1. Introduction

Problem 1. Prove that $H_1(S^3\backslash K;\mathbb{Z}) = \mathbb{Z}$ and $H_2(S^3\backslash K;\mathbb{Z}) = 0$ without using the Alexander duality.

Problem 2. Compute the knot group of the trefoil. Show that it is not trivial.

Problem 3. Generalize it to computing the knot group of an arbitrary torus knot.

Problem 4. Show that the meridian of a knot is the generator of H_1 of the complement.

Problem 5. Let T be a tubular neighborhood of the knot K. Show that there exists a curve λ on ∂T, such that λ is trivial in $H_1(S^3\backslash T)$ and λ is parallel to K. Prove that the linking number of K with λ is zero.

Problem 6. Suppose T is a tubular neighborhood of a non-trivial knot. Prove that $i_*\colon \pi_1(\partial T) \to \pi_1(S^3\backslash T)$ is injective.

Problem 7. Compute the knot group of the square knot and of the granny knot. Show that they are isomorphic. Prove that their peripheral structure is not the same (the peripheral subgroup is the image of π_1 of the boundary).

Problem 8. Show that a link complement is a $K(\pi, 1)$ space.

Problem 9. Prove that $g(K_1 \# K_2) = g(K_1) + g(K_2)$.

Problem 10. Let $\alpha, \beta \in S^3$ be two simple closed curves with $\alpha \cap \beta = \emptyset$. Assume that A, B are two compact oriented surfaces in B^4, such that $\partial A = \alpha$, $\partial B = \beta$. If A, B intersect transversally, show that $\operatorname{lk}(\alpha, \beta) = A \cdot B$.

Problem 11. Is the statement of Problem 10 true if B^4 is replaced by $S^3 \times [0, 1]$? What about other spaces?

2. Alexander polynomials

Problem 12. Calculate the Alexander polynomial of a torus knot using Fox differential calculus.

Problem 13. Compute the Alexander polynomial of a figure-eight knot using incidence matrices.

Problem 14. Prove the formula $\Delta_K(t) = \Delta_J(t^q)\Delta_{K'}(t)$ if K is a satellite on J with companion K' going around J q times.

Problem 15. Prove the connected sum formula for Alexander polynomials.

Problem 16. Show that if K is fibered, then Δ is monic. Give an example that the converse is not true.

Problem 17. Show that if K_1 and K_2 are fibered, then so is $K_1 \# K_2$.

Problem 18. Prove that if K is fibered then the monodromy operator h is $V \cdot (V^t)^{-1}$.

Problem 19. Prove that torus knots are fibered.

Problem 20. Show that the Alexander polynomial of a fibered knot is monic.

Problem 21. Find a knot whose Alexander polynomial is monic but which is not fibered.

Problem 22. Does there exist a knot that is not concordant to any fibered knot?

Problem 23. Show that $\deg \Delta_K \leqslant g_3(K)$, where g_3 is the three-genus.

Problem 24. Show that $\deg \Delta_K \leqslant c(K)$, where c is the crossing number.

Problem 25. Compute the Alexander polynomial of the twisted Whitehead double of a knot.

Problem 26. Suppose V is a Seifert matrix of size $2g$. For any $k > 0$ define Δ_k to be the *gcd* of all the minors of $tV - V^t$ of size $2g - k$. Is it always defined over $\mathbb{Z}[t, t^{-1}]$? Show that if it is defined, it is an invariant of the S-equivalence.

3. Signatures

Problem 27. Show that if K_+ and K_- differ by a crossing change, then $\sigma(K_+) - \sigma(K_-) \in \{0, -2\}$.

Problem 28. For $z \in \mathbb{C}\backslash\{1\}$ define $\sigma(z)$ as the signature of the hermitian form $(1 - z)A + (1 - \overline{z})A^t$. Prove that $\sigma(z)$ is a link invariant.

Problem 29. Show that the values of $\sigma(z)$ are determined by the values of σ at the unit circle.

Problem 30. Prove that σ is a piecewise constant function on a unit circle. Show that if it has a discontinuity at a point z_0, then $\Delta(z_0) = 0$. Moreover, the amount of the jump is at most twice as large as the order of Δ at z_0.

4. Branched covers

Problem 31. Prove rigorously that the presentation matrix for $H_1(\Sigma(K); \mathbb{Z})$ is $V + V^T$.

