Gromov—Witten invariants

introduction to results of A. Okounkov

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The moduli problem consist of finding a universal family of curves such that any other family is induced from $\mathcal{M}$ by a unique morphism b.

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- In most cases it does not exists in the category of schemes (need to use stacks)

Examples

$\mathcal{M}_{0,3}$ is obviously a point.

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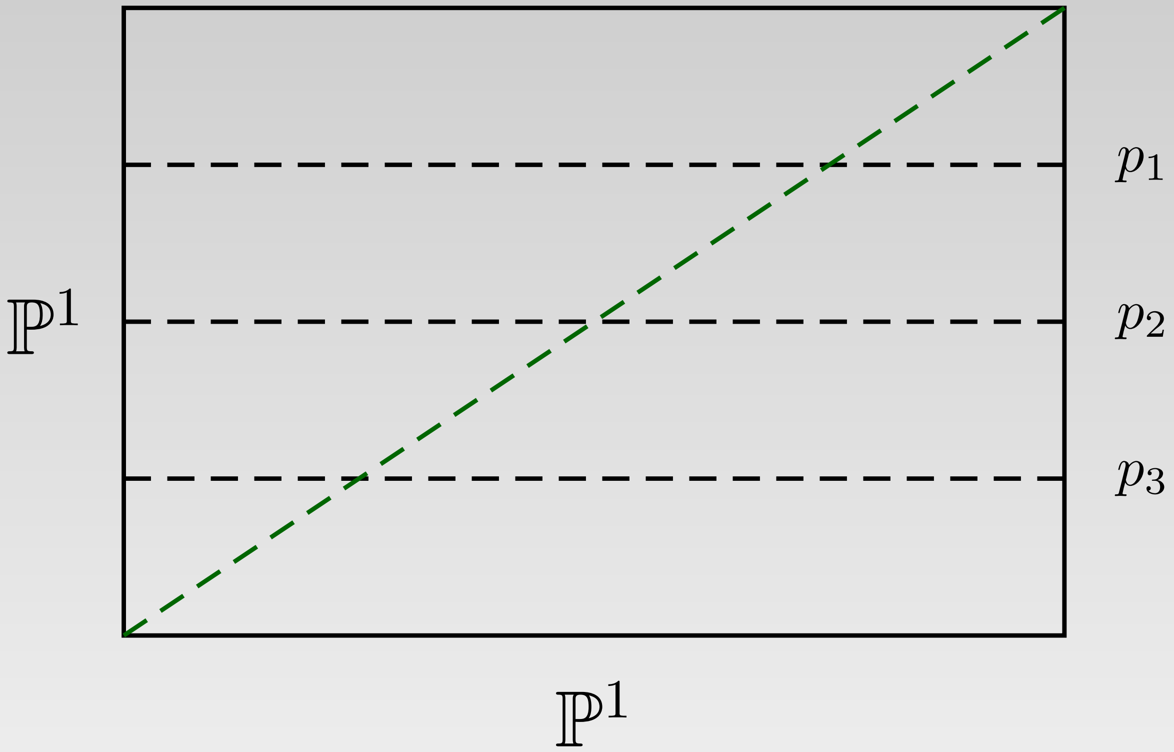
Now $\mathcal{M}_{0,4}$.



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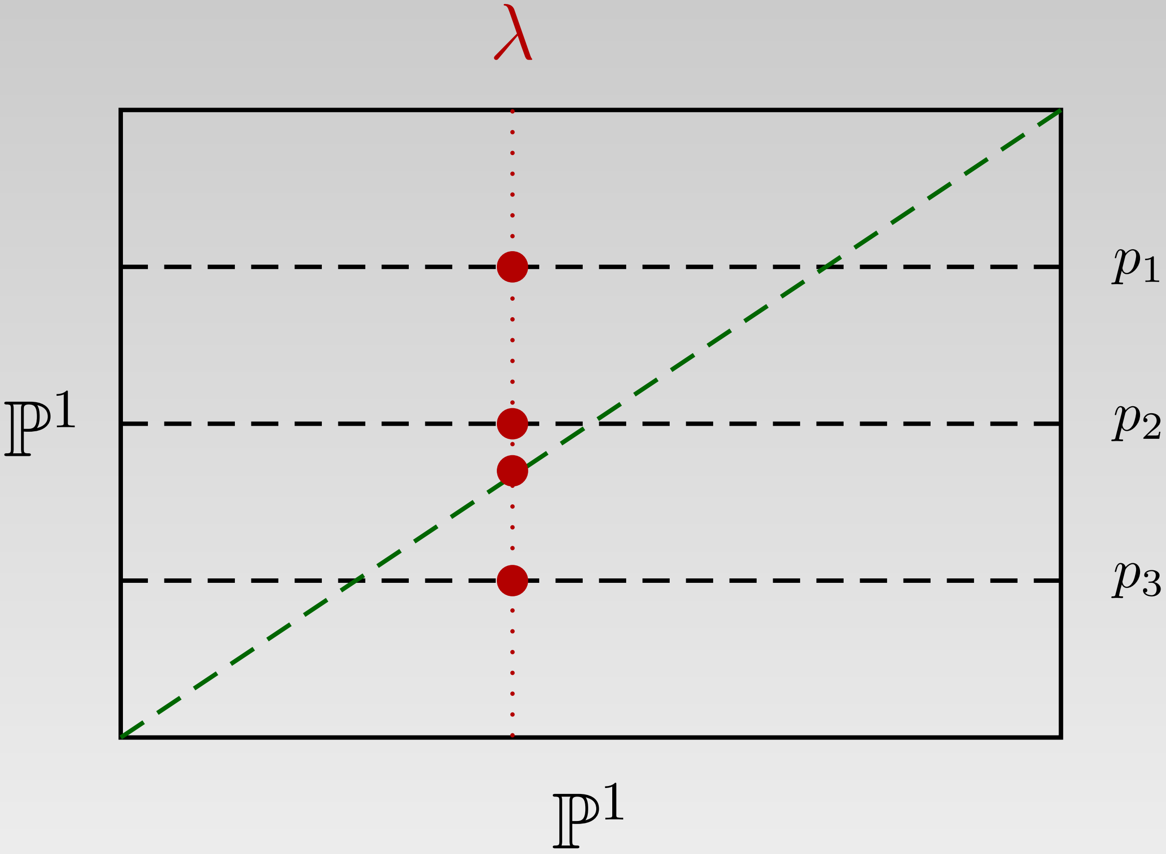

