1984.

Polynomial invariants

- Assign a polynomial to a knot.
- Alexander polynomial defined in 1928.
- Jones polynomial discovered in 1984.
- HOMFLYPT polynomial constructed in 1985 by Hoste, Ocneanu, Millet, Freyd, Yetter, and independently by Przytycki and Traczyk in 1986.

Polynomial invariants

- Assign a polynomial to a knot.
- Alexander polynomial defined in 1928.
- Jones polynomial discovered in 1984.
- HOMFLYPT polynomial constructed in 1985 by Hoste, Ocneanu, Millet, Freyd, Yetter, and independently by Przytycki and Traczyk in 1986.

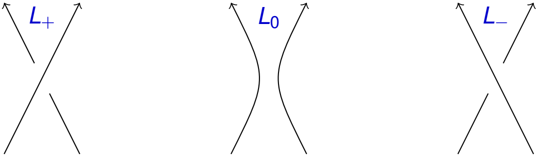
Alexander and Jones polynomials are polynomials in one variable (formally in $t^{1/2}$ and $t^{-1/2}$, so Laurent polynomials. HOMLYPT is a two-variable polynomial.

Polynomial invariants

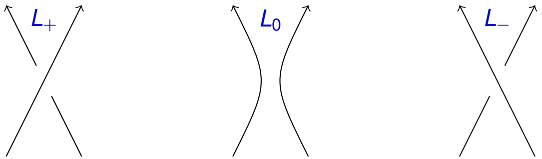
- Assign a polynomial to a knot.
- Alexander polynomial defined in 1928.
- Jones polynomial discovered in 1984.
- HOMFLYPT polynomial constructed in 1985 by Hoste, Ocneanu, Millet, Freyd, Yetter, and independently by Przytycki and Traczyk in 1986.

Alexander and Jones polynomials are polynomials in one variable (formally in $t^{1/2}$ and $t^{-1/2}$, so Laurent polynomials. HOMLYPT is a two-variable polynomial.
There are many more polynomial invariants, but these are the most basic. They have a special property.

Skein relation



Skein relation

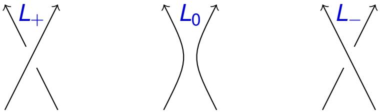


Definition (Informal)

A *skein relation* is a relation between the polynomials for links differing at a single place of the diagram.

Skein relation for Alexander and Jones polynomial

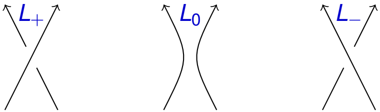
Let A be the Alexander polynomial and J be the Jones polynomial.



Skein relation for Alexander and Jones polynomial

Let A be the Alexander polynomial and J be the Jones polynomial.

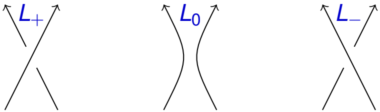
- We have: $A_{L_+}(t) - A_{L_-}(t) = (t^{1/2} - t^{-1/2})A_{L_0}(t)$.



Skein relation for Alexander and Jones polynomial

Let A be the Alexander polynomial and J be the Jones polynomial.

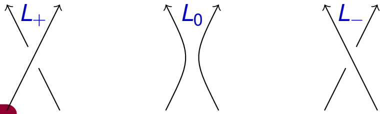
- We have: $A_{L_+}(t) - A_{L_-}(t) = (t^{1/2} - t^{-1/2})A_{L_0}(t)$.
- For Jones: $t^{-1}J_{L_+}(t) - tJ_{L_-}(t) = (t^{1/2} - t^{-1/2})J_{L_0}(t)$.



Skein relation for Alexander and Jones polynomial

Let A be the Alexander polynomial and J be the Jones polynomial.

- We have: $A_{L_+}(t) - A_{L_-}(t) = (t^{1/2} - t^{-1/2})A_{L_0}(t)$.
- For Jones: $t^{-1}J_{L_+}(t) - tJ_{L_-}(t) = (t^{1/2} - t^{-1/2})J_{L_0}(t)$.



Remark

There are various normalizations of the Alexander and Jones polynomials, which lead to different looking formulas.

Jones vs. Alexander

Alexander polynomial	Jones polynomial

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
Multiplicative for connected sums	

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
Multiplicative for connected sums	as well

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
Multiplicative for connected sums	as well
Belongs to $\mathbb{Z}[t, t^{-1}]$ for knots	as well

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
Multiplicative for connected sums	as well
Belongs to $\mathbb{Z}[t, t^{-1}]$ for knots	as well

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
Multiplicative for connected sums	as well
Belongs to $\mathbb{Z}[t, t^{-1}]$ for knots	as well
Satisfies $A(t^{-1}) = A(t)$	

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
Multiplicative for connected sums	as well
Belongs to $\mathbb{Z}[t, t^{-1}]$ for knots	as well
Satisfies $A(t^{-1}) = A(t)$	No such relation

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
Multiplicative for connected sums	as well
Belongs to $\mathbb{Z}[t, t^{-1}]$ for knots	as well
Satisfies $A(t^{-1}) = A(t)$	No such relation
$\Delta_K(t) = \pm 1$ for knots	

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
Multiplicative for connected sums	as well
Belongs to $\mathbb{Z}[t, t^{-1}]$ for knots	as well
Satisfies $A(t^{-1}) = A(t)$	No such relation
$\Delta_K(t) = \pm 1$ for knots	J determined on roots of unity of order $2, 3, 4$

Alexander polynomial	**Jones polynomial**
Belongs to $\mathbb{Z}[t, t^{-1}]$ for knots	as well
Satisfies $A(t^{-1}) = A(t)$	No such relation
$\Delta_K(t) = \pm 1$ for knots	J determined on roots of unity of order $2, 3, 4$
All polynomials satisfying above points can be realized	

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
Belongs to $\mathbb{Z}[t, t^{-1}]$ for knots	as well
Satisfies $A(t^{-1}) = A(t)$	No such relation
$\Delta_K(t) = \pm 1$ for knots	J determined on roots of unity of order $2, 3, 4$
All polynomials satisfying above points can be realized	??

