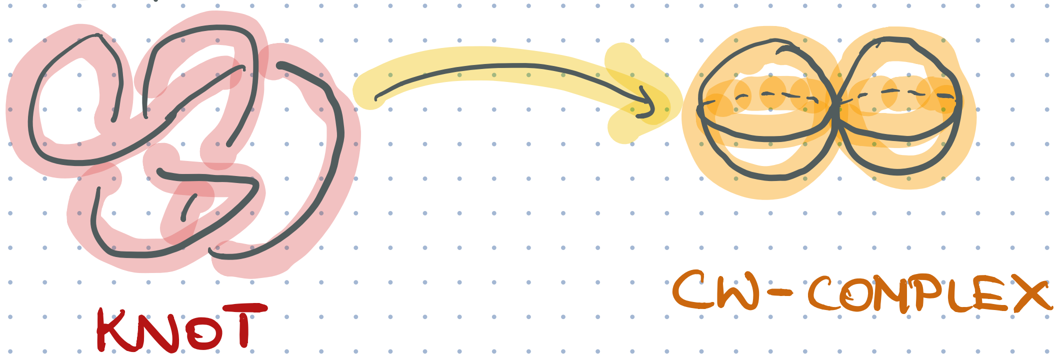
UPSHOT:



REFERENCES:

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"A KHOVANOV HOMOTOPY TYPE"

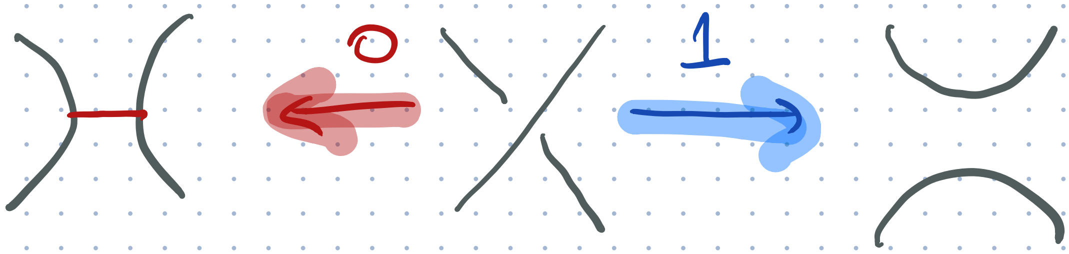
2) M. Borodzik, W. Politarczyk, M. Silvero

"KHOVANOV HOMOTOPY TYPE, PERIODIC LINKS AND LOCALIZATIONS"

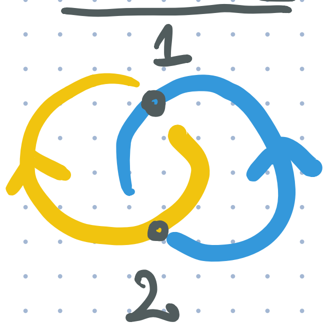
3) D. Jones, A. Lobb, D. Schütz

"MORSE MOVES IN FLOW CATEGORIES"

