Heegaard Floer homologies and rational cuspidal curves

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Warsaw, February 2016

Heegaard decomposition of 3-manifolds

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Let Y be a closed oriented and connected 3–manifold. A *Heegaard decomposition* is the presentation of Y as a union $Y = H_1 \cup_\Sigma \cup H_2$, where H_1 and H_2 are handlebodies and Σ is a closed connected surface.

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Sketch of proof.

Use Morse theory. □

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Problem

Prove the last statement.

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A pointed Heegaard diagram is a quadruple $(\Sigma, \alpha, \beta, z)$, where $z \in \Sigma \setminus (\alpha \cup \beta)$.

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Show that there is a 1–1 correspondence between points $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $g-$tuples of points $(x_1, \dots, x_g) \in \Sigma$ such that there exists a permutation $\sigma \colon \{1, \dots, g\} \to \{1, \dots, g\}$ and $x_i \in \alpha_i \cap \beta_{\sigma(i)}$.

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Let $D_z = \{z\} \times \Sigma \times \ldots \times \Sigma$. Then D_z is a divisor in $Sym^g(\Sigma)$.

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- The differential counts holomorphic maps $\phi\colon D \to Sym^g$ such that $\phi(-1) = x$, $\phi(1) = y$, $\phi_{\partial_- D} \subset \mathbb{T}_\alpha$ and $\phi_{\partial_+ D} \subset \mathbb{T}_\beta$. We make it precise.

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- The differential for $\widehat{CF}$ is

$$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x,y)\colon \mu(\phi)=1, n_z(\phi)=0} \#\mathcal{M}(\phi) y.$$

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A remark. The complexes have a relative grading, called the Maslov grading, with $M(x) - M(y) = \mu(\phi) - 2n_z(\phi)$.

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Theorem (Turaev)

Suppose $\dim M = 3$. *Consider the set of non-vanishing vector fields on* M. *Consider two vector fields equivalent if they are homotopic through vector fields non-vanishing outside of a point. Then the set of abstraction classes is in a bijective correspondence with the set of spin-c structures.*

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Given that, the chain complexes split as direct sums of subcomplexes corresponding to different spin-c structures.

Independence

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The homologies HF^+, HF^-, HF^∞ and $\widehat{HF}$ are independent of the choices made and are invariants of $(Y, \mathfrak{s})$. Moreover, if Y is a rational homology sphere, then $HF^\infty(Y, \mathfrak{s}) = \mathbb{Z}_2[U, U^{-1}]$.

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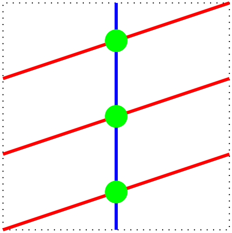

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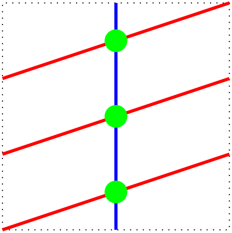

The complex for $L(3, 1)$ has three generators and no differentials. Each generator corresponds to another spin-c structure. We get that $CF^{-}(L(3, 1), \mathfrak{s}) \cong \mathbb{Z}_2[U]$ for each $\mathfrak{s}$.

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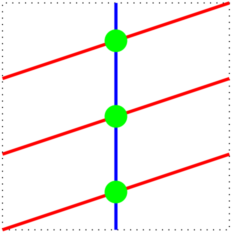

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The same statement hold for every lens space.

L–spaces

Definition

A rational homology three–sphere is called an *L–space* if for every spin-c structure $\mathfrak{s}$ we have $\widehat{HF}(Y, \mathfrak{s}) = \mathbb{Z}_2$ and $HF^-(Y, \mathfrak{s}) = \mathbb{Z}_2[U]$ (these two are equivalent to each other and also equivalent to saying that $HF^+ = \mathbb{Z}_2[U, U^{-1}]/Z_2[U]$).

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Künneth formula

Problem

Suppose $(Y_1, \mathfrak{s}_1)$ *and* $(Y_2, \mathfrak{s}_2)$ *are two three–manifolds. Prove the following Künneth formula for* CF^- *and* CF^∞*:*

$$CF^-(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) \cong CF^-(Y_1, \mathfrak{s}_1) \otimes CF^-(Y_2, \mathfrak{s}_2).$$

Adjunction inequality

Theorem (Ozsváth–Szabo 2003)

Suppose Y has $b_1(Y) > 0$. Let $Z \subset Y$ be a closed oriented surface in Y. If $HF^+(Y, \mathfrak{s}) \neq 0$, then $|\langle c_1(\mathfrak{s}), Z\rangle| \leq 2g(Z) - 2$.

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This is the one of the two main technical tools in dealing with Heegaard Floer theory. This is also one of the sources of its power.

Surgery exact sequence

Theorem

Suppose Y is a homology three–sphere and $K \subset Y$ is a knot. Then there exists an exact sequence

$$\ldots \to HF^+(Y) \to HF^+(Y_0) \to HF^+(Y_1) \to HF^+(Y) \to \ldots,$$

where Y_1 is the $+1$ surgery and Y_0 is the 0-surgery.

Idea of proof.

Construct a suitable triple Heegaard diagram and define various maps by counting holomorphic triangles. □

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- If W is negative definite, then the red arrow is an isomorphism so we obtain the fundamental inequality between d-invariants:

$$d(Y_1, \mathfrak{s}_1) \geq d(Y_2, \mathfrak{s}_2) + \deg F_W.$$

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The inequality for d-invariants is strong enough to:

- Reprove the Donaldson's diagonalization theorem.
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- many other things.

A glimpse into the future

Corollary

If $(Y,\mathfrak{s})$ bounds a rational homology ball W (that is $H_k(W;\mathbb{Q})=0$ for $k\geq 1$) and the spin-c structure $\mathfrak{s}$ extends over W, then $d(Y,\mathfrak{s})=0$.