Alexander polynomial	**Jones polynomial**
Satisfies $A(t^{-1}) = A(t)$	No such relation
$\Delta_K(t) = \pm 1$ for knots	J determined on roots of unity of order $2, 3, 4$
All polynomials satisfying above points can be realized	??
There are knots with $A(t) \equiv 1$	

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
Satisfies $A(t^{-1}) = A(t)$	No such relation
$\Delta_K(t) = \pm 1$ for knots	J determined on roots of unity of order $2, 3, 4$
All polynomials satisfying above points can be realized	??
There are knots with $A(t) \equiv 1$	??

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
$\Delta_K(t) = \pm 1$ for knots	J determined on roots of unity of order $2, 3, 4$
All polynomials satisfying above points can be realized	??
There are knots with $A(t) \equiv 1$	??
Topological meaning perfectly understood	

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
$\Delta_K(t) = \pm 1$ for knots	J determined on roots of unity of order $2, 3, 4$
All polynomials satisfying above points can be realized	??
There are knots with $A(t) \equiv 1$	??
Topological meaning perfectly understood	We know very little beyond combinatorics

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
$\Delta_K(t) = \pm 1$ for knots	J determined on roots of unity of order $2, 3, 4$
All polynomials satisfying above points can be realized	??
There are knots with $A(t) \equiv 1$	??
Topological meaning perfectly understood	We know very little beyond combinatorics
Computable in polynomial time	

Jones vs. Alexander

Alexander polynomial	**Jones polynomial**
$\Delta_K(t) = \pm 1$ for knots	J determined on roots of unity of order $2, 3, 4$
All polynomials satisfying above points can be realized	??
There are knots with $A(t) \equiv 1$	??
Topological meaning perfectly understood	We know very little beyond combinatorics
Computable in polynomial time	Most likely exponential time needed

Cube of resolutions. Part 1.

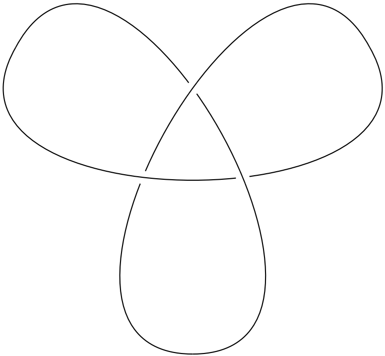
- We specify *resolutions* of a knot diagram.

Cube of resolutions. Part 1.

- We specify *resolutions* of a knot diagram.

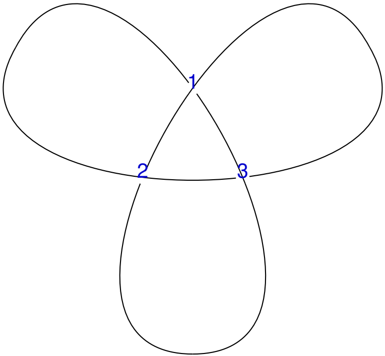
Cube of resolutions. Part 1.

- We specify *resolutions* of a knot diagram.
- Take a knot.



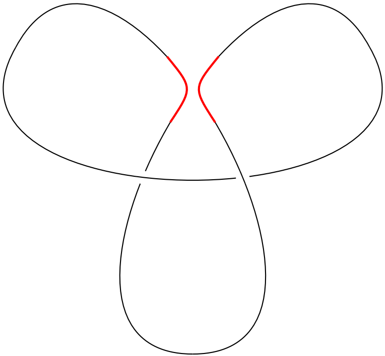
Cube of resolutions. Part 1.

- We specify *resolutions* of a knot diagram.
- Take a knot. Enumerate its crossings.

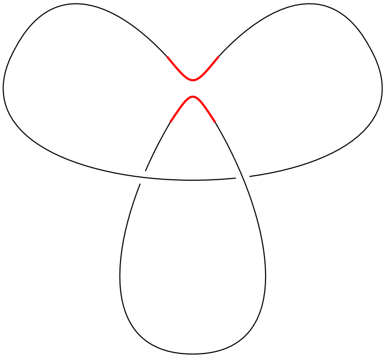


Cube of resolutions. Part 1.

- We specify *resolutions* of a knot diagram.
- Take a knot. Enumerate its crossings.
- 0-resolution of the first crossing.

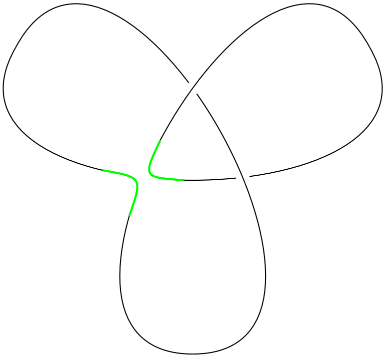


- We specify *resolutions* of a knot diagram.
- Take a knot. Enumerate its crossings.
- 1-resolution of the first crossing.