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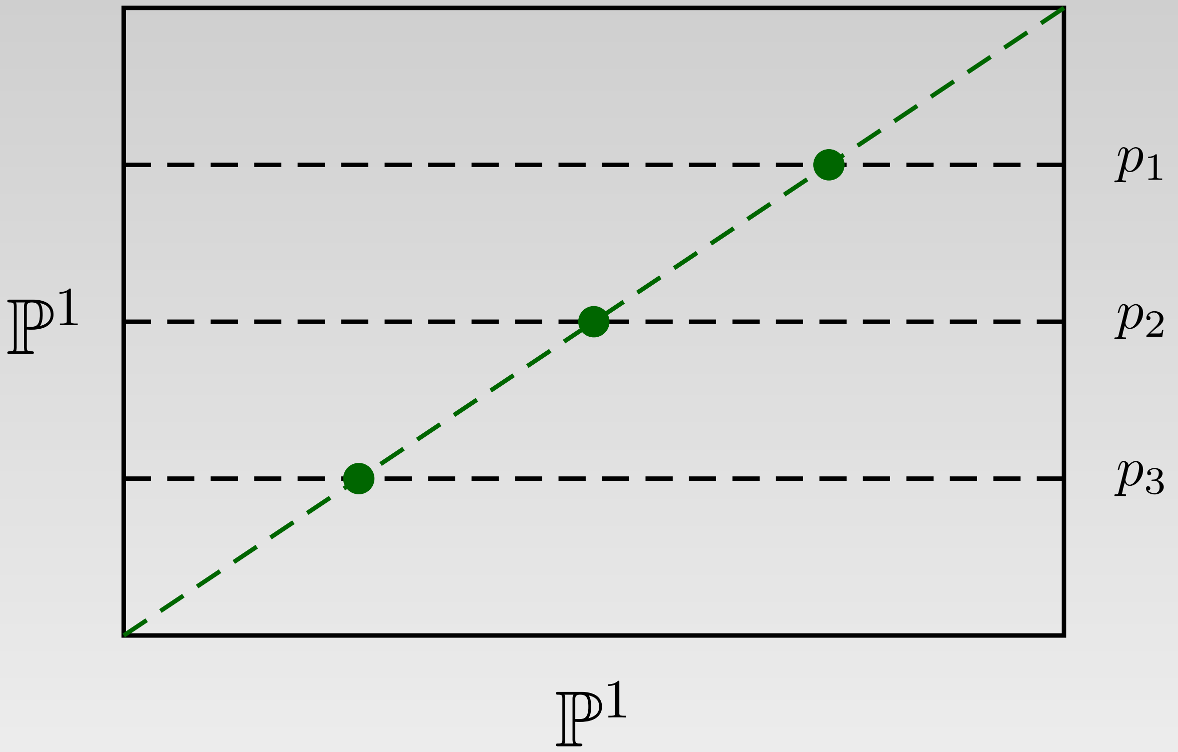

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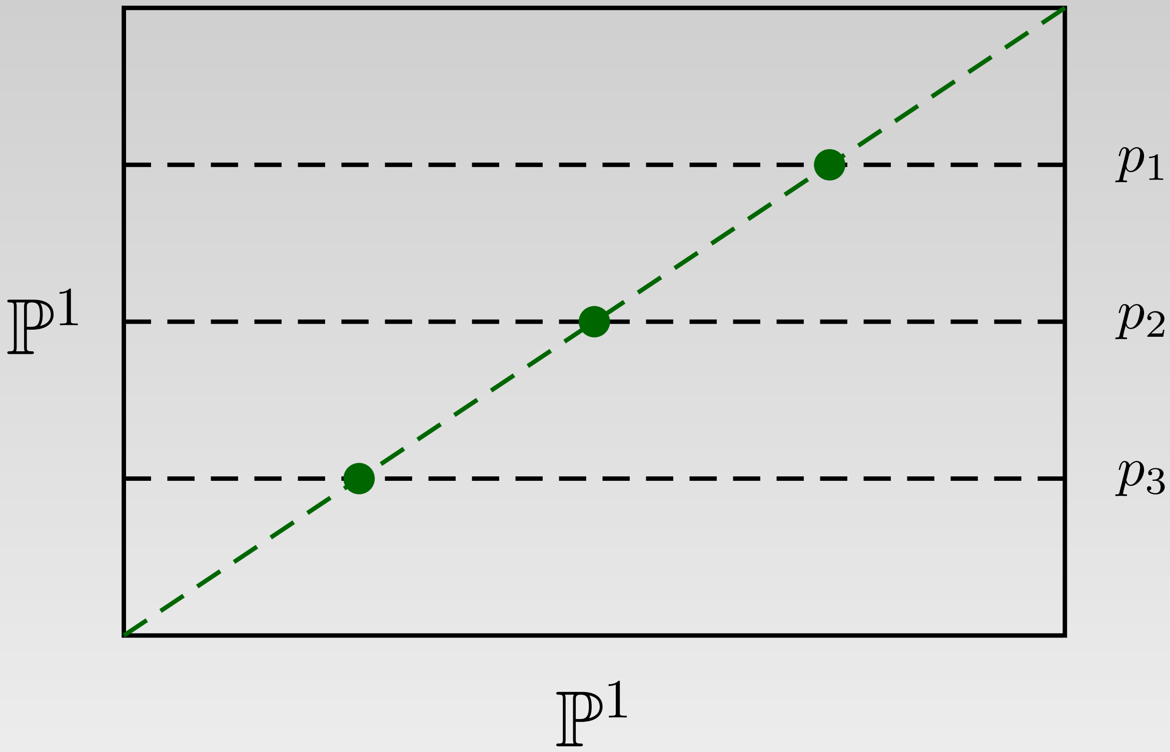

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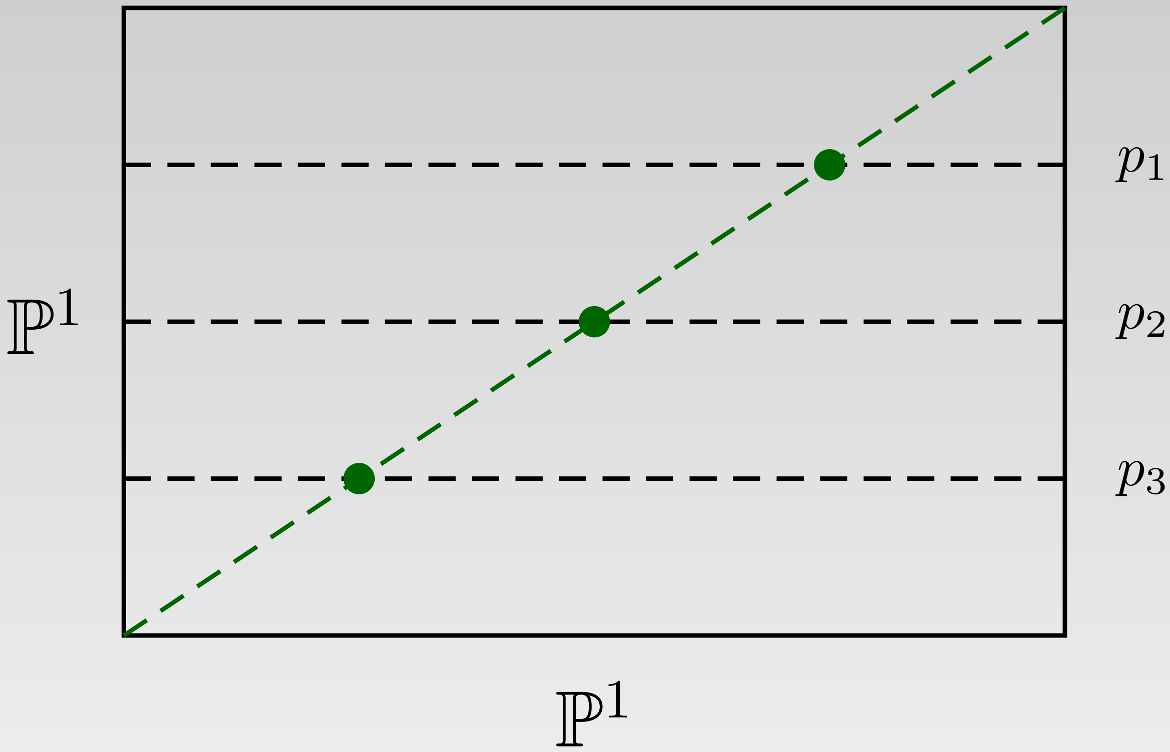