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Being a rational homology ball is the same as being a $\mathbb{Q}$-acyclic surface. In particular, a complement of a rational cuspidal curve C in $\mathbb{C}P^2$ is a rational homology ball.

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Question

How to calculate d-invariants?

Knots and Heegaard diagrams

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We think of a knot as a of a doubly pointed Heegaard diagram.

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Lemma

We have $\sum_{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} (-1)^{M(x)} q^{A(x)} = \Delta(q)$.

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- Detects the genus. That is, $g(K) = \max\{i \colon \widehat{HFK}_*(K,i) \neq 0\}$.

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- The τ-invariant, $\tau(K) = -\max\{s\colon \exists x \in HFK^-_*(K,s)\colon U^j x \neq 0\}$ is a concordance invariant, equal to $2g(K)$ for all positive knots, detecting the unknotting number of positive knots.

Surgeries and spin-c structures

Let $K \subset S^3$ be a knot. Take ball B^4 and glue to it a two–handle along K with framing q. We obtain a 4–manifold N with boundary $S^3_q(K)$. The core of the handle and a Seifert surface for K form a closed surface F that generates $H_2(N; \mathbb{Z})$.

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For every $m \in [-q/2, q/2) \cap \mathbb{Z}$ there exists a unique spin-c structure $\mathfrak{s}_m$ on Y that extends to a spin-c structure $\mathfrak{t}_m$ on N characterized by the property that $\langle c_1(\mathfrak{t}_m), F\rangle + 2m = q$

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The bottom line: think of spin-c structures as of integers in some interval!

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Large surgeries

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Suppose $K \subset S^3$ and $q > 2g(K)$. Let $Y = S^3_q(K)$. Then $CF^-(Y, \mathfrak{s}_m) \cong CFK^\infty(K)(i < 0, j < m)$ and $CF^+(Y, \mathfrak{s}_m) \cong CFK^\infty/(i < 0, j < m)$.

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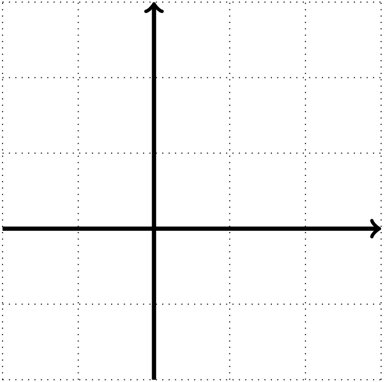

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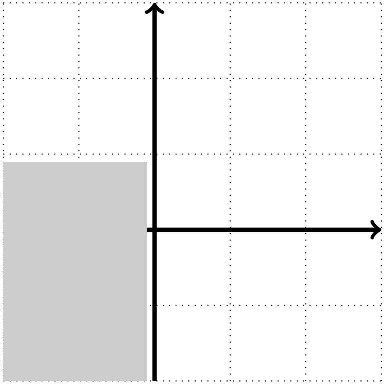

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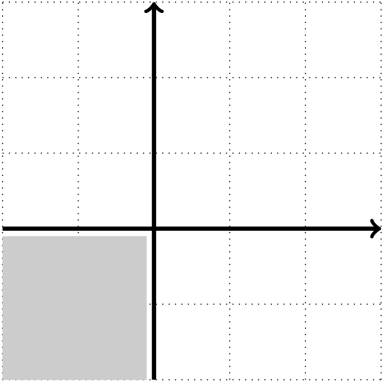

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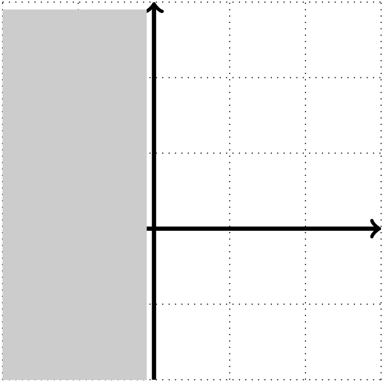

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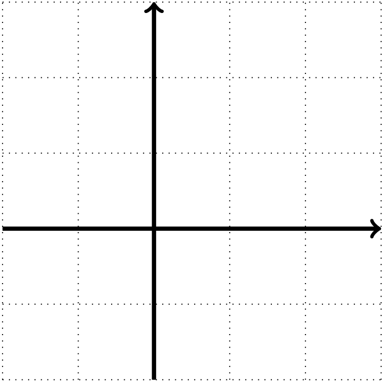

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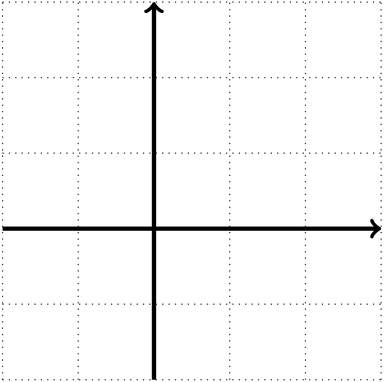

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- All needed data is derived from the CFK^∞

L–space knots

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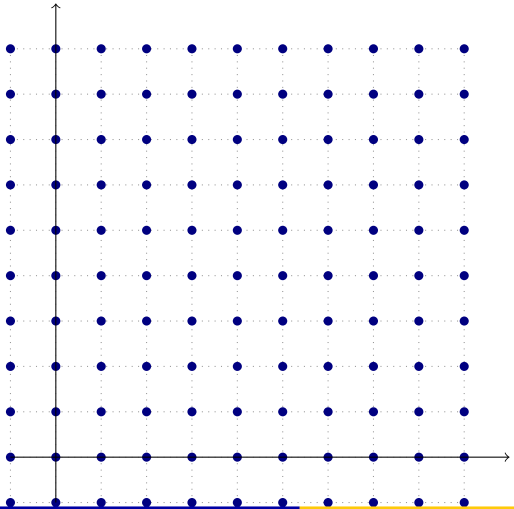
L–space knots have the CFK^∞ determined from the Alexander polynomial.