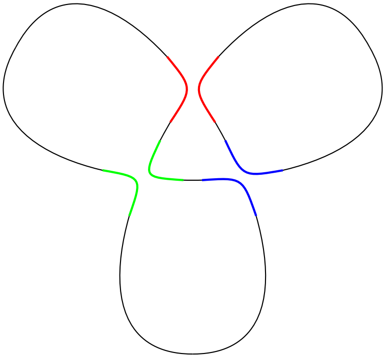
Cube of resolutions. Part 1.

- We specify *resolutions* of a knot diagram.
- Take a knot. Enumerate its crossings.
- 0-resolution of the second crossing.



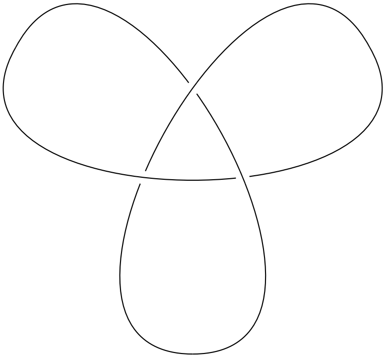
Cube of resolutions. Part 1.

- We specify *resolutions* of a knot diagram.
- Take a knot. Enumerate its crossings.
- 010 resolution.

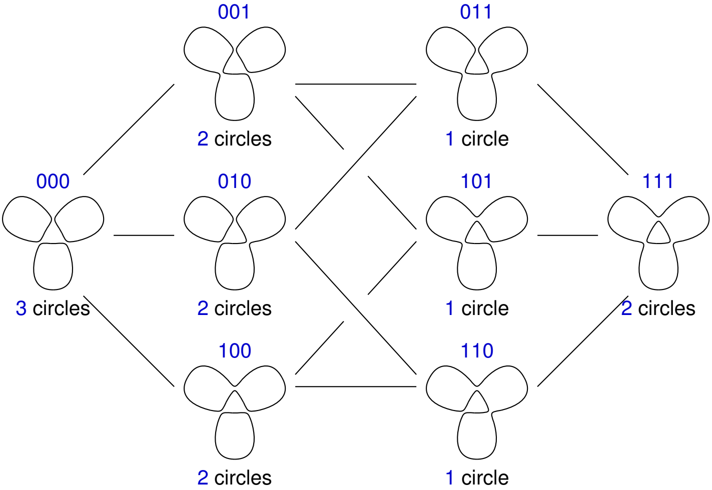


Cube of resolutions. Part 1.

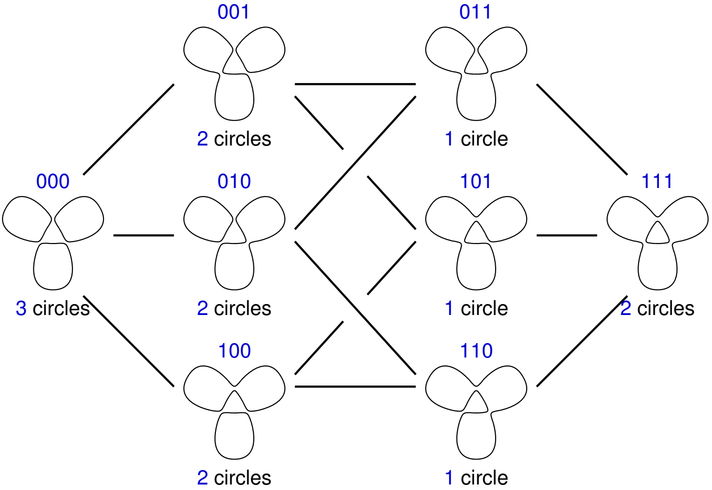
- We specify *resolutions* of a knot diagram.
- Take a knot. Enumerate its crossings.
- 010 resolution.
- Any triple $\{0, 1\}^3$ gives a resolution.