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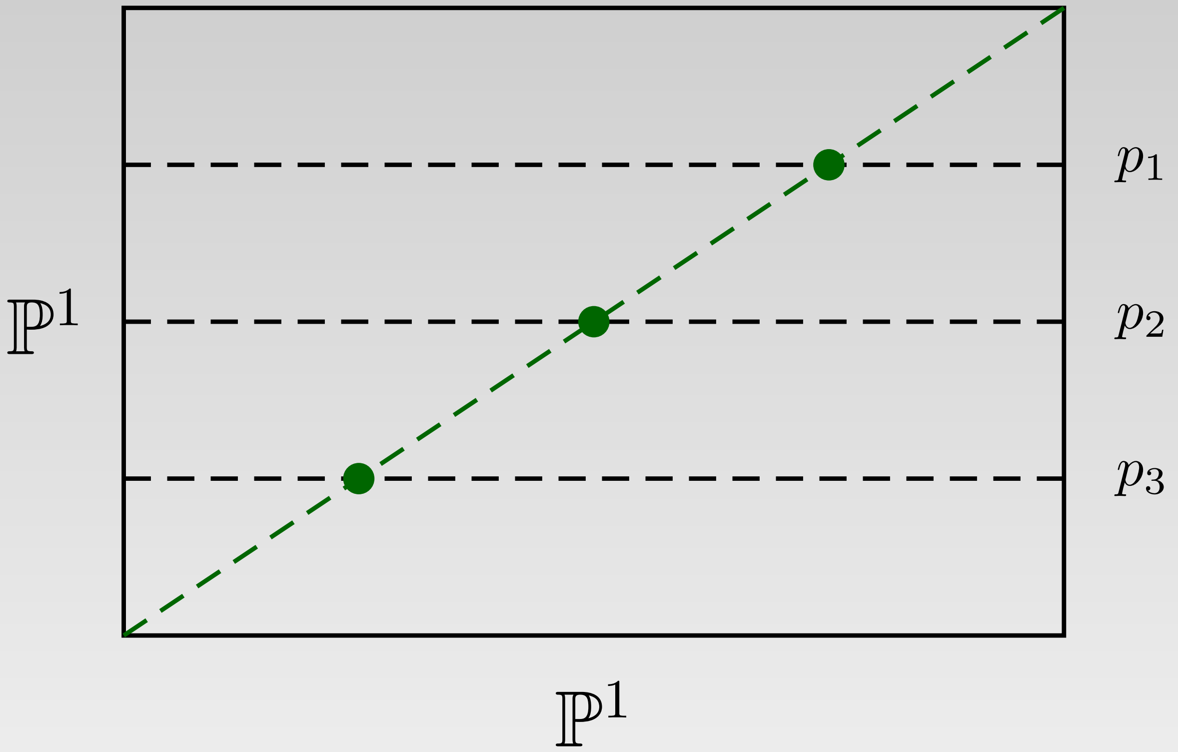

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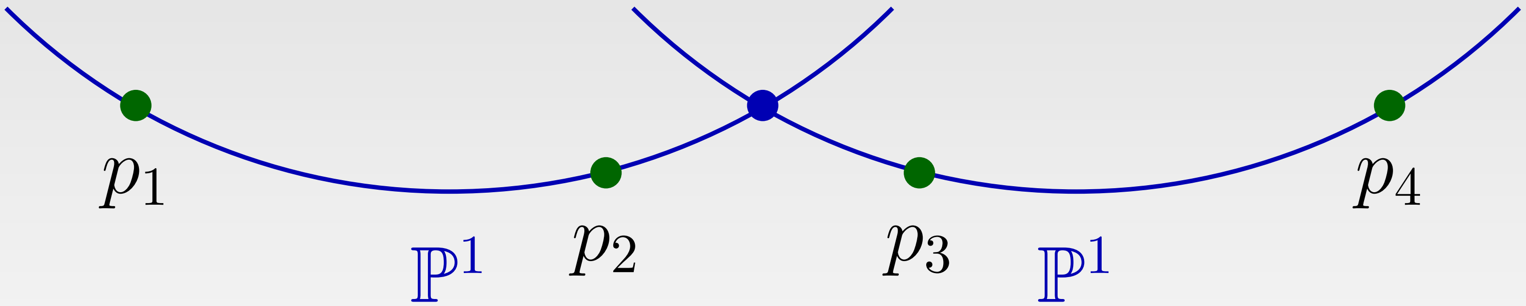

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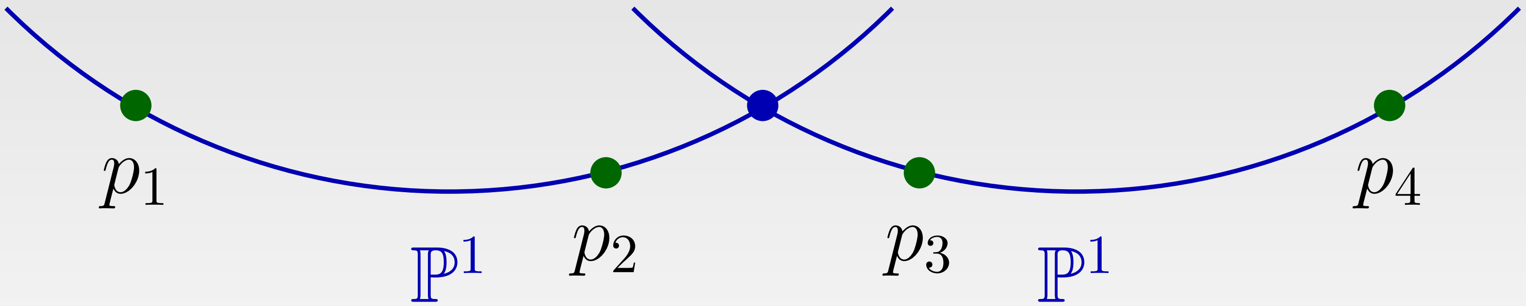