The staircase

$$\Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^{9} + t^{7} - t^{5} + t^{4} - t + 1.$$



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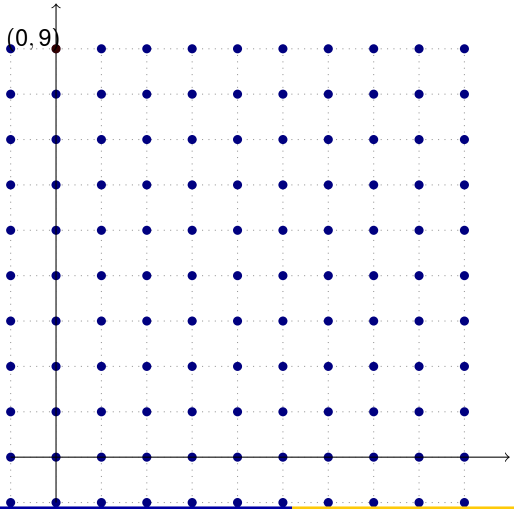

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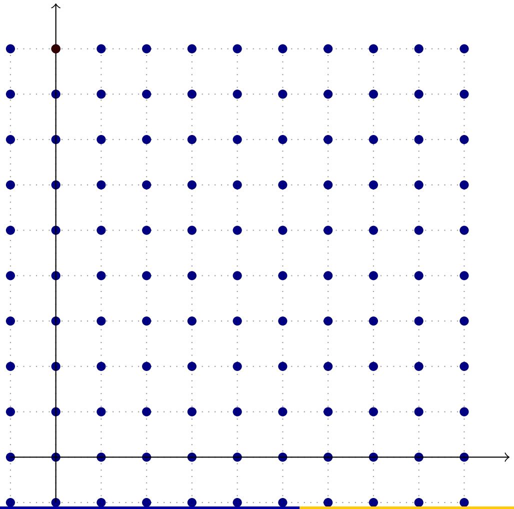

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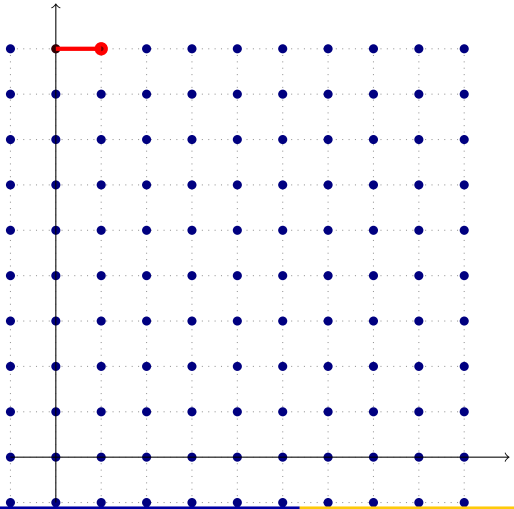

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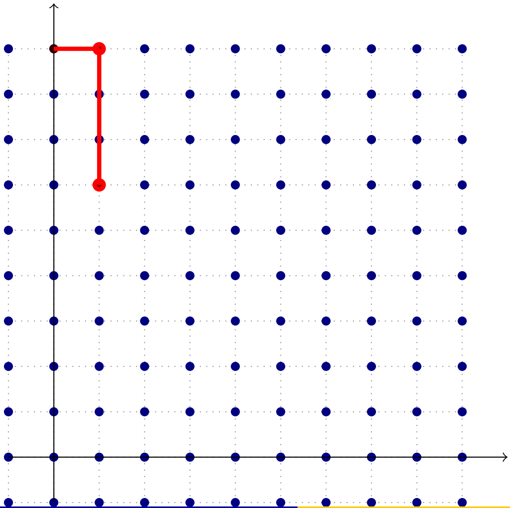

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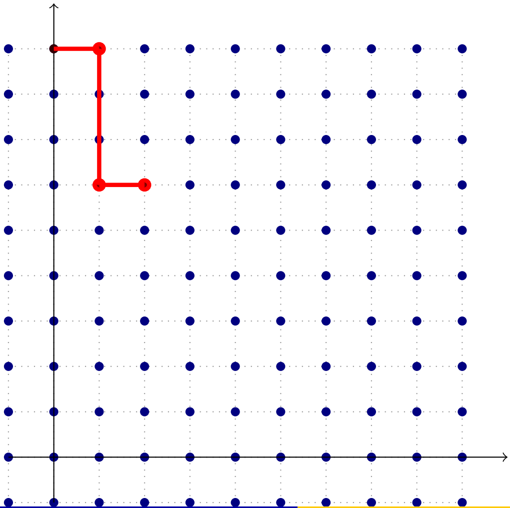

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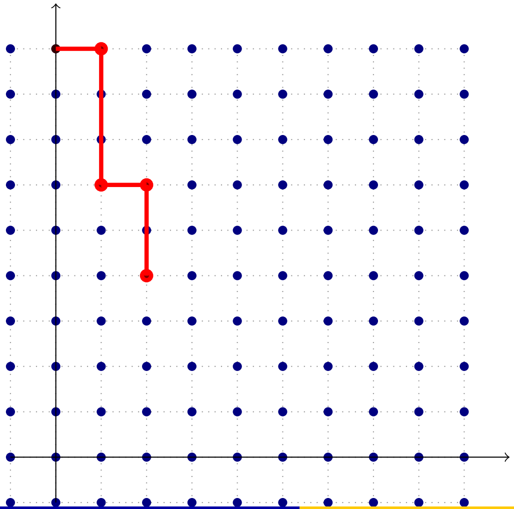