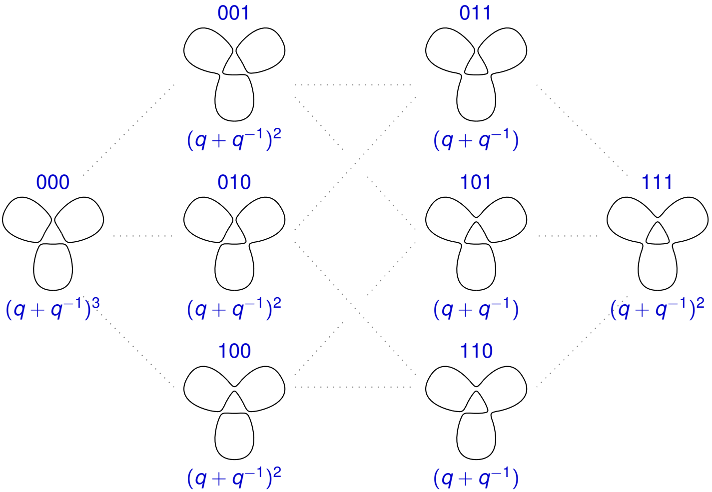
Cube of resolution



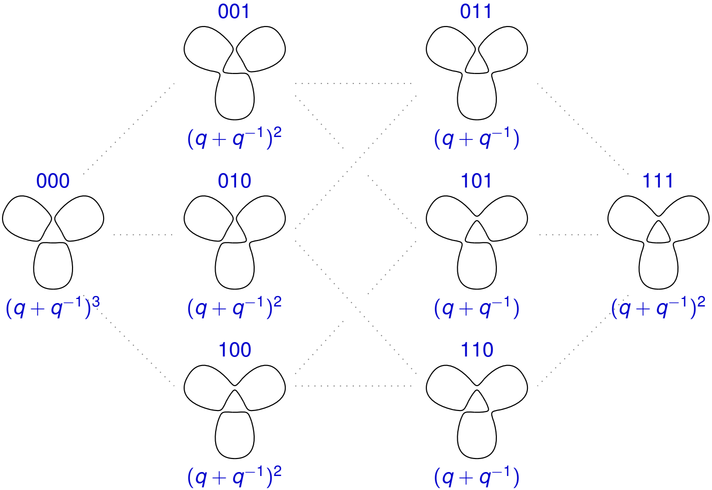
Cube of resolution



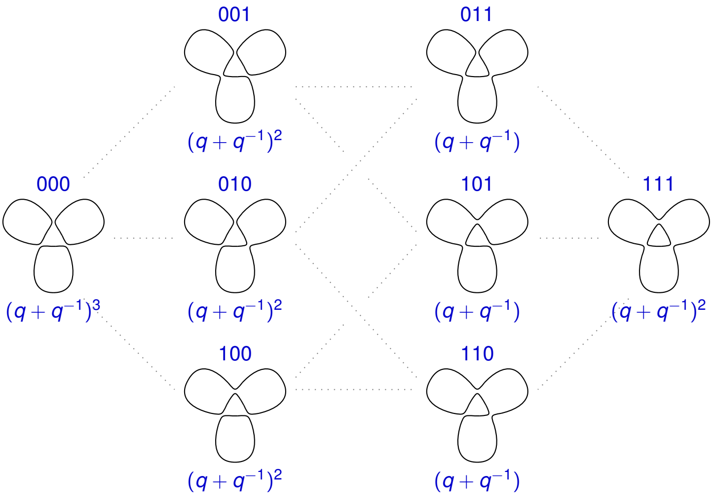
Cube of resolution



Cube of resolution

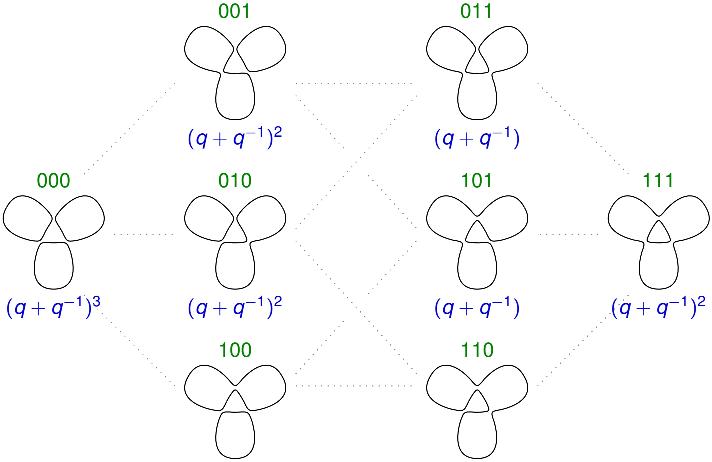


Cube of resolution



$(q+q^{-1})^3$ $\quad 3(q+q^{-1})^2$ $\quad 3(q+q^{-1})$ $\quad (q+q^{-1})^2$

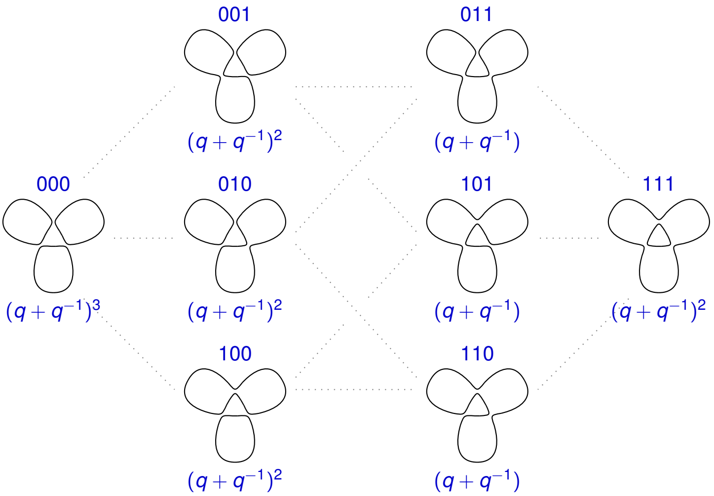
Cube of resolution



$(q+q^{-1})^2$ (001), $(q+q^{-1})$ (011), $(q+q^{-1})^3$ (000), $(q+q^{-1})^2$ (010), $(q+q^{-1})$ (101), $(q+q^{-1})^2$ (111), $(q+q^{-1})^2$ (100), $(q+q^{-1})$ (110)

$(q+q^{-1})^3q^0 \quad 3(q+q^{-1})^2q^1 \qquad 3(q+q^{-1})q^2 \quad (q+q^{-1})^2q^3$

Cube of resolution



$$(q+q^{-1})^3q^0 - 3(q+q^{-1})^2q^1 + 3(q+q^{-1})q^2 - (q+q^{-1})^2q^3$$

We have

$$(q^{-1}+q)^3 - 3q(q^{-1}+q)^2 + 3q^2(q^{-1}+q) - q^3(q^{-1}+q) = \\ -q^6(q^{-2}-q^{-3}+q^{-4}-q^{-9})$$

We have

$$(q^{-1}+q)^3 - 3q(q^{-1}+q)^2 + 3q^2(q^{-1}+q) - q^3(q^{-1}+q) = \\ -q^6(q^{-2}-q^{-3}+q^{-4}-q^{-9})$$

In this way we obtain the Jones polynomial for the (negative) trefoil. Factor $-q^{-6}$ is a normalization.