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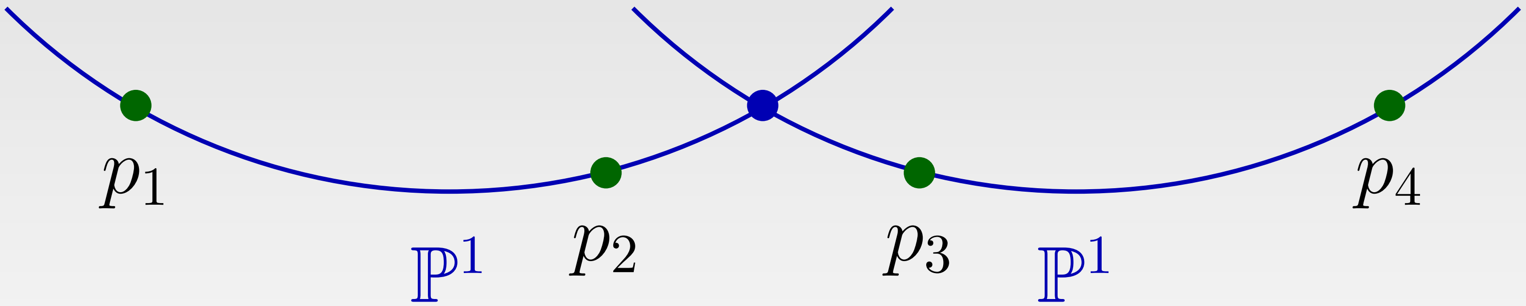

Important, albeit trivial! All divisors corresponding to different splittings are linearly equivalent.

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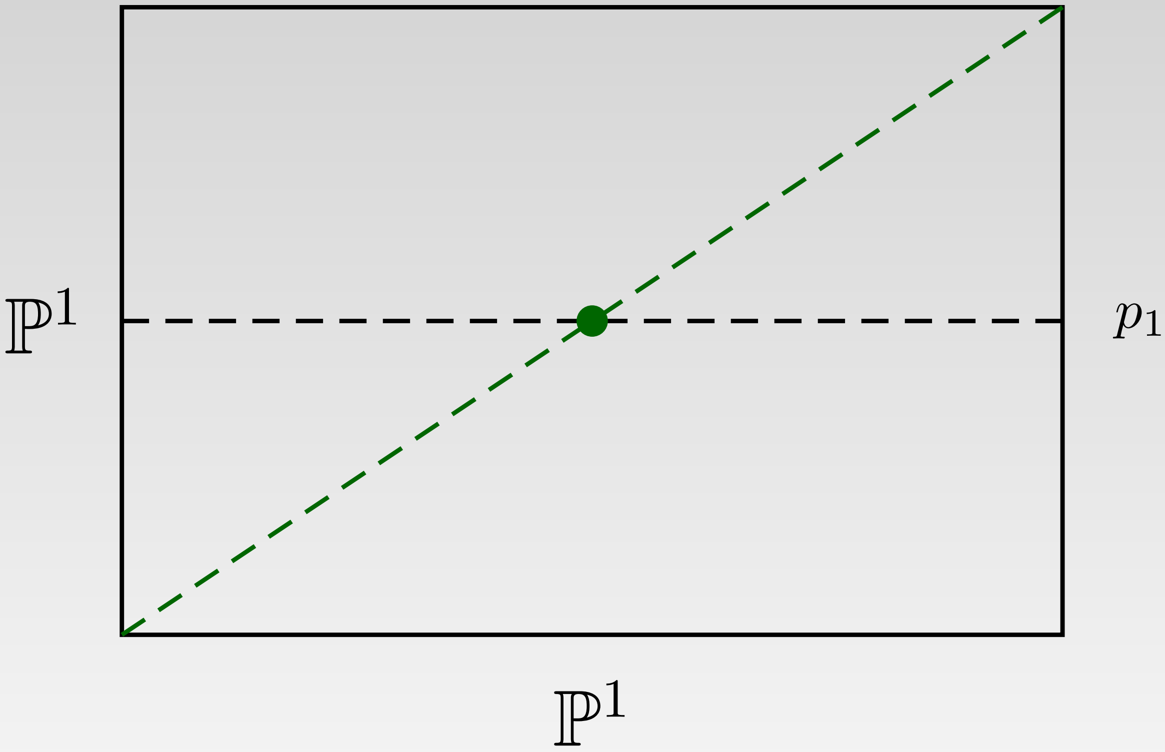

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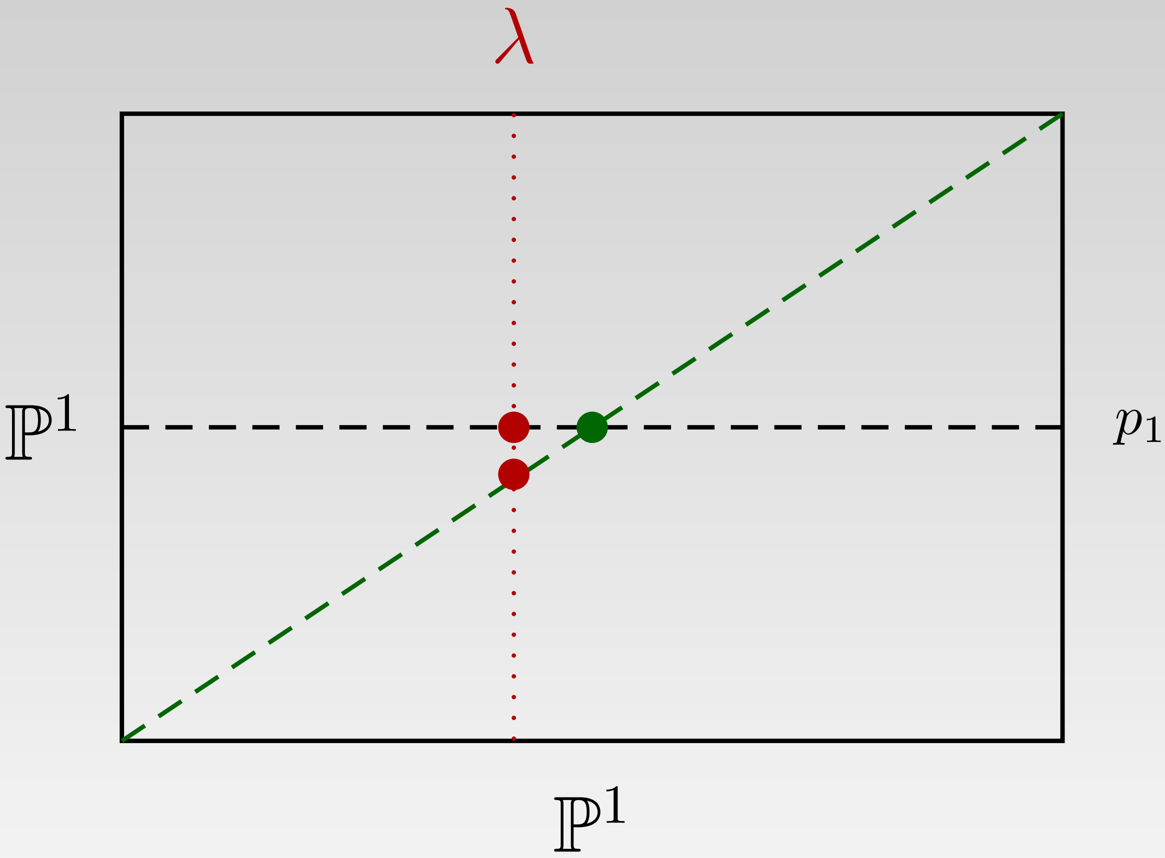