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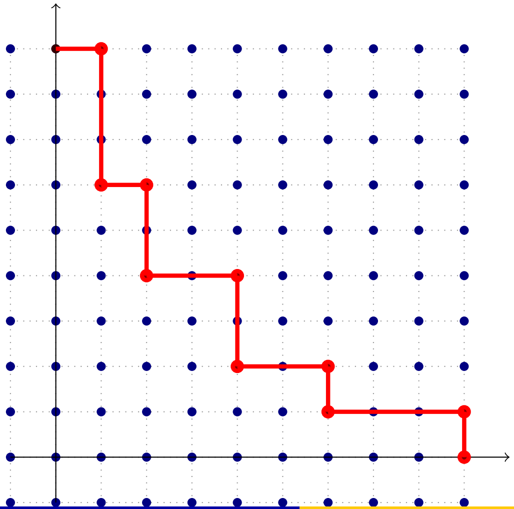

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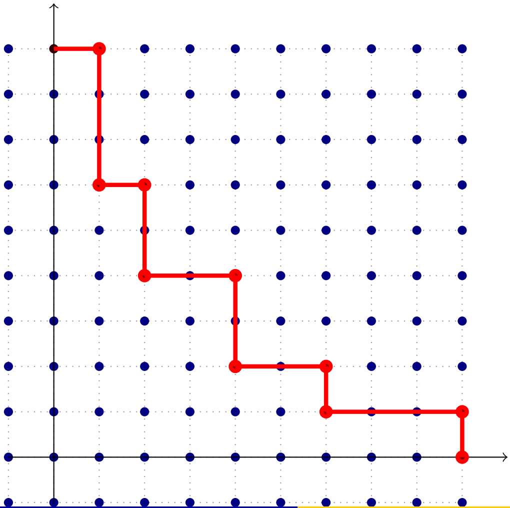

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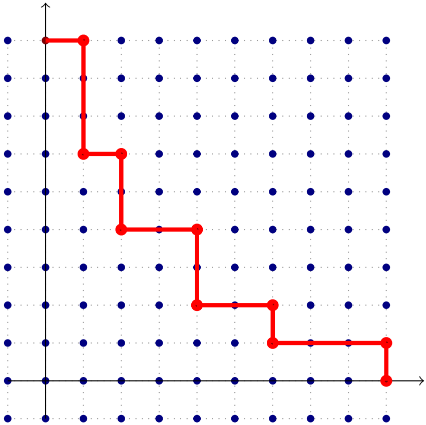

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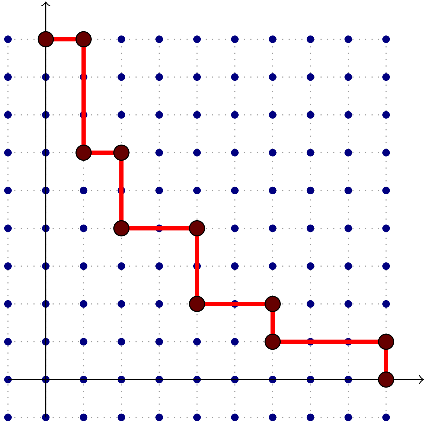

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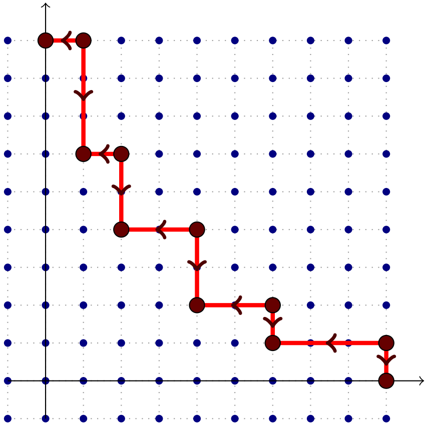

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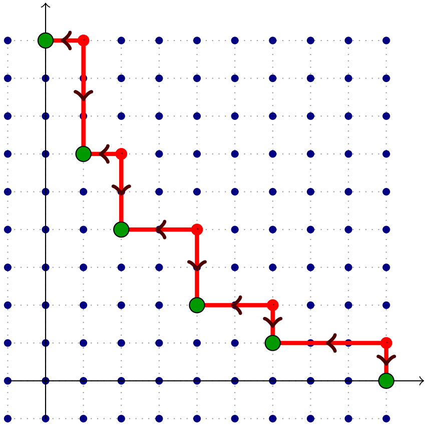

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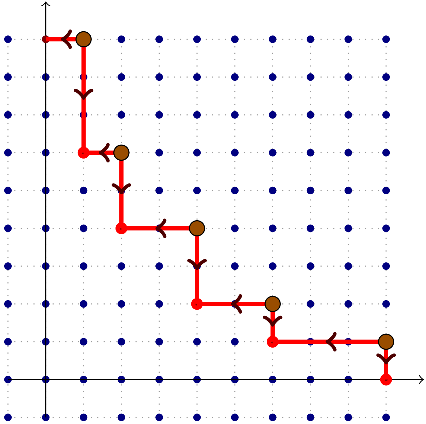

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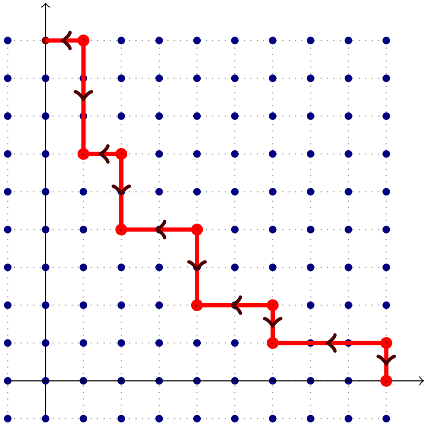