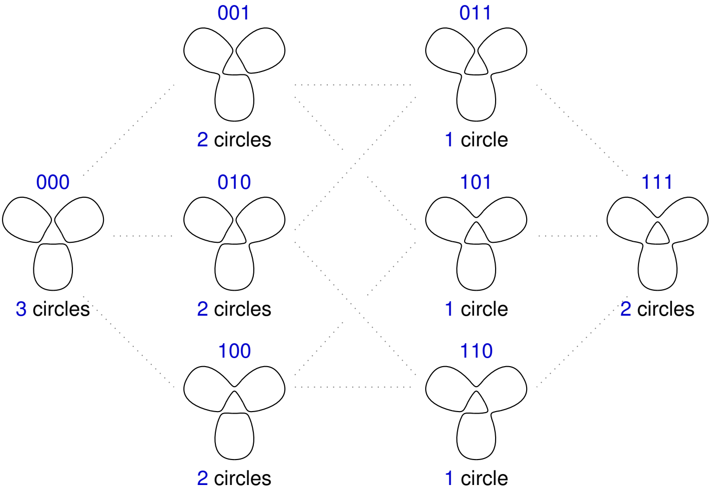
Khovanov's approach

Khovanov's approach

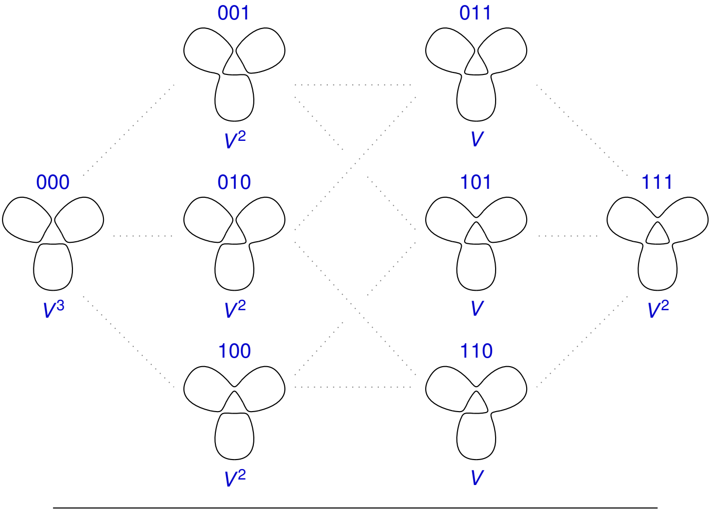
Main Idea

Replace factor $q + q^{-1}$ in the cube of resolution by a two-dimensional vector space V.

Khovanov's approach



Khovanov's approach

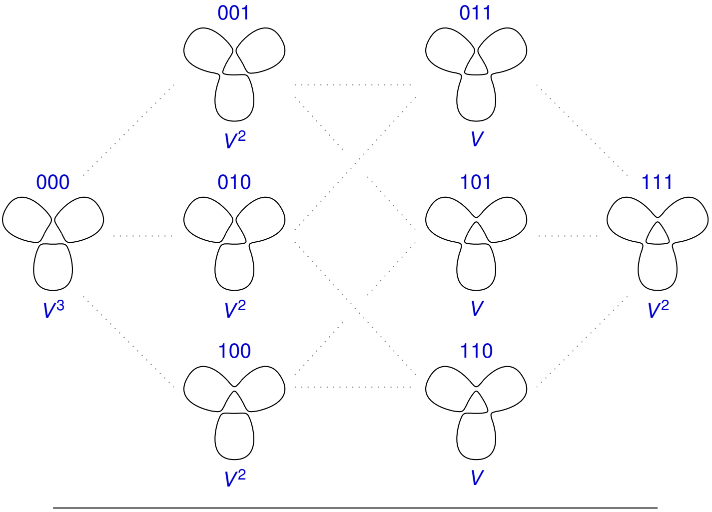


Khovanov's approach

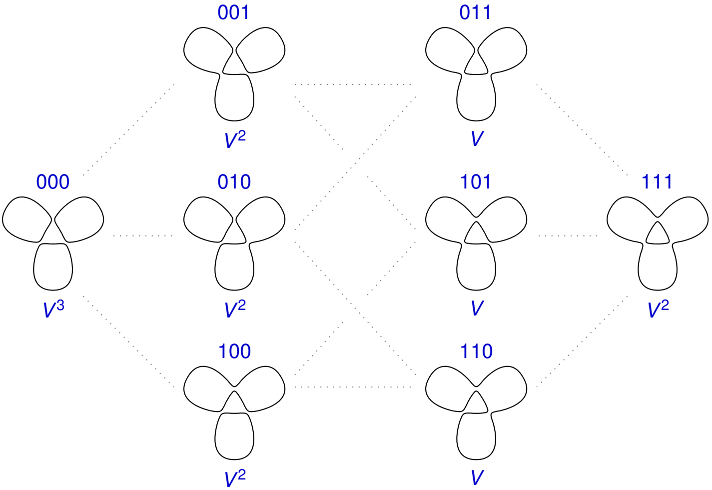
Explanation

The meaning of V^3 *is the tensor product. An element in* V^3 *is a linear combination of triples* (a, b, c) *(written usually* $a \otimes b \otimes c$*). We have* $a_1 \otimes b \otimes c + a_2 \otimes b \otimes c = (a_1 + a_2) \otimes b \otimes c$*, but not* $a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 = (a_1 + a_2) \otimes (b_1 + b_2) \otimes (c_1 + c_2)$*.* $\dim V^{\otimes 3} = (\dim V)^3$ *and not* $3 \dim V$*!*