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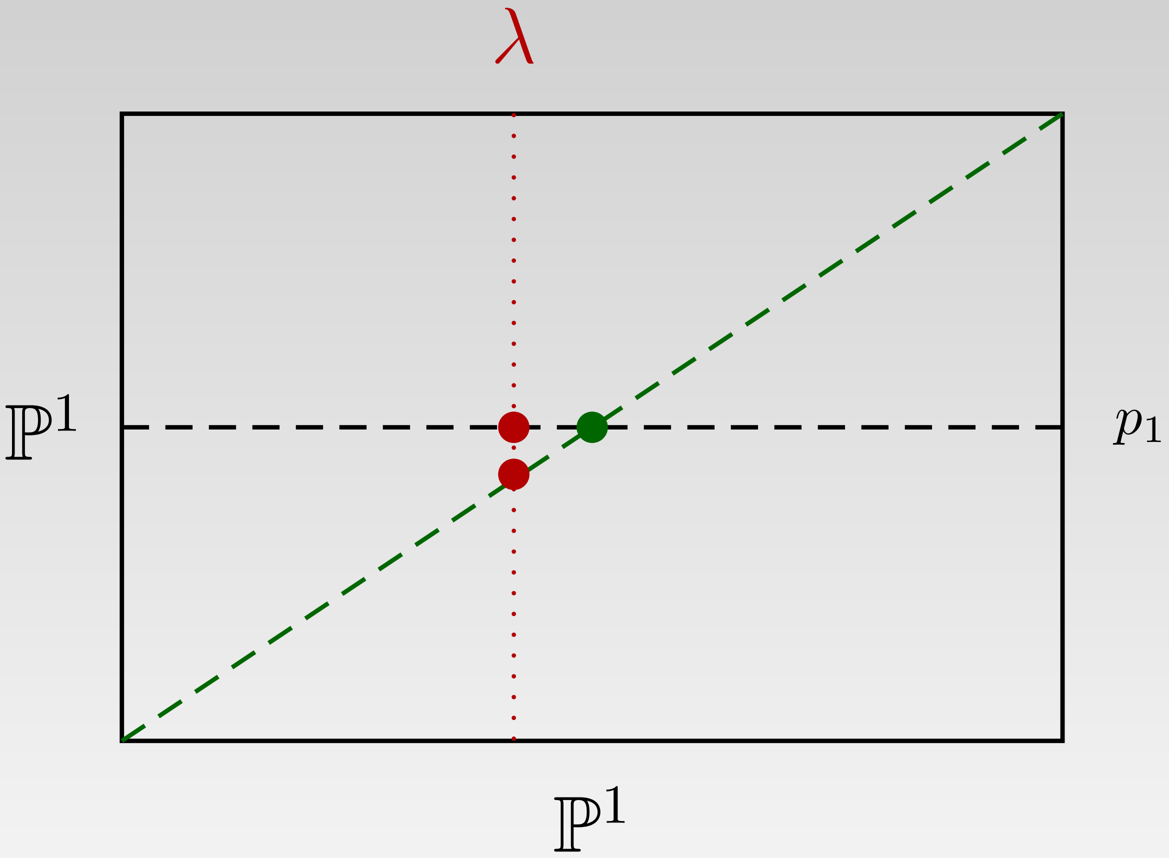

All curves corresponing to different values of $\lambda \neq p_1$ are equivalent.

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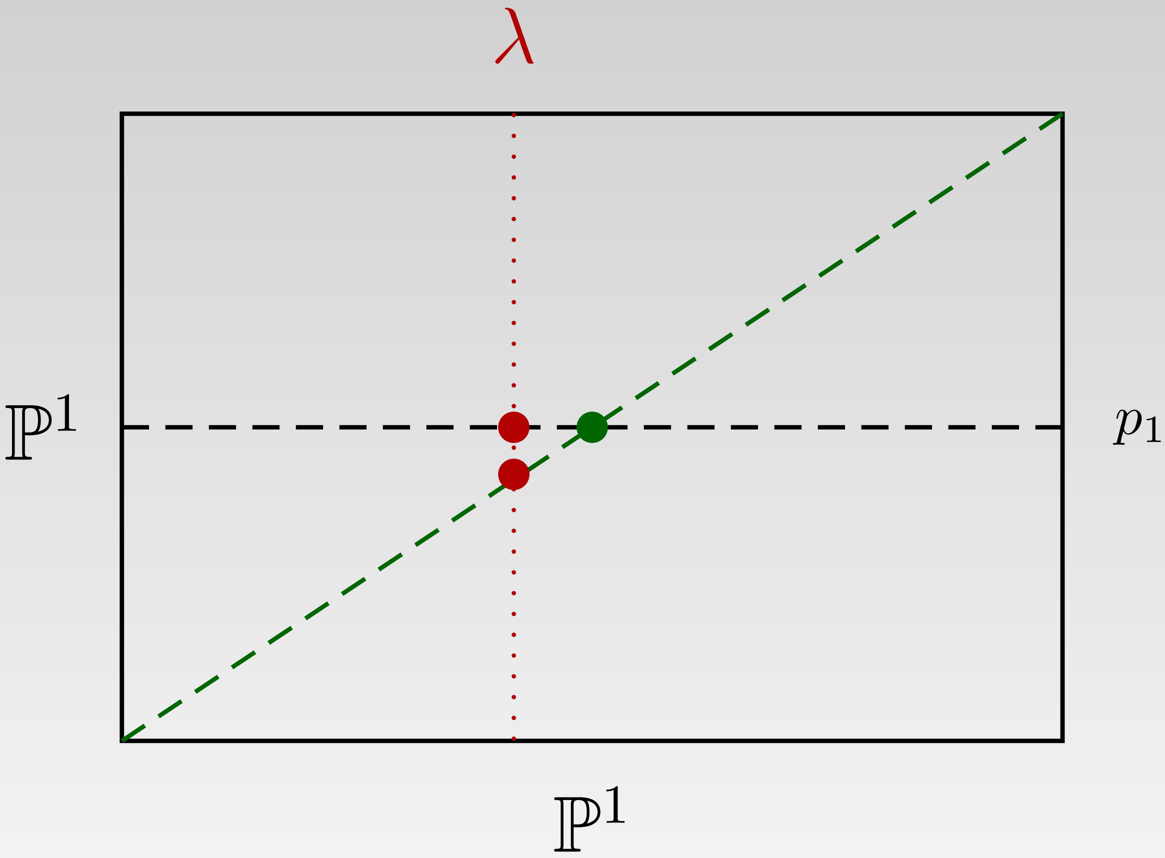

Blow up the green point. We get a family of identical curves tending to a different curve.

