Algebraic unknotting number and 4-manifolds

joint with S. Friedl

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- u_a depends only on the Seifert matrix. For example, if $\Delta(K) \equiv 1$, then $u_a = 0$.

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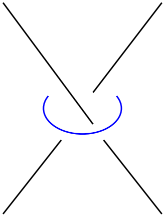

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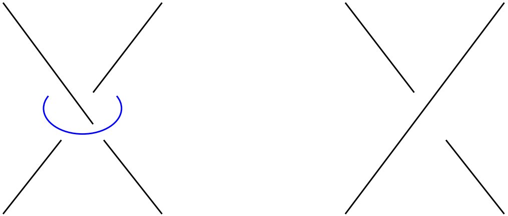

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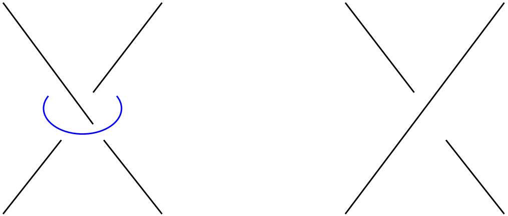

A surgery presentation is a collection of such circles and numbers ± 1, such that a simultaneous surgery transforms the knot into the unknot.

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- These surgeries induce a cobordism of $S^3_0(K)$ with $S^2 \times S^1$ with only 2-handles. We glue $D^3 \times S^1$ at the end.
- We obtain W with $\partial W = S^3_0(K)$.

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Definition

If W is a manifold as above, then we shall say that it *strictly cobounds* $M(K)$.

Off-topic

This formula appears in almost every talk here, so I will write it.

$$\dots \mathcal{F}_{(n.5)} \subset \mathcal{F}_{(n)} \subset \dots \subset \mathcal{F}_{(0)} \subset \mathcal{C}$$

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The construction resembles the standard construction of a linking form on a rational homology sphere.

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Lemma (Kearton 1975)

A Seifert matrix gives rise to a presentation matrix of the same size.

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Theorem (–,Friedl 2013)

$u_a = n(K)$. Thus $u_a(K) = \min b_2(W)$ over all topological manifolds strictly cobounding $M(K)$.

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- The minimal $b_2(W)$ for a manifold W strictly cobounding $M(K)$;

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Upper bounds.

- the unknotting number;
- algebraic unknotting on matrices. Can be implemented on a computer;

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- Is the condition $A(1)$ is diagonal important?
- What if we require W to be smooth?
- Does this generalize to higher dimensions? We have a notion of a zero-surgery on $S^{n-2} \subset S^n$.