Khovanov's approach

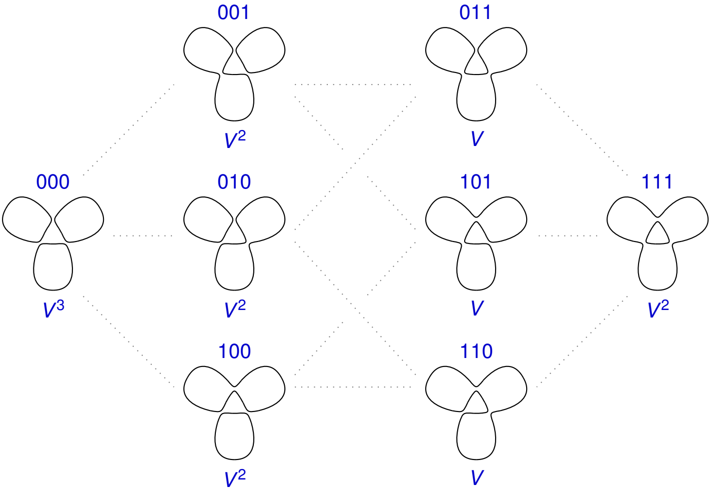


Khovanov's approach



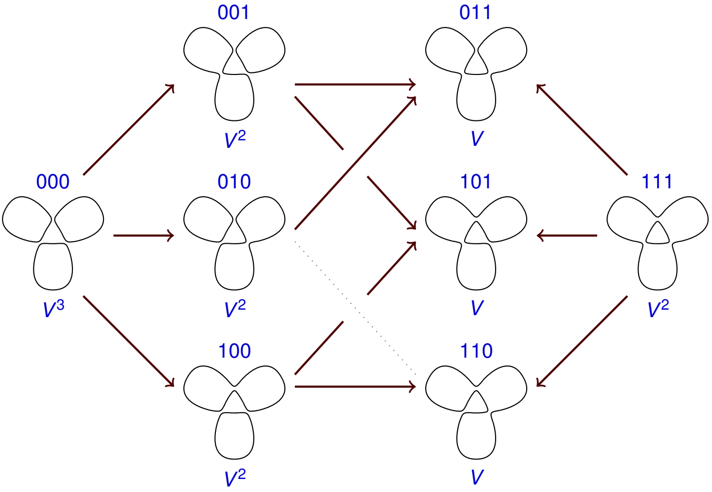
$V^{\otimes 3}$ $\quad$ $V^{\otimes 2} \oplus V^{\otimes 2} \oplus V^{\otimes 2}$ $\quad$ $V \oplus V \oplus V$ $\quad$ $V^{\otimes 2}$

Khovanov's approach



$V^{\otimes 3}$? $V^{\otimes 2} \oplus V^{\otimes 2} \oplus V^{\otimes 2}$? $V \oplus V \oplus V$? $V^{\otimes 2}$

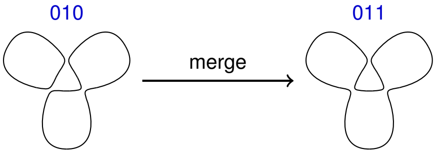
Khovanov's approach



$V^{\otimes 3}$? $V^{\otimes 2} \oplus V^{\otimes 2} \oplus V^{\otimes 2}$? $V \oplus V \oplus V$? $V^{\otimes 2}$

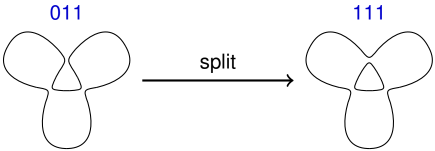
Maps in Khovanov's approach

- An arrow can either merge two circles into one.



Maps in Khovanov's approach

- An arrow can either merge two circles into one.
- Or split one into two circles.



Maps in Khovanov's approach

- An arrow can either merge two circles into one.
- Or split one into two circles.
- In the first case we need a map $V \otimes V \to V$.

Maps in Khovanov's approach

- An arrow can either merge two circles into one.
- Or split one into two circles.
- In the first case we need a map $V \otimes V \to V$.
- In the second case we need a map $V \to V \otimes V$.

Maps in Khovanov's approach

- An arrow can either merge two circles into one.
- Or split one into two circles.
- In the first case we need a map $V \otimes V \to V$.
- In the second case we need a map $V \to V \otimes V$.
- Without extra structure, it is hard to define such maps consistently.

- Think of V as a space of affine functions $ax + b$ with $a, b \in \mathbb{Z}$.

Maps in Khovanov homology

- Think of V as a space of affine functions $ax + b$ with $a, b \in \mathbb{Z}$.
- The map $V \otimes V \to V$ is the linear part of the product: $1 \otimes 1 \mapsto 1$, $x \otimes 1$, $1 \otimes x \mapsto x$, $x \otimes x \mapsto 0$.

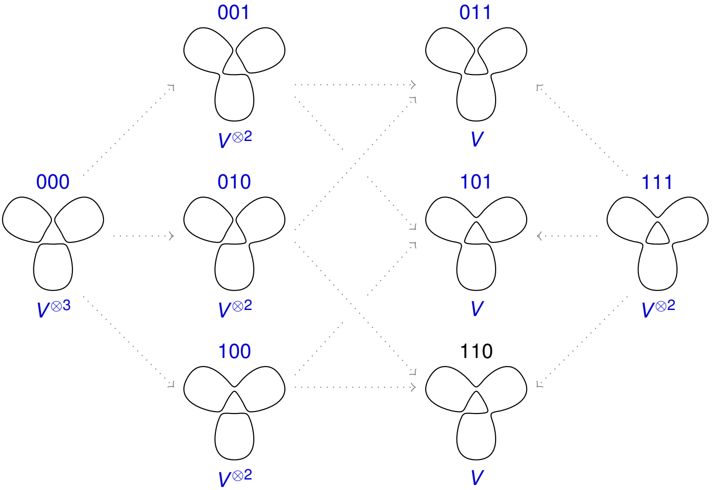
Maps in Khovanov homology

- Think of V as a space of affine functions $ax + b$ with $a, b \in \mathbb{Z}$.
- The map $V \otimes V \to V$ is the linear part of the product: $1 \otimes 1 \mapsto 1$, $x \otimes 1$, $1 \otimes x \mapsto x$, $x \otimes x \mapsto 0$.
- The map from $V \to V \otimes V$ 'copies' the function on the generators: $x \mapsto x \otimes x$, $1 \mapsto 1 \otimes 1$.

