Morse theory for plane algebraic curves

Jaca, 2009

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Jaca, June 2009

Setup

Our setup is the following

- $C \subset \mathbb{C}^2$ algebraic curve
- Intersect C with a sphere S_r of radius r. Get a link L_r.
- Links for small r are understood.
- Links at infinity are understood.
- Study properties of C by these links.

What happens with L_r if we change r?

Motto

C introduces a "cobordism" between links of singular points and the link at infinity.

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Classical Morse theory

Classical arguments from Morse theory

Lemma

If for all $r \in [r_1, r_2]$, C is transverse to S_r, then L_{r_1} is isotopic to L_{r_2}.

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Crossing a non transversality point is either 0, or 1 or 2 – handle addition to the link.

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Handles

Adding handles mean

- 0-handles: adding an unknot to L_r.
- 2-handles: deleting an unknot to L_r.
- 1-handles: adding a band.

Lemma

If C is a complex curve, there are no 2-handles.

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What about singularities?

Crossing a singular point of multiplicity m can be viewed as follows

- Take a disconnected sum of L_r with a link of singularity...
- And then join them with precisely m one handles.

Example

Passing through a double point corresponds to changing an undercrossing to an overcrossing on some planar diagram of the link.

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Examples

Now, please, hold Your breath, I will try to show some real pictures.

Knot invariants

Take your favourite link invariant such that

- It is computable for many algebraic knots
- You can control its changes when adding a handle
- It is not too good. It is not equal to genus for positive links.

And this invariant yields obstruction for the existence of a plane curve with given singularities.

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Knot invariants II

My favourite invariant up to now is. . . is. . .

Tristram–Levine signature

Definition

If S is Seifert matrix of the link L and $|\zeta| = 1$, then $\sigma_L(\zeta)$ is the signature of the form

$$(1-\zeta)S + (1-\bar{\zeta})S^T.$$

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Corollaries

Theorem

If $L_1, \dots, L_n$ are links of singular points of C, L_∞ is a link at infinity, then for almost all ζ

$$\left| \sigma_{L_\infty}(\zeta) - \sum_{k=1}^{n} \sigma_{L_k}(\zeta) \right| \leq b_1(C),$$

where $b_1(C)$ is the first Betti number.

In the proof we use the absence of 2–handles, but this can be done in general, i.e. non-complex case, too (Kawauchi et al.)

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Applications

- A polynomial curve of bidegree (m, n), having an A_{2k} singularity at the origin, has $k \lesssim \frac{1}{4}mn$.
- If a knot bounds an algebraic rational curve, almost all its signatures must be non-positive.
- Find maximal number of cusps on a curve in $\mathbb{C}P^2$ of degree d. We reprove Varchenko's result $s(d) \lesssim \frac{23}{72}d^2$, which is very close to the best known $\frac{125+\sqrt{73}}{432}d^2$.
- Possible proof of Zajdenberg–Lin theorem using the fact that $b_1(C) = 0$ and relations among signatures of torus knots.
- BMY-like inequality for polynomial curves.
- Studying deformations of singular points: we get new relations.
- Possible ways to improve everything if we apply better invariants.

There is one more thing I'd like to say at the end.

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