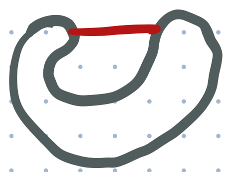
RESOLUTION CONFIGURATION



EXAMPLE HOPF LINK H_2

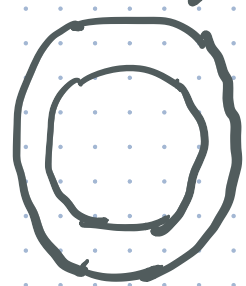


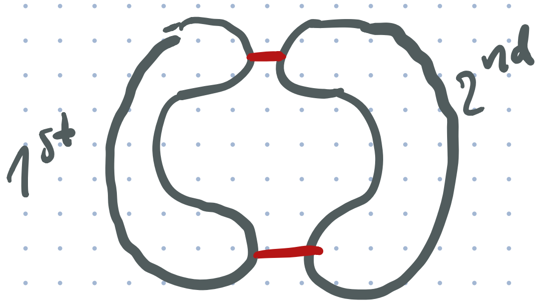
2^n



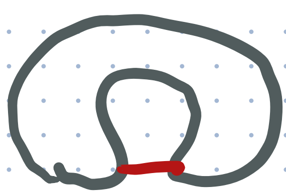
$D_{H_2}(0,1)$

$D_{H_2}(1,1)$





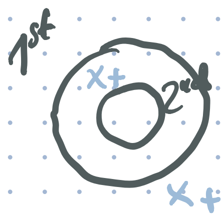
$D_{H_2}(0,0)$



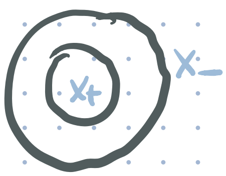
$D_{H_2}(1,0)$

LABELED RESOLUTION CONFIGURATION

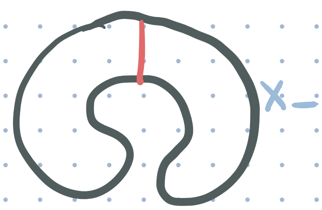
Definition 2.9. A *labeled resolution configuration* is a pair (D, x) of a resolution configuration D and a labeling x of each element of $Z(D)$ by either x_+ or x_-.



$(D_{H_2}(1,1), x_+ x_+)$



$(D_{H_2}(1,1), x_- x_+)$



$(D_{H_2}(1,0), x_-)$

Name	Generator
a	$(D_H(00), x_+x_+)$
v	$(D_H(00), x_+x_-)$
w	$(D_H(00), x_-x_+)$
m	$(D_H(00), x_-x_-)$
b	$(D_H(10), x_+)$
x	$(D_H(10), x_+)$

Name	Generator
c	$(D_H(01), x_+)$
y	$(D_H(01), x_-)$
e	$(D_H(11), x_-x_+)$
z	$(D_H(11), x_-x_-)$
l	$(D_H(11), x_+x_+)$
d	$(D_H(11), x_+x_-)$

Let's introduce something more!

→ PARTIAL ORDER

Definition 2.10. There is a partial order $\prec$ on labeled resolution configurations defined as follows. We declare that $(E, y) \prec (D, x)$ if:

(1) The labelings x and y induce the same labeling on $D \cap E = E \cap D$.

(2) D is obtained from E by surgering along a single arc of $A(E)$. In particular, either:

  (a) $Z(E \setminus D)$ contains exactly one circle, say Z_i, and $Z(D \setminus E)$ contains exactly two circles, say Z_j and Z_k, or

  (b) $Z(E \setminus D)$ contains exactly two circles, say Z_i and Z_j, and $Z(D \setminus E)$ contains exactly one circle, say Z_k.

(3) In Case (2a), either $y(Z_i) = x(Z_j) = x(Z_k) = x_-$ or $y(Z_i) = x_+$ and $\{x(Z_j), x(Z_k)\} = \{x_+, x_-\}$.

In Case (2b), either $y(Z_i) = y(Z_j) = x(Z_k) = x_+$ or $\{y(Z_i), y(Z_j)\} = \{x_-, x_+\}$ and $x(Z_k) = x_-$.

surgery along upper red arc

We have situation 2b) from the definition

KHOVANOV CHAIN COMPLEX

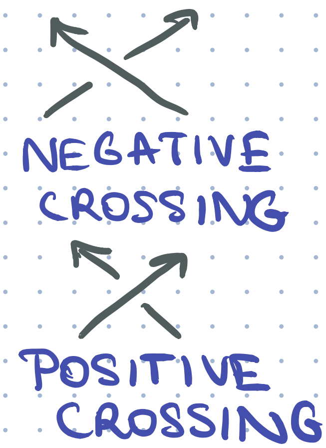
Definition 2.15. Given an oriented link diagram L with n crossings and an ordering of the crossings in L, the *Khovanov chain complex* is defined as follows.