Maps in Khovanov homology

- Think of V as a space of affine functions $ax + b$ with $a, b \in \mathbb{Z}$.
- The map $V \otimes V \to V$ is the linear part of the product: $1 \otimes 1 \mapsto 1$, $x \otimes 1$, $1 \otimes x \mapsto x$, $x \otimes x \mapsto 0$.
- The map from $V \to V \otimes V$ 'copies' the function on the generators: $x \mapsto x \otimes x$, $1 \mapsto 1 \otimes 1$.
- Combining these maps (and after some sign adjustments) we obtain maps replacing $+$ and $-$ signs.

Global maps. Revised



$$V^{\otimes 3} \xrightarrow{d_0} V^{\otimes 2} \oplus V^{\otimes 2} \oplus V^{\otimes 2} \xrightarrow{d_1} V \oplus V \oplus V \xrightarrow{d_2} V^{\otimes 2}$$

Khovanov invariant

Theorem (Khovanov 2000)

The maps d_0, d_1 *and* d_2 *satisfy* $d_2 \circ d_1 = 0$ *and* $d_1 \circ d_0 = 0$. *The abelian groups* $\ker d_i / \operatorname{im} d_{i-1}$ *are independent of the knot diagram.*

Theorem (Khovanov 2000)

The maps d_0, d_1 *and* d_2 *satisfy* $d_2 \circ d_1 = 0$ *and* $d_1 \circ d_0 = 0$. *The abelian groups* $\ker d_i / \operatorname{im} d_{i-1}$ *are independent of the knot diagram.*

Remark

In mathematics, a sequence of vector spaces $V_0, \dots, V_s$ *together with linear maps* $d_i \colon V_i \to V_{i+1}$ *satisfying* $d_i \circ d_{i-1} = 0$ *for all* i *is called a cochain complex. The groups* $\ker d_i / \operatorname{im} d_{i-1}$ *are called cohomology groups.*

Khovanov invariant

Theorem (Khovanov 2000)

The maps d_0, d_1 and d_2 satisfy $d_2 \circ d_1 = 0$ and $d_1 \circ d_0 = 0$. The abelian groups $\ker d_i / \operatorname{im} d_{i-1}$ are independent of the knot diagram.

Remark

In mathematics, a sequence of vector spaces $V_0, \dots, V_s$ together with linear maps $d_i \colon V_i \to V_{i+1}$ satisfying $d_i \circ d_{i-1} = 0$ for all i is called a cochain complex. The groups $\ker d_i / \operatorname{im} d_{i-1}$ are called cohomology groups.

Yes, I know, saying ‘a vector space over $\mathbb{Z}$’ is an abuse.

Properties of Khovanov invariant

- Detects the unknot (Kronheimer, Mrowka 2011).

Properties of Khovanov invariant

- Detects the unknot (Kronheimer, Mrowka 2011).
- Detects the Hopf link and the trefoil.

Properties of Khovanov invariant

- Detects the unknot (Kronheimer, Mrowka 2011).
- Detects the Hopf link and the trefoil.
- Specifies to and generalizes the Jones polynomial.

Properties of Khovanov invariant

- Detects the unknot (Kronheimer, Mrowka 2011).
- Detects the Hopf link and the trefoil.
- Specifies to and generalizes the Jones polynomial.
- Can be used to prove the Milnor's conjecture (on the unknotting number of torus knots).

Properties of Khovanov invariant

- Detects the unknot (Kronheimer, Mrowka 2011).
- Detects the Hopf link and the trefoil.
- Specifies to and generalizes the Jones polynomial.
- Can be used to prove the Milnor's conjecture (on the unknotting number of torus knots).
- Computational complexity is daunting.

Making it better

- We said that Khovanov invariant is cohomology of a chain complex.

Making it better

- We said that Khovanov invariant is cohomology of a chain complex.
- Many people are familiar with cohomology of topological spaces.

Making it better

- We said that Khovanov invariant is cohomology of a chain complex.
- Many people are familiar with cohomology of topological spaces.

Question

Given a knot K can one construct a topological space X such that the cohomology of X is the Khovanov invariant of K? Is there a consistent construction?

Lipsitz and Sarkar construction

- First construction of Khovanov homotopy type using flow categories and Cohen-Jones-Segal (2012).

Lipsitz and Sarkar construction

- First construction of Khovanov homotopy type using flow categories and Cohen-Jones-Segal (2012).
- New invariants of knots coming from cohomological operations (2013).

Lipsitz and Sarkar construction

- First construction of Khovanov homotopy type using flow categories and Cohen-Jones-Segal (2012).
- New invariants of knots coming from cohomological operations (2013).
- Another construction of flow categories using cubical flow categories and Burnside categories (2014, jointly with Lawson).