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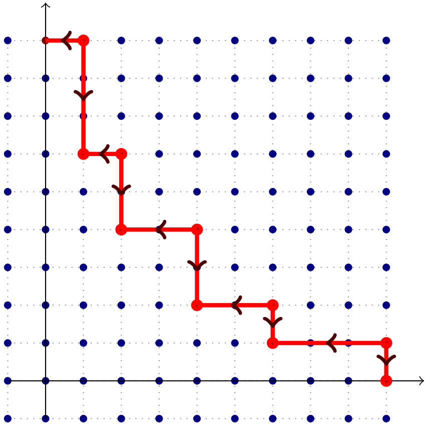
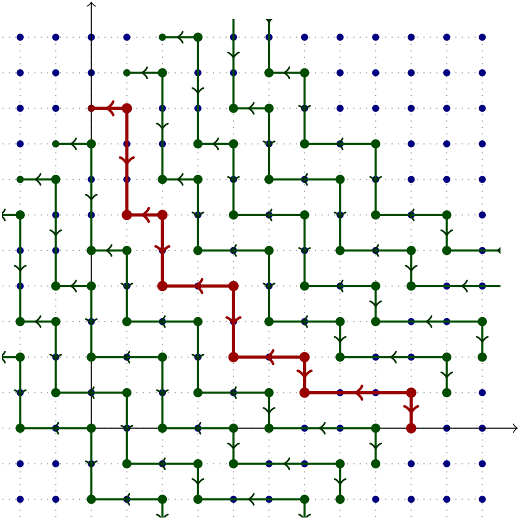


- Place $\mathbb{Z}_2$ for each vertex.
- Differential is given by lines as depicted.
- Type A vertices.
- Type B vertices.
- Bifiltration is given by coordinates.
- Absolute grading of a type A vertex is 0, of type B is 1.

Tensoring



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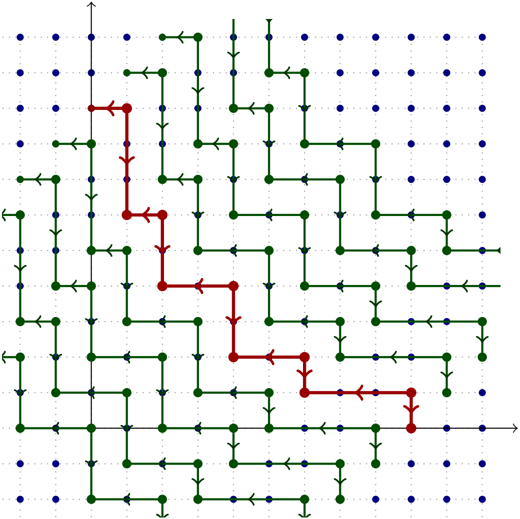

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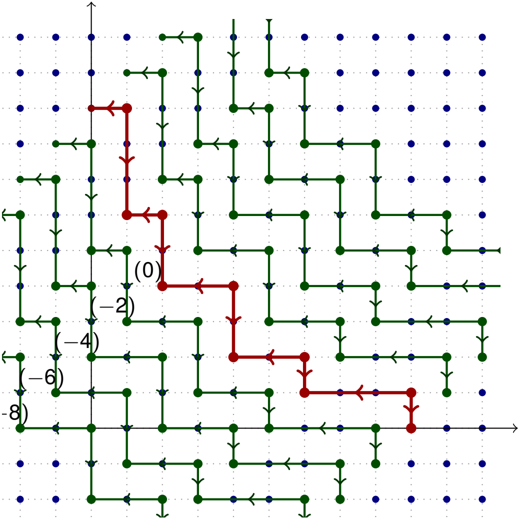

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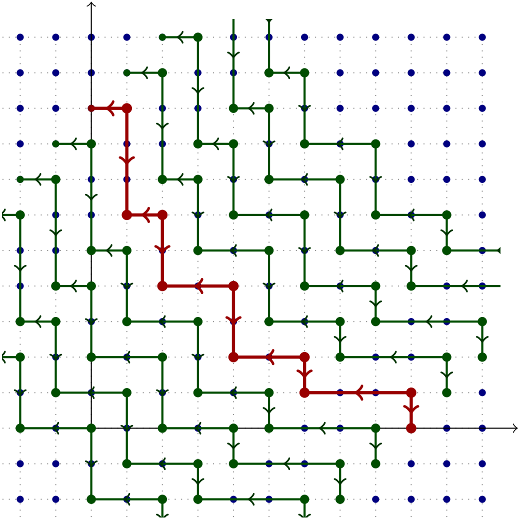
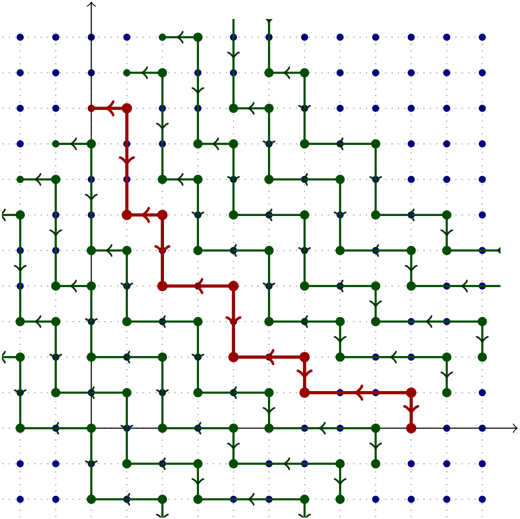


- Tensor St(K) by $\mathbb{Z}_2[U, U^{-1}]$.
- U changes the filtration level by $(-1, -1)$ and the absolute grading by -2.
- The resulting complex is $CFK^\infty(K)$ if K is an algebraic knot.