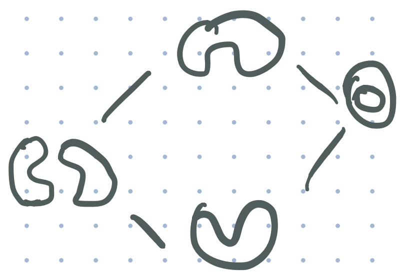
The chain group $KC(L)$ is the $\mathbb{Z}$-module freely generated by labeled resolution configurations of the form $(D_L(u), x)$ for $u \in \{0,1\}^n$. The chain group $KC(L)$ carries two gradings, a *homological grading* gr_h and a *quantum grading* gr_q, defined as follows:

$$\mathrm{gr}_h((D_L(u), x)) = -n_- + |u|,$$
$$\mathrm{gr}_q((D_L(u), x)) = n_+ - 2n_- + |u| + \#\{Z \in Z(D_L(u)) \mid x(Z) = x_+\} - \#\{Z \in Z(D_L(u)) \mid x(Z) = x_-\}.$$

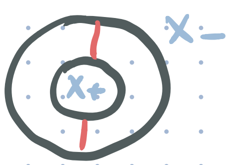
$n_+ = 2$

$n_- = 0$





- Example:



$(D_{H_2}(1,1), x_- x_+)$

$\mathrm{gr}_h((D_{H_2}(1,1), x_- x_+)) = 0 + 2$

$\mathrm{gr}_q((D_{H_2}(1,1), x_- x_+)) = 2 - 0 + 2 + 1 - 1 = 4$

Name	Generator	gr_h	gr_q
a	$(D_H(00), x_+x_+)$	0	4
v	$(D_H(00), x_+x_-)$	0	2
w	$(D_H(00), x_-x_+)$	0	2
m	$(D_H(00), x_-x_-)$	0	0
b	$(D_H(10), x_+)$	1	4
x	$(D_H(10), x_+)$	1	2

Name	Generator	gr_h	gr_q
c	$(D_H(01), x_+)$	1	4
y	$(D_H(01), x_-)$	1	2
e	$(D_H(11), x_-x_+)$	2	4
z	$(D_H(11), x_-x_-)$	2	2
l	$(D_H(11), x_+x_+)$	2	6
d	$(D_H(11), x_+x_-)$	2	4

WHAT ELSE DO WE NEED TO GET A CHAIN COMPLEX?

DIFFERENTIAL

The differential preserves the quantum grading, increases the homological grading by 1, and is defined as

$$\delta(D_L(v), y) = \sum_{\substack{(D_L(u),x) \\ |u|=|v|+1 \\ (D_L(v),y) \prec (D_L(u),x)}} (-1)^{s_0(\mathcal{C}_{u,v})} (D_L(u), x),$$

DEF

where $s_0(\mathcal{C}_{u,v}) \in \mathbb{F}_2$ is defined as follows: if $u = (\epsilon_1, \ldots, \epsilon_{i-1}, 1, \epsilon_{i+1}, \ldots, \epsilon_n)$ and $v = (\epsilon_1, \ldots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \ldots, \epsilon_n)$, then $s_0(\mathcal{C}_{u,v}) = (\epsilon_1 + \cdots + \epsilon_{i-1})$; see also Definition 4.5.

T(2,3)

THIS WE KNOW:

KNOT

$\cdots \to A_n \to A_{n+1} \to \cdots$

BIGRADED CHAIN COMPLEX

WHAT'S NEXT?

→ FRAMED FLOW CATEGORIES

→ CW COMLEXES

finally

FRAMED FLOW CATEGORIES

Why do we need them?

3.3. **Framed flow categories to CW complexes.** We are interested in framed flow categories because one can build a CW complex $|\mathscr{C}|$ from a framed flow category $\mathscr{C}$ in such a way that if $\mathscr{C}$ refines a chain complex C^* then C^* is the cellular cochain complex of $|\mathscr{C}|$.