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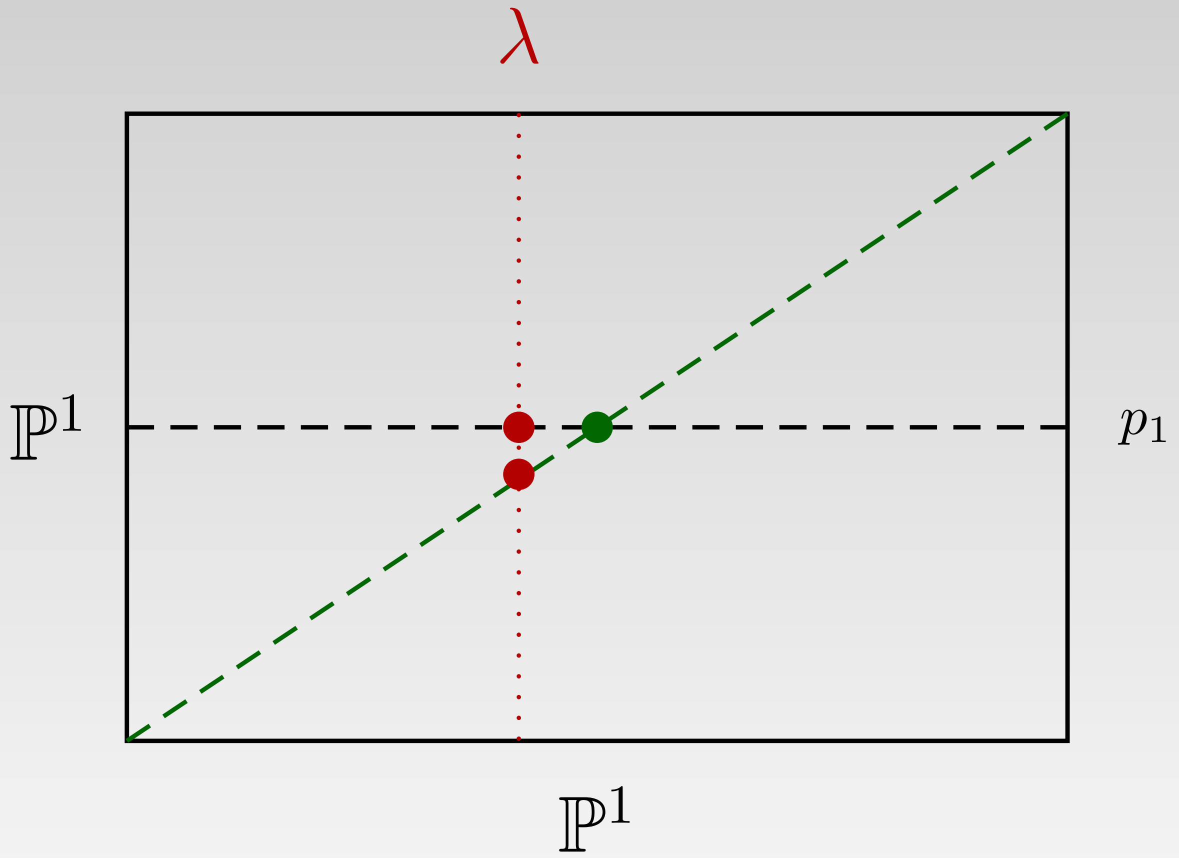

The space of curves with such topology is not Hausdorff.

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- Exercise. A smooth curve of genus g is stable if $3g - 3 + n \geq 0$.
- A curve is called nodal (or, misleadingly, cuspidal) if it has only double points as its singularities.

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$$\text{stability} \rightarrow \chi(C_0) < 0.$$

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- For $\delta > 0$ space $\mathcal{G}_\delta$ is compact (relatively easy).

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- The metrics are convergent on C_0^{thick}.
- Annulus. Small r_i — large modulus (ratio $\frac{R}{r}$).

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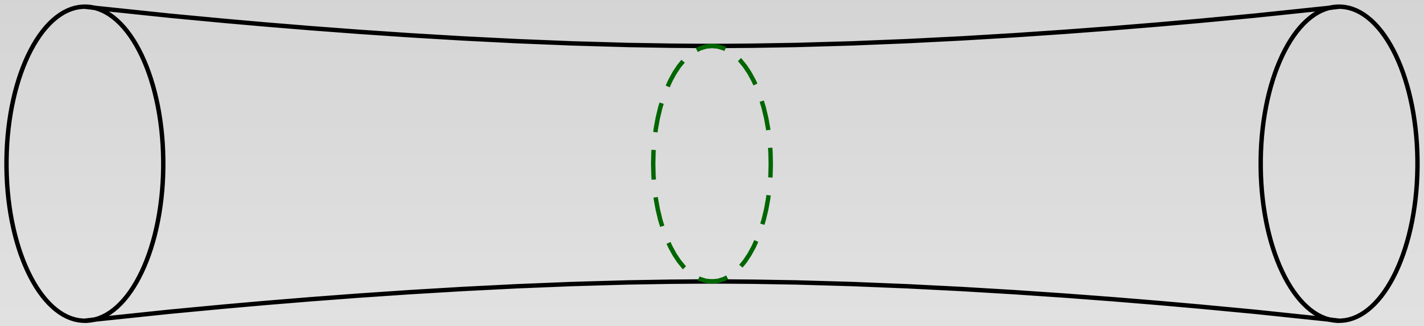

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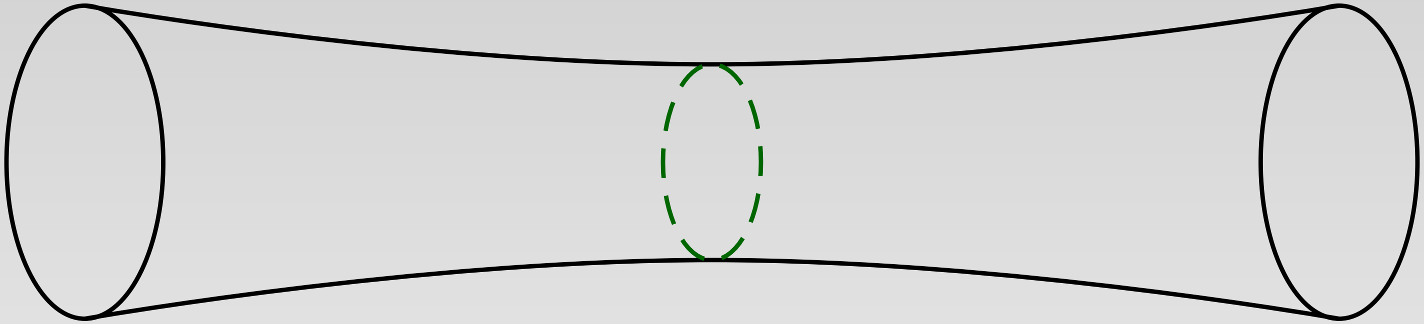