The function $J(m)$



- $m \in \mathbb{Z}$. Here $m = 3$.

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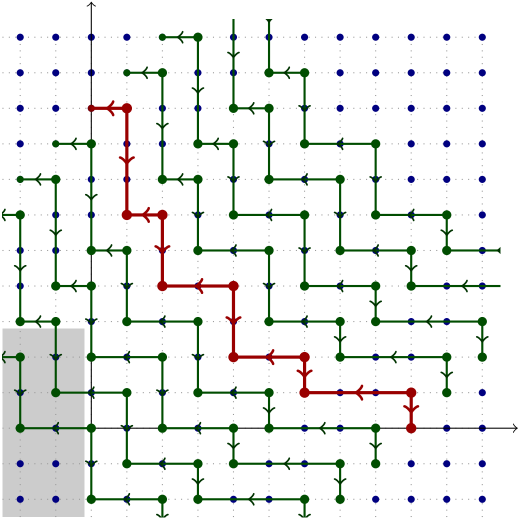

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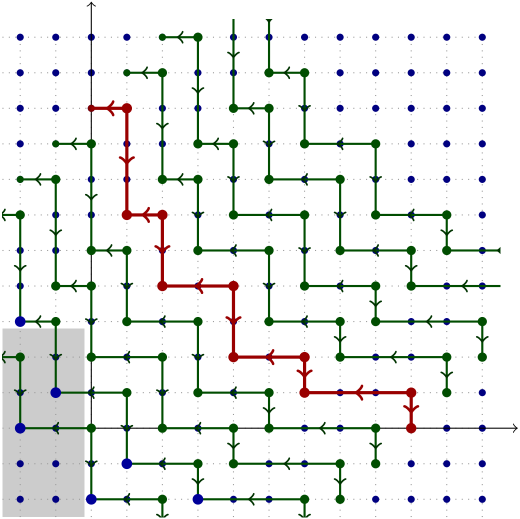

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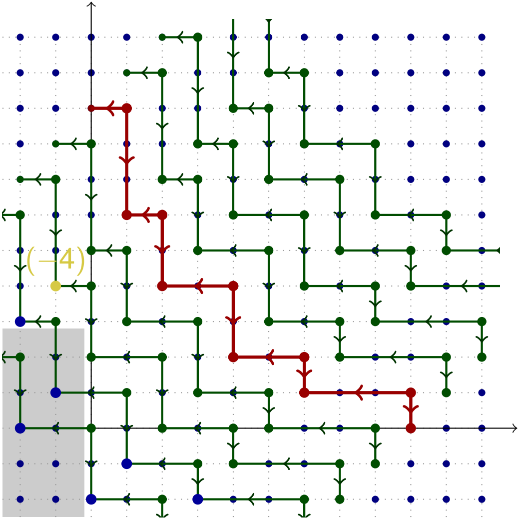

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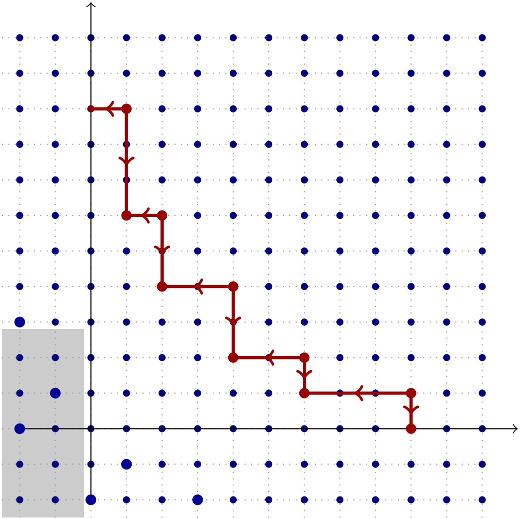

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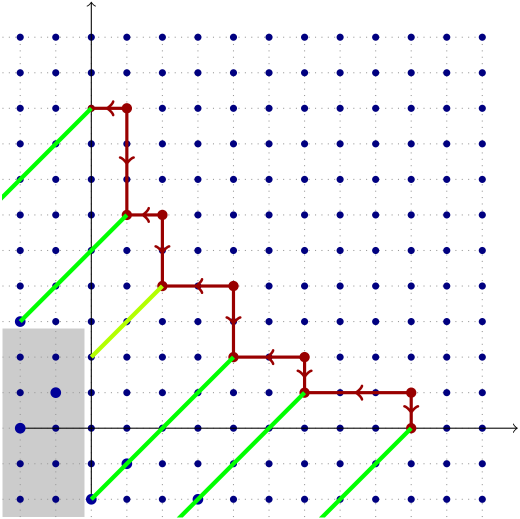

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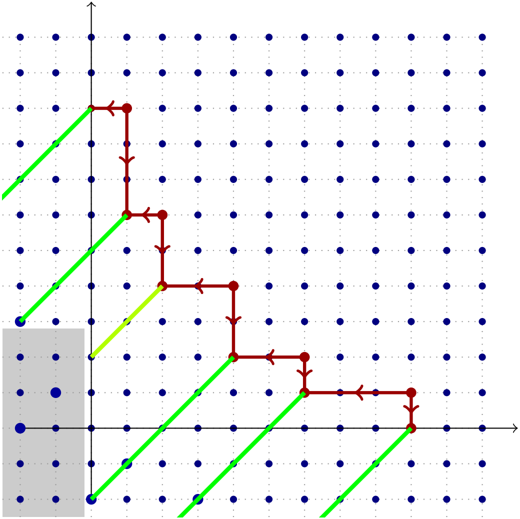

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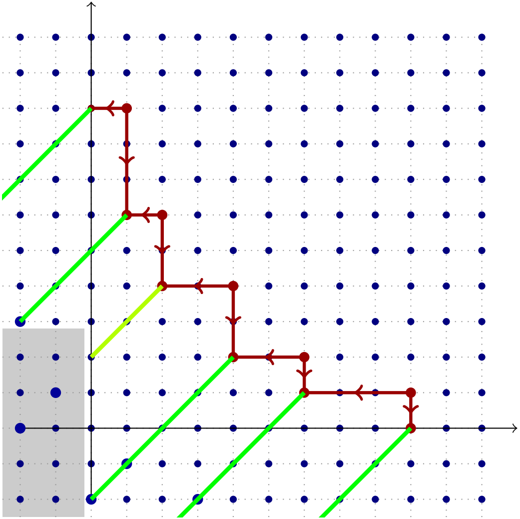

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- We will show yet another description of J.