$|\mathscr{C}|$

BUT FIRST WE NEED FEW DEFINITIONS

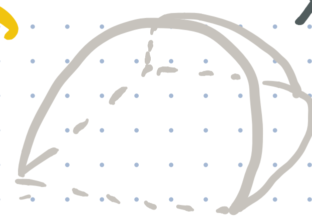
• SMOOTH MANIFOLD WITH CORNERS

$\mathbb{R}_+^k$ $([0,\infty))^k$

$\downarrow$

$(x_1, \dots, x_k)$

smooth structure



X

$x \in X$ is an image of some $(x_1, \dots, x_k)$

Let's define the codimension-i-boundary

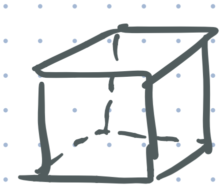
$$\partial^i X = \{x \in X \mid c(x) = i\}$$

number of coordinates in $(x_1, \dots, x_k)$ s.t. $x_i = 0$

Remark: x belongs to $\leq c(x)$ different components $\partial^1 X$.

• SMOOTH MANIFOLD WITH FACES

if every $x \in X$ is contained in exactly $c(x)$ components of $\partial^1 X$



any polytope

→ connected face = closure of a component of $\partial^1 X$

→ face := any union of pairwise disjoint connected faces (including $\emptyset$)

• SMOOTH $\langle n \rangle$ - MANIFOLD

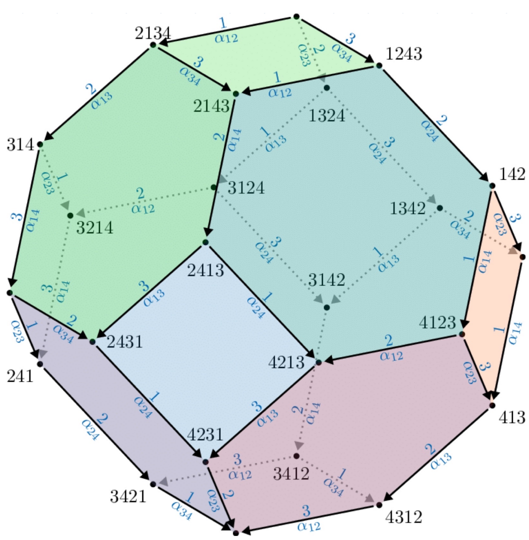
X - smooth manifold with faces with additional structure:

n-face structure = ordered tuple $(\partial_1 X, \ldots, \partial_n X)$ of faces s.t.

1) $\partial_1 X \cup \ldots \cup \partial_n X = \partial X$
2) $\partial_i X \cap \partial_j X$ is a face of both $\partial_i X$ and $\partial_j X$ for $i \neq j$

EXAMPLE

PERMUTOHEDRON



Def $\underline{d = (d_0, \dots, d_n) \in \mathbb{N}_+^{n+1}}$

Define

$$\mathbb{E}^d := \underbrace{\mathbb{R}^{d_0}} \times \underbrace{[0,\infty)}_{} \times \mathbb{R}^{d_1} \times [0,\infty) \times \dots \times [0,\infty)$$

$\mathbb{R}_+$

turn into $\langle n \rangle$-manifold defining

$$\partial_i \mathbb{E}^d := \mathbb{R}^{d_0} \times \dots \times \mathbb{R}^{d_{i-1}} \times \underbrace{\{0\}}_{} \times \mathbb{R}^{d_{i+1}} \times \dots \times \mathbb{R}^{d_n}$$

$[0,\infty)$

ith place

Def A NEAT IMMERSION $\imath$ of $\langle n \rangle$-manifold is a

→ smooth immersion $\imath : X \hookrightarrow \mathbb{E}^d$ for some d

+

→ $\forall_i \; \imath^{-1}(\partial_i \mathbb{E}^d) = \partial_i X$

→ the intersection of $X(a) := \bigcap_{i \in \{j \mid a_j = 0\}} \partial_i X$ and $\mathbb{E}^d(b)$ is $\perp$ for all $b < a$

HOW DOES IT LOOK LIKE?