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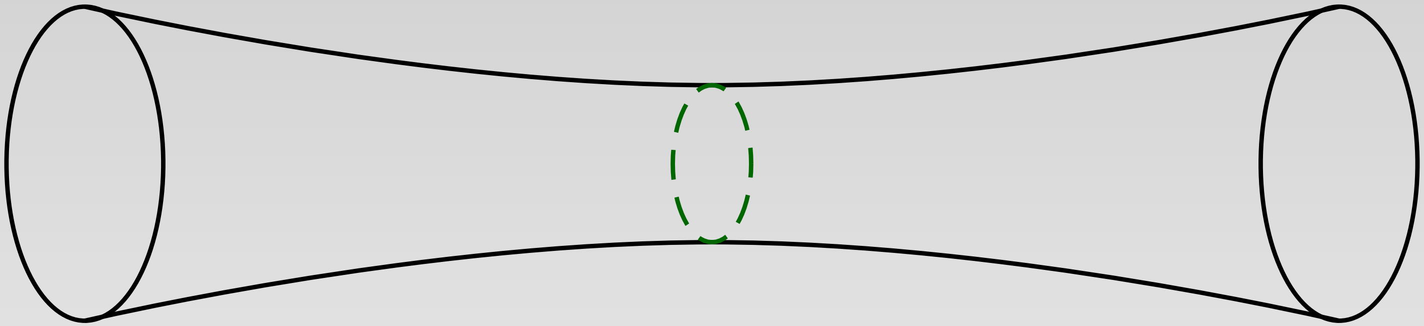

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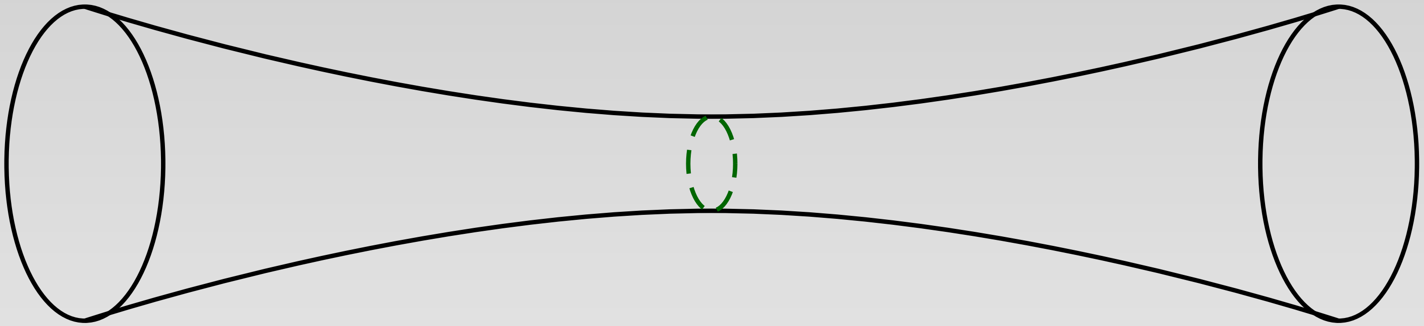

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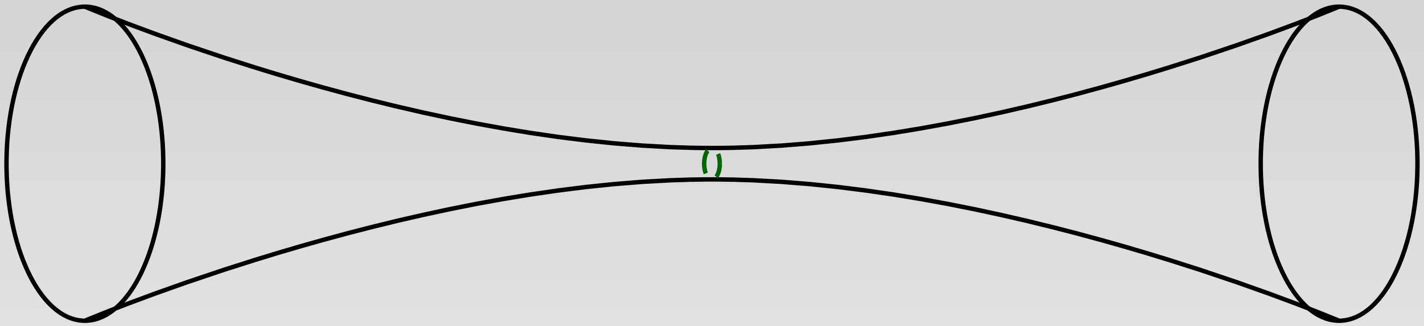

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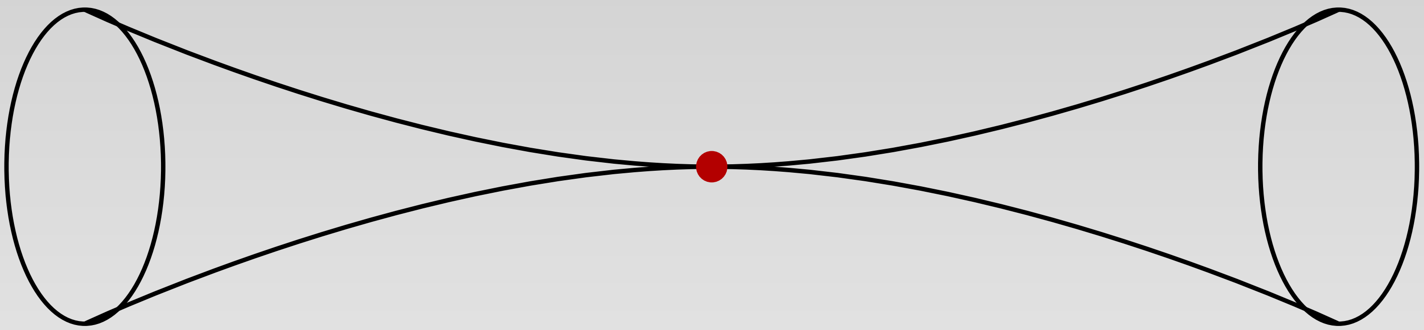

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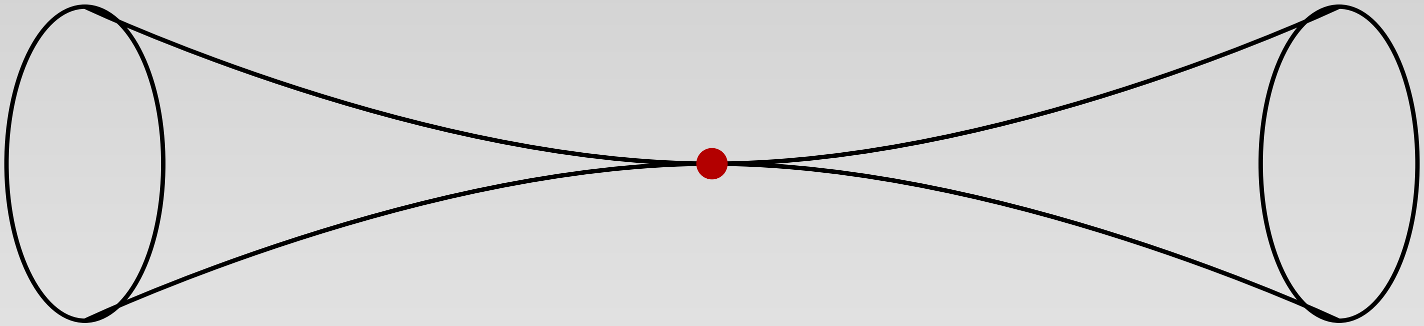

Annuli degenerate to two discs with one common point.

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- the orbifold is the most basic non–trivial example of a stack.