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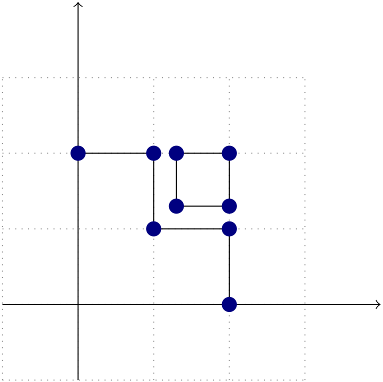

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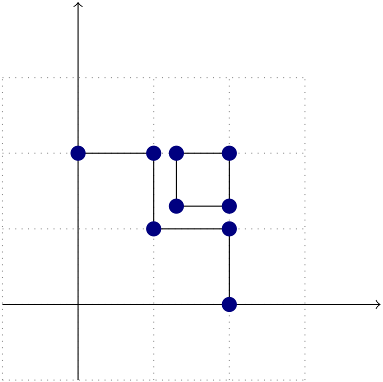

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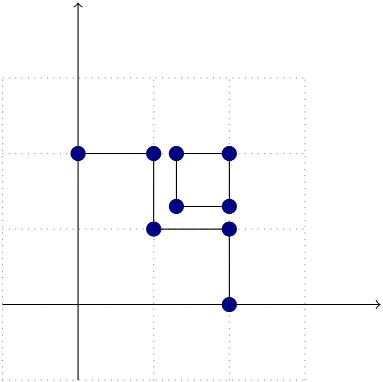

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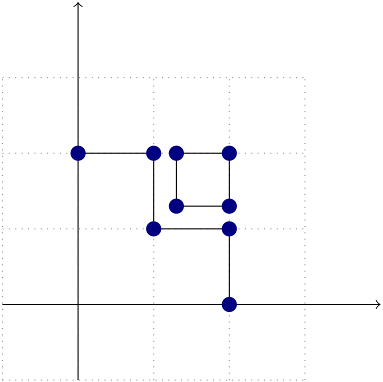
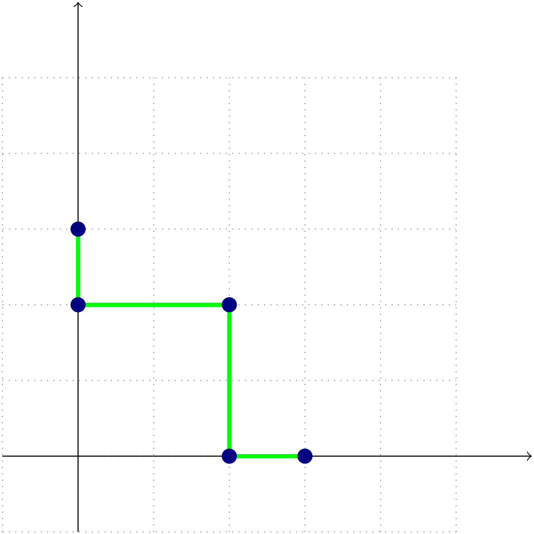


- The whole picture must be tensored by $\mathbb{Z}_2[U, U^{-1}]$.
- We have a staircase plus an acyclic complex.
- This is not always true, for example for $T(4,5)\#T(4,5)$.

CFK^∞ for $-T(3,4)$



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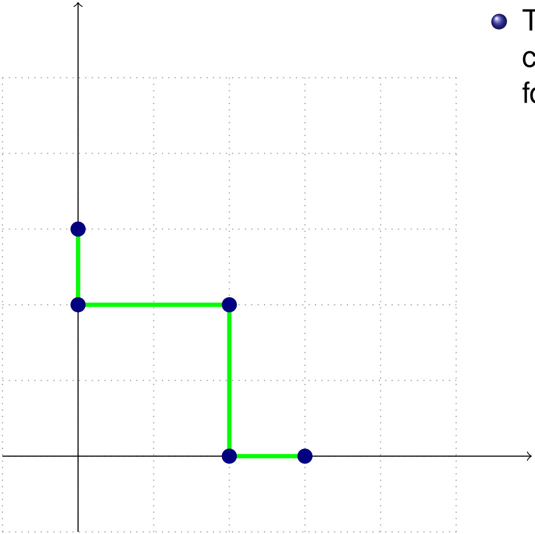

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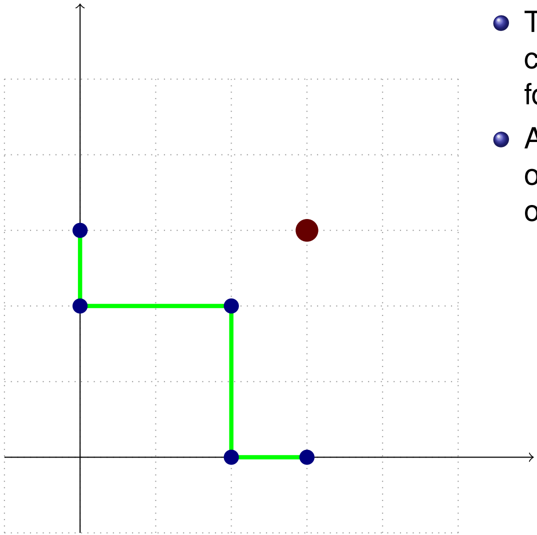