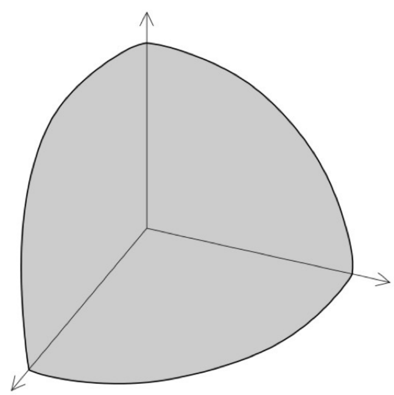


FIGURE 3.1. **A neat embedding of the 2-dimensional $\langle 3\rangle$-manifold 'triangle' in $(\mathbb{R}_+)^3$.**

Definition 2.3. A *framed flow category* consists of a category $\mathscr{C}$ with finitely many objects $\mathrm{Ob} = \mathrm{Ob}(\mathscr{C})$, a function $|\cdot|\colon \mathrm{Ob} \to \mathbb{Z}$, called the *grading*, an $(n+1)$-tuple of non-negative integers $\mathbf{d} = (d_k, \dots, d_{n+k})$ and a collection $\varphi_.$ of immersions satisfying the following:

(1) $k = \min\{|x| : x \in \mathrm{Ob}(\mathscr{C})\}$ and $n = \max\{|x| : x \in \mathrm{Ob}(\mathscr{C})\} - k$.

(2) $\mathrm{Hom}(x,x) = \{\mathrm{id}\}$ for all $x \in \mathrm{Ob}$, and for $x \neq y \in \mathrm{Ob}$, $\mathrm{Hom}(x,y)$ is a smooth, compact $(|x|-|y|-1)$-dimensional $\langle |x|-|y|-1 \rangle$-manifold which we denote by $\mathcal{M}(x,y)$, and whose immersions are functions $\imath_{x,y}\colon \mathcal{M}(x,y) \to \mathbb{E}^{\mathbf{d}}[|y| : |x|]$.

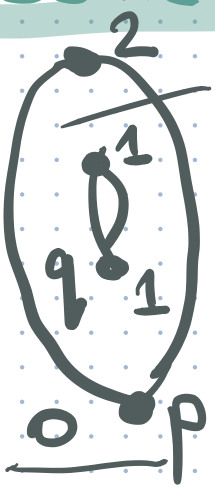
(3) For $x, y, z \in \mathrm{Ob}$ with $|z| - |y| = m$, the composition map
$$\circ\colon \mathcal{M}(z,y) \times \mathcal{M}(x,z) \to \mathcal{M}(x,y)$$
is an embedding into $\partial_m \mathcal{M}(x,y)$. Furthermore,
$$\circ^{-1}(\partial_i \mathcal{M}(x,y)) = \begin{cases} \partial_i \mathcal{M}(z,y) \times \mathcal{M}(x,z) & \text{for } i < m \\ \mathcal{M}(z,y) \times \partial_{i-m}\mathcal{M}(x,z) & \text{for } i > m \end{cases}$$
and
$$\imath_{x,y}(p \circ q) = (\imath_{z,y}(p), 0, \imath_{x,z}(q)).$$

(4) For $x \neq y \in \mathrm{Ob}$, $\circ$ induces a diffeomorphism
$$\partial_i \mathcal{M}(x,y) \cong \coprod_{z,\, |z| = |y|+i} \mathcal{M}(z,y) \times \mathcal{M}(x,z).$$

(5) The immersions $\imath_{x,y}$ for $x, y \in \mathrm{Ob}(\mathscr{C})$ extend to immersions
$$\varphi_{x,y}\colon \mathcal{M}(x,y) \times [-\varepsilon, \varepsilon]^{\mathbf{d}_{|y|:|x|}} \looparrowright \mathbb{E}^{\mathbf{d}}[|y| : |x|]$$
which satisfy
$$\varphi(x,y)(p \circ q, t_1, \dots, t_{\mathbf{d}_{|y|,|x|}}) = \\ (\varphi_{z,y}(p, t_1, \dots, t_{\mathbf{d}_{|y|:|z|}}), 0, \varphi_{x,z}(q, t_{\mathbf{d}_{|y|:|z|}+1}, \dots, t_{\mathbf{d}_{|y|:|x|}}))$$
for all $p \in \mathcal{M}(z,y)$, $q \in \mathcal{M}(x,z)$ where $z \in \mathrm{Ob}(\mathscr{C})$.