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- the group G is $Aut(C, \{p_1, \ldots, p_n\})$ generically trivial.

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- for $g=0$, G is always trivial, so $\bar{\mathcal{M}}_{g,n}$ is a smooth manifold.

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- Glueing map: $\bar{\mathcal{M}}_{g_1,I_1 \cup p} \times \bar{\mathcal{M}}_{g_2,I_2 \cup q} \to \mathcal{M}_{g,I}$. Glue p with q.

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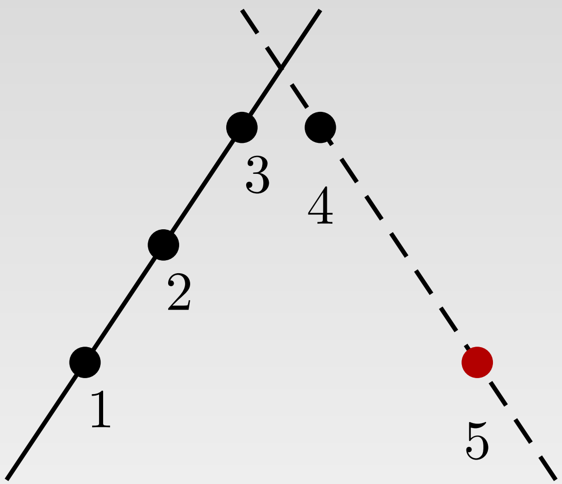

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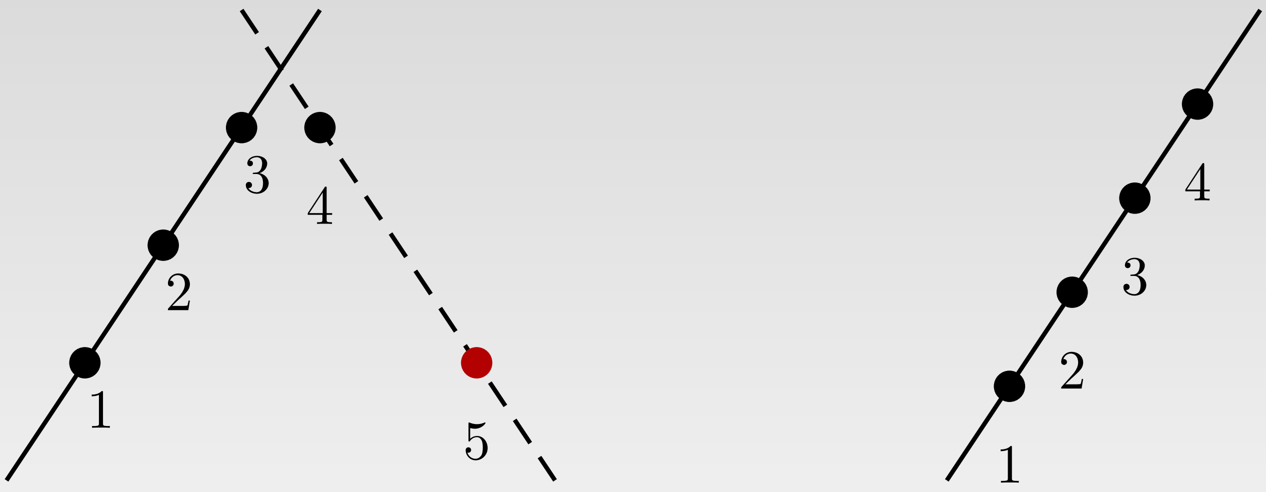

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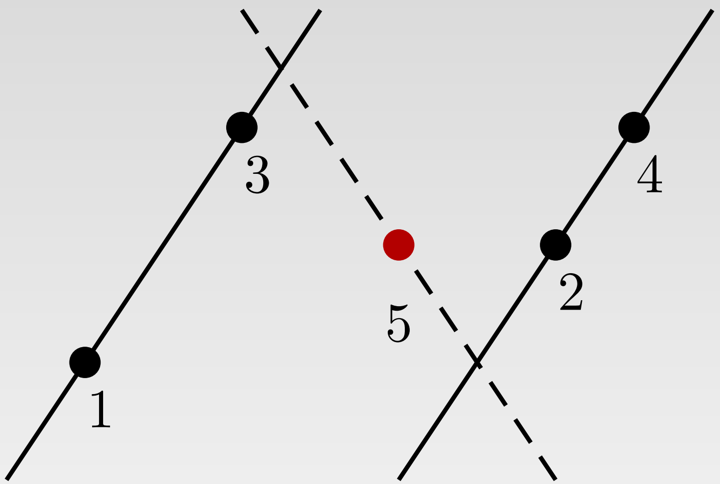

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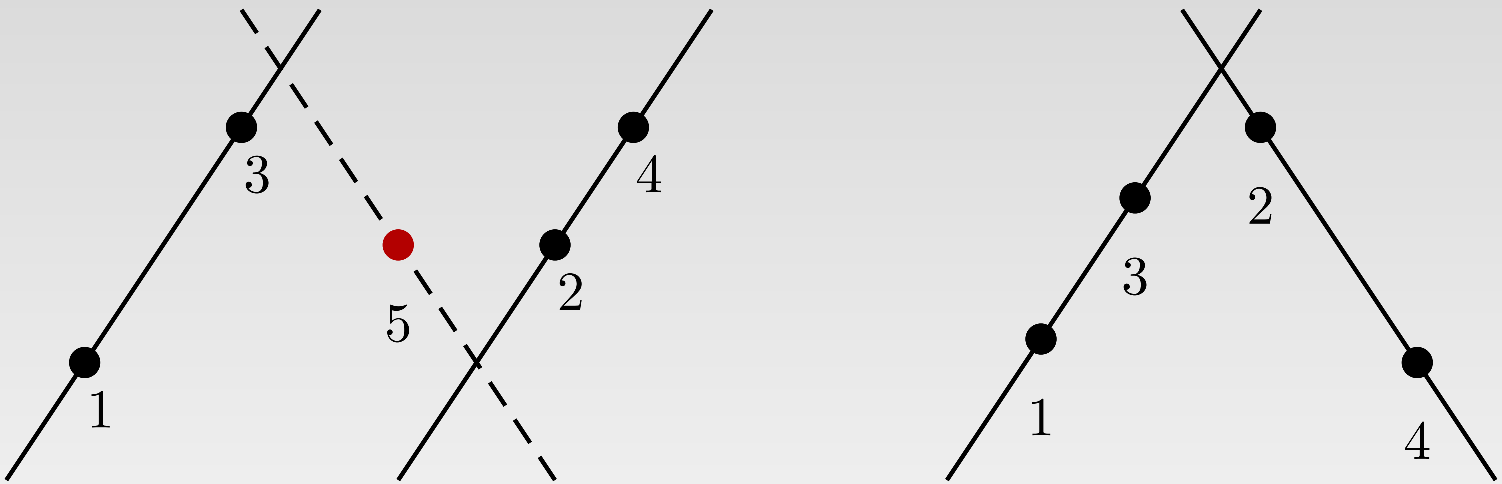

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- **Exercise** for $k = 1$ obtain KdV.

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