Lipsitz and Sarkar construction

- First construction of Khovanov homotopy type using flow categories and Cohen-Jones-Segal (2012).
- New invariants of knots coming from cohomological operations (2013).
- Another construction of flow categories using cubical flow categories and Burnside categories (2014, jointly with Lawson).
- Invited to the ICM in 2018.

Periodic knots

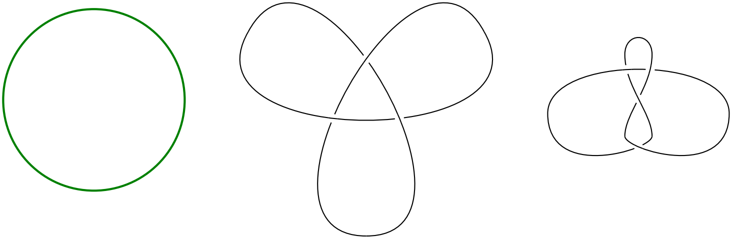
- A knot is p-periodic if it admits a diagram invariant under rotation by $\mathbb{Z}/p$.

Periodic knots

- A knot is p-periodic if it admits a diagram invariant under rotation by $\mathbb{Z}/p$.
- Which knots are p-periodic?

Periodic knots

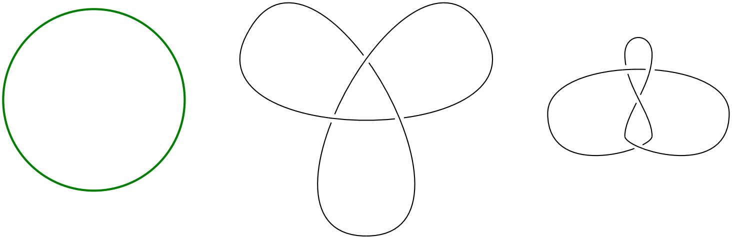
- A knot is p-periodic if it admits a diagram invariant under rotation by $\mathbb{Z}/p$.
- Which knots are p-periodic?



Periodic knots

- A knot is p-periodic if it admits a diagram invariant under rotation by $\mathbb{Z}/p$.
- Which knots are p-periodic?

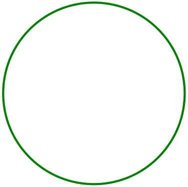
periodic



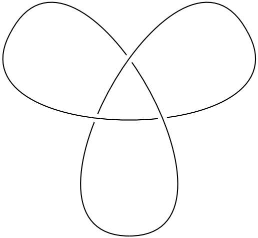
Periodic knots

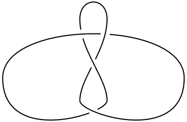
- A knot is p-periodic if it admits a diagram invariant under rotation by $\mathbb{Z}/p$.
- Which knots are p-periodic?

periodic



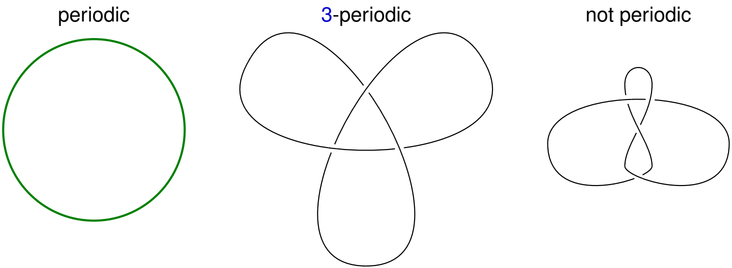
3-periodic





Periodic knots

- A knot is p-periodic if it admits a diagram invariant under rotation by $\mathbb{Z}/p$.
- Which knots are p-periodic?



Equivariant Khovanov invariants

- Politarczyk 2014: Construction of equivariant Khovanov invariants;

Equivariant Khovanov invariants

- Politarczyk 2014: Construction of equivariant Khovanov invariants;
- Politarczyk 2015: New periodicity criterion based on equivariant Jones polynomial;

Equivariant Khovanov invariants

- Politarczyk 2014: Construction of equivariant Khovanov invariants;
- Politarczyk 2015: New periodicity criterion based on equivariant Jones polynomial;
- Borodzik, Politarczyk 2018: Another, much stronger, periodicity criterion based on equivariant Khovanov invariants.

Equivariant Khovanov invariants

- Politarczyk 2014: Construction of equivariant Khovanov invariants;
- Politarczyk 2015: New periodicity criterion based on equivariant Jones polynomial;
- Borodzik, Politarczyk 2018: Another, much stronger, periodicity criterion based on equivariant Khovanov invariants.

Question

Does there exists equivariant Khovanov homotopy type?

Theorem (B. — Politarczyk — Silvero 2018, Stoffregen — Zhang 2018)

There exists equivariant Khovanov homotopy type.

Equivariant Khovanov homotopy type

Theorem (B. — Politarczyk — Silvero 2018, Stoffregen — Zhang 2018)

There exists equivariant Khovanov homotopy type.

BPS approach proves also that equivariant cohomology of this space is Politarczyk's equivariant Khovanov invariant.

Perspectives

- Construct HOMLYPT homotopy type;

Perspectives

- Construct HOMLYPT homotopy type;
- Construct a homotopy type that reflects and intertwines the quantum grading.

Perspectives

- Construct HOMLYPT homotopy type;
- Construct a homotopy type that reflects and intertwines the quantum grading.
- Understand, why Khovanov invariants work.

Perspectives

- Construct HOMLYPT homotopy type;
- Construct a homotopy type that reflects and intertwines the quantum grading.
- Understand, why Khovanov invariants work.
- Find a simpler way to calculate Khovanov invariants.