The manifold $\mathcal{M}(x,y)$ is called the *moduli space from x to y*, and we also set $\mathcal{M}(x,x) = \emptyset$.

Some intuition: MORSE FLOW CATEGORY



$\xrightarrow[\text{function}]{\text{Morse}} \mathbb{R}$

$f: M \to \mathbb{R}$

index = dim of a space where hessian matrix is negative def.

objects: critical points

grading: index of each point

$|p| = 0$.

If we have two critical points of a relative index $n+1$

$M(x,y)$ from x to y

$(n+1)$-dim subspace of all points that flow up to x and down to y

Where is the promised CW-complex?

Here

$|a| = i$

Definition 2.4. Let $\mathscr{C}$ be a framed flow category embedded into $\mathbb{E}^{\mathbf{d}}$ for some $\mathbf{d} = (d_k, \dots, d_{k+n})$. For an arbitrary object a in $\mathrm{Ob}(\mathscr{C})$ of degree i, recall that for each object b in $\mathrm{Ob}(\mathscr{C})$ of degree $j < i$, we have the embedding

$$\varphi_{a,b} : \mathcal{M}(a,b) \times [-\varepsilon, \varepsilon]^{\mathbf{d}_{j:i}} \to [-R,R]^{d_j} \times [0,R] \times \cdots \times [0,R] \times [-R,R]^{d_{i-1}}$$

where R is chosen to be large enough that all moduli spaces $\mathcal{M}(a,b)$ can be embedded in this way. The CW complex $|\mathscr{C}|$ consists of one 0-cell (the basepoint) and one $(d_k + \cdots + d_{n+k-1} - k + i)$-cell $\mathcal{C}(a)$ for every object a with $|a| = i$ defined as

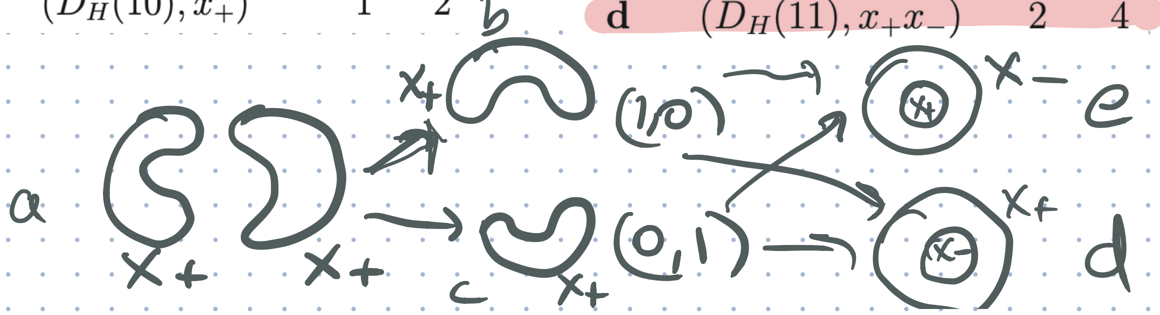
$$[0,R] \times [-R,R]^{d_k} \times \cdots \times [-R,R]^{d_{i-1}} \times \{0\} \times [-\varepsilon,\varepsilon]^{d_i} \times \{0\} \times \cdots \times \{0\} \times [-\varepsilon,\varepsilon]^{d_{n+k-1}}.$$