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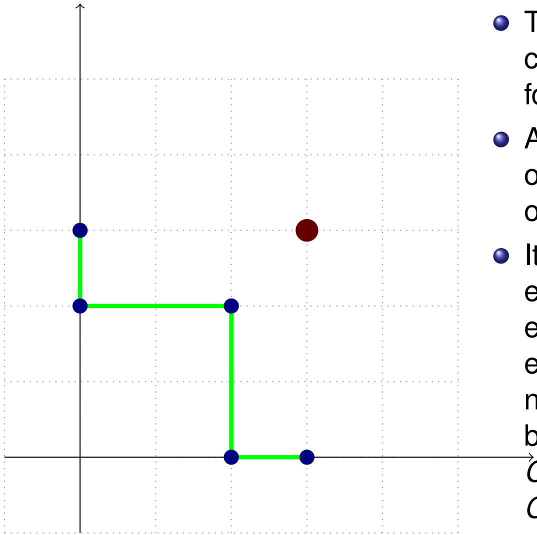

- The situation is completely different than for positive $T(3,4)$.
- A generator of homology of the complex is a sum of filtered elements.
- It is not a filtered element, that is an element at bifiltration element (x,y) that is non-zero in the quotient by $CFK^\infty(i \le x-1, j \le y) + CFK^\infty(i \le x, j \le y-1)$.

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The semigroup of a singular point on a plane curve is the set of possible local intersection with the curve.

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This is the Alexander polynomial of the knot of the singularity.

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Theorem

For an algebraic knot $J(m) = -2I(m+g)$, where $g = \mu/2$ is the genus.

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Let $K = K_1 \# \ldots \# K_n$ be a connected sum of algebraic knots. Gap functions are $I_1, \ldots, I_n$. Set $I = I_1 \diamond \ldots \diamond I_n$. Then $J(m) = -2I(m+g)$, where J is the minimal grading . . .

d-invariants again

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Proposition

The spin-c structure $\mathfrak{s}_m$ extends over $\mathbb{C}P^2 \setminus N$ if $m = jd$ for $j \in \mathbb{Z}$ if d is odd and $m = (j + \frac{1}{2})d$ if d is even.

The FLMN conjecture

Combining results we obtain the following result.

Theorem (—,Livingston, 2013)

For $j = 0, \dots, d-3$ *we have*

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Theorem (–, 2015)

Generalization for rcc in surfaces with $p_g = 0$. The condition implies that the complement of a rcc is a negative definite manifold.

If time permits

Theorem (FLMN)

Suppose that C is a curve in $\mathbb{C}P^2$ of degree d. Let $z \in C$ be a singular point and S its semigroup. Then for $j = 1, \ldots, d-1$ we have

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- The quantity $\#S \cap [0, k)$ is the number of conditions of a curve D to intersect C at z with multiplicity at least k.

If time permits

Theorem (FLMN)

Suppose that C is a curve in $\mathbb{C}P^2$ of degree d. Let $z \in C$ be a singular point and S its semigroup. Then for $j = 1, \ldots, d-1$ we have

$$\#S \cap [0, jd+1) \geq \frac{1}{2}(j+1)(j+2).$$

- This is one part of the FLMN conjecture.
- The right hand side is the dimension of space of polynomials of degree j, $H^0(\mathbb{C}P^2, \mathcal{O}(jH))$.
- The quantity $\#S \cap [0, k)$ is the number of conditions of a curve D to intersect C at z with multiplicity at least k.
- If the inequality is violated, then there exists a curve D of degree j intersecting C with multiplicity $jd+1$ or higher. Contradicition.

If time permits 2

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Prove the FLMN inequalities using the line of FLMN for almost complex manifolds replacing $H^0(\mathbb{C}P^2, \mathcal{O}(jH))$ by some moduli space of J-holomorphic curves.

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Prove the FLMN inequalities using the line of FLMN for almost complex manifolds replacing $H^0(\mathbb{C}P^2, \mathcal{O}(jH))$ by some moduli space of J-holomorphic curves. Explain the similarity between the two approaches as a variant of GW–SW correspondence.

Work in progress

Joint project with Hom and Schinzel.

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Use Involutive Floer homology for finer obstruction.

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Definition

For an algebraic knot K with a staircase, the *stretch* is the length of the middle step of a staircase. A knot is called *even* or *odd* if the staircase has an even or odd number of steps.

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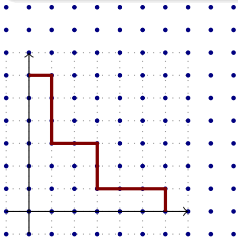

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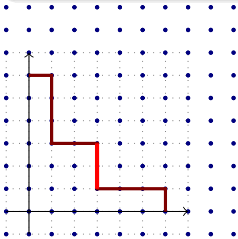

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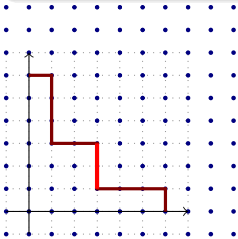

The stretch of the staircase for $T(4,5)$ is 2. This knot is odd.

Bound from the *IH*

Theorem (Hom, Schinzel, –)

Let p, q be coprime. Write the continuous fraction expansion $q/p = [a_0; a_1; \dots; a_k]$. Then the stretch of $T(p, q)$ is equal to $[\frac{a_k - 1}{2}] + 1$.

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We have only this result for curves of odd degree.

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Remark

This obstructs some cases with one 'big' singularity and some small.