Problem 32. Show that if k is odd, then $H_1(\Sigma^k(K); \mathbb{Z}) = G \bigoplus G$ for some finite abelian group G.

Problem 33. Prove that multiplication by $(t - 1)$ is an isomorphism of the Alexander module of a knot. Show it is not true for links.

Problem 34. Let K be a knot and V a Seifert matrix for K. Prove that the Alexander module $H_1(S^3\backslash K; \mathbb{R}[t, t^{-1}])$ is cyclic if and only if the higher Alexander polynomials of K are trivial.

Problem 35. Find examples of two knots with the same determinant and different $H_1(\Sigma(K); \mathbb{Z})$.

Problem 36. Find examples of two knots with the same $H_1(\Sigma(K); \mathbb{Z})$ and different linking forms on the double cover.

Problem 37. Define rigorously the linking form on the torsion part of a closed 3–manifold. Show it is nondegenerate.

Problem 38. Suppose W is a 4-fold such that $\partial W = M$ is a rational homology 3–sphere. Assume that $H_1(W; \mathbb{Z}) = 0$. Show that the intersection form $H_2(W; \mathbb{Z}) \times H_2(W; \mathbb{Z}) \to \mathbb{Z}$ is well defined. Assume it is represented by a matrix A of size $n = rk_{\mathbb{Z}} H_2(W; \mathbb{Z})$. Prove that the linking form on M is represented by A, that is $H_1(M; \mathbb{Z}) = \mathit{coker}\, A$ and if $a, b \in H_1(M; \mathbb{Z})$, then $lk(a, b) = a^T A^{-1} b$.

5. Slice genus

Problem 39. Prove that if K is slice, its has a choice of a Seifert matrix which has a block structure $\begin{smallmatrix} 0 & B \\ C & D \end{smallmatrix}$.

The result is much easier to prove if one assumes that K bounds a smooth disk. However, it is true if K is only topologically slice (ie bounds a locally flat disk).

Problem 40. Let K be a knot and $-K$ be its mirror. Show that $K\#-K$ is smoothly slice.

Problem 41. Let K be a slice knot. Show that the signature function $z \mapsto \sigma(z)$ as a function on S^1 is zero for all but finitely many values.

Again it is much easier to prove for smoothly slice knots. The case of non-smoothly slice knot was only recently written up by Mark Powell, even though the result had been used before.

Problem 42. Show that if K is slice, then its Alexander polynomial is of form $f(t)f(t^{-1})$.

Problem 43. Show that the figure-eight knot is not slice.

Problem 44. Give an example of a smoothly slice knot, whose signature function is not constant.

Problem 45. Consider the set of knots with relation $K \sim L$ if $K\#-L$ is smoothly (resp. topologically) slice. Show that the connected sum operation endows this set with a structure of an abelian group. It is called the concordance group.

Problem 46. Construct a non-trivial map from the topological concordance group to the Witt group over the ring $\mathbb{Z}[t, t^{-1}]$.

Problem 47. Let X be a topologcal space and U_i a finite cover. Suppose for each pair of indices i, j there is defined a map $g_{ij} \colon U_i \cap U_j \to Aut(V)$, where V is some vector space and Aut the group of linear automorphisms. Prove that the cocycle condition $g_{ij}g_{jk}g_{ki} = 1$ is a sufficient condition for the existence of a vector bundle E over X, whose fiber is V and the transition functions are g_{ij}.

Problem 48. Prove that every vector bundle over a compact manifolds is a subbundle of a trivial bundle.

Problem 49. Use Problem 48 to show that if L is a line bundle over a compact manifold M, then there is a map $\phi \colon M \to \mathbb{C}P^N$ (for some large N) such that $L = \phi^*H$, where H is the tautological line bundle over M. Show that the element $c_1(L) \in H^2(M;\mathbb{Z})$ defined as a pull-back of the generator of $H^2(\mathbb{C}P^N;\mathbb{Z}) = \mathbb{Z}$ does not depend on presenting L as a subbundle of a trivial bundle.

Problem 50. Prove that $c_1(L \otimes L') = c_1(L) + c_1(L')$, where c_1 is the first Chern class defined in Problem 49.

Problem 51. Calculate the first Chern class of the tangent bundle to S^2.

Problem 52. Prove that a vector bundle over a compact manifolds admits a riemannian metric, that is, a scalar product in the fiber.