EXAMPLE

generators with quantum grading 4

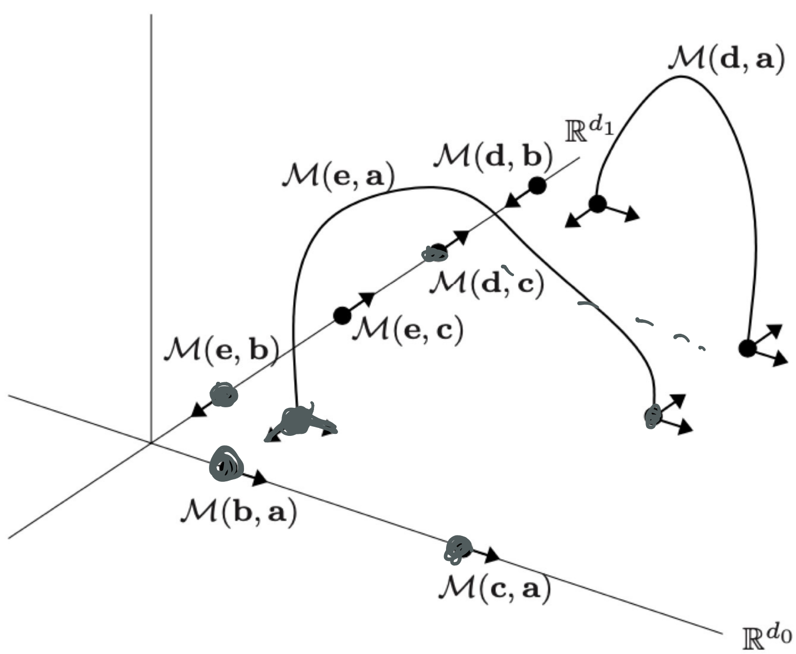
Name	Generator	gr_h	gr_q
a	$(D_H(00), x_+x_+)$	0	4
v	$(D_H(00), x_+x_-)$	0	2
w	$(D_H(00), x_-x_+)$	0	2
m	$(D_H(00), x_-x_-)$	0	0
b	$(D_H(10), x_+)$	1	4
x	$(D_H(10), x_+)$	1	2

Name	Generator	gr_h	gr_q
c	$(D_H(01), x_+)$	1	4
y	$(D_H(01), x_-)$	1	2
e	$(D_H(11), x_-x_+)$	2	4
z	$(D_H(11), x_-x_-)$	2	2
l	$(D_H(11), x_+x_+)$	2	6
d	$(D_H(11), x_+x_-)$	2	4



Corresponding chain complex

$$\begin{tikzcd} & b \arrow[r, "-1"] \arrow[rdd, "-1"'] & e \\ a \arrow[ru] \arrow[rd] & & \\ & c \arrow[r] \arrow[ruu] & d \end{tikzcd}$$



a: Embedding of the subcategory in quantum grading 4.

Definition 2.4. Let $\mathscr{C}$ be a framed flow category embedded into $\mathbb{E}^{\mathbf{d}}$ for some $\mathbf{d} = (d_k, \ldots, d_{k+n})$. For an arbitrary object a in $\mathrm{Ob}(\mathscr{C})$ of degree i, recall that for each object b in $\mathrm{Ob}(\mathscr{C})$ of degree $j < i$, we have the embedding

$$\varphi_{a,b} : \mathcal{M}(a,b) \times [-\varepsilon, \varepsilon]^{\mathbf{d}_{j:i}} \to [-R, R]^{d_j} \times [0, R] \times \cdots \times [0, R] \times [-R, R]^{d_{i-1}}$$

where R is chosen to be large enough that all moduli spaces $\mathcal{M}(a,b)$ can be embedded in this way. The CW complex $|\mathscr{C}|$ consists of one 0-cell (the basepoint) and one $(d_k + \cdots + d_{n+k-1} - k + i)$-cell $\mathcal{C}(a)$ for every object a with $|a| = i$ defined as

$$[0, R] \times [-R, R]^{d_k} \times \cdots \times [-R, R]^{d_{i-1}} \times \{0\} \times [-\varepsilon, \varepsilon]^{d_i} \times \{0\} \times \cdots \times \{0\} \times [-\varepsilon, \varepsilon]^{d_{n+k-1}}.$$

Let's embed relative $d = (1,1)$

How do cells look like?

$|a| = 0$

2-cell

$C(a) = \{0\} \times [-\varepsilon, \varepsilon] \times \{0\} \times [-\varepsilon, \varepsilon]$

$|b| = 1 = |c|$

3-cell

$C(b) = [0,R] \times [-R,R] \times \{0\} \times [-\varepsilon, \varepsilon] = C(c)$

$|d| = 2 = |e|$

$$C(d) = [0,R] \times [-R,R] \times \ldots \times [-R,R] = C(e)$$

quantum grading 4