Problem 53. Prove that a real vector bundle of rank n over a compact manifold is orientable if and only if the transition fuctions g_{ij} can be chosen to sit in $GL(n;\mathbb{R})_0$ (the connected component of GL containing the identity).

Problem 54. Show that a real vector bundle admits a riemannian metric if and only if g_{ij} can be chosen to be orthogonal matrices.

Problem 55. Show that a real vector bundle admits a complex structure if and only if g_{ij} can be chosen as complex matrices.

Problem 56. Show that a real vector bundle over a finite CW-complex is oriented if and only if it is trivial on the 1-skeleton.

Problem 57. Prove that $SO(3)$ is homeomorphic to the real projective plane.

Problem 58. Let G be the universal cover of $SO(3)$. Show that if M is a closed oriented 3-manifold which is an integer homology sphere, then there is a finite cover U_j on which TM is trivial and such that each of the transition functions $g_{ij}\colon U_i \cap U_j \to SO(3)$ can be lifted to $\widetilde{g}_{ij}\colon U_i \cap U_j \to G$ in such a way that $\widetilde{g}$ satisfy the cocycle condition.

Problem 59. Generalize Problem 58. Let G_n be the universal (double cover) of $SO(n)$ for $n > 2$ and let M be a smooth closed n-dimensional manifold. Prove that the cocycle $g_{ij}\colon U_i \cap U_j \to SO(n)$ for TM can be lifted to $\widetilde{g}_{ij}\colon U_i \cap U_j \to G_n$ if and only if TM is trivial when restricted to the 2-skeleton of M.

Problem 60. Suppose L is a line bundle over X and g_{ij} is the cocycle that trivializes it. Show that g_{ij} defines an element g in the Cech cohomology $H^1(X; \mathbb{C}^*)$. Prove that $c_1(L)$ is equal to the image of g under the map $H^1(X; \mathbb{C}^*) \to H^2(X; \mathbb{Z})$, which is the connecting homomorphism in the long exact sequence of cohomology associated to the short exact sequence $0 \mapsto \mathbb{Z} \mapsto \mathbb{C} \mapsto \mathbb{C}^* \to 0$.

Problem 61. As a corollary prove that if M is a simply connected manifold, then each isomorphism class of a complex line bundle over M is determined by its first Chern class $c_1(L)$ and each element in $H^2(M; \mathbb{Z})$ determines a line bundle, whose first Chern class is L.

Problem 62. Prove that a function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is holomorphic if and only if $J \cdot DF = DF \cdot J$, where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Problem 63. Prove that if a closed manifold M is a complex manifold (admits an atlas such that the transition functions are holomorphic), then there exists a bundle map $J\colon TM \to TM$ such that $J^2 = -id$.

Problem 64. Prove that for $M = S^4$ there is no map $J\colon TM \to TM$ such that $J^2 = -id$.

Problem 65. Let Σ be a closed surface. Consider $Sym^n\Sigma$ defined as the quotient of $\Sigma \times \cdots \times \Sigma$ by the symmetric group S_n. Prove that $Sym^n\Sigma$ is a smooth manifold.

Problem 66. Calculate homology groups of $Sym^2 T$, where T is a torus $S^1 \times S^1$.

Problem 67. Find the α and the β curve for a Heegaard decomposition of $L(p, q)$.

Problem 68. Suppose L is a complex line bundle on a closed oriented surface of genus g. Let SL be the associated circle bundle (choose a metric on L and SL is the set of points of norm 1). Find a Heegaard decomposition of SL.

Problem 69. Calculate $\pi_2(Sym^g\Sigma)$ if the genus of Σ is $g > 1$.

References

[1] Snappy, http://www.math.uic.edu/t3m/SnapPy/index.html, best used with Sage
[2] KnotTheory, Mathematica package, http://katlas.org/wiki/Setup
[3] Sage has a surprisingly good knot theory package, http://www.sagemath.org
[4] Knotinfo is a great source of knot invariants, http://www.indiana.edu/~knotinfo/
[5] Knotorious gives a good amount on informations on the algebraic unknotting number, https://www.mimuw.edu.pl/~mcboro/knotorious.php
[6] Knotkit calculates Khovanov homology. It is much faster than sage and KnotTheory, https://github.com/cseed/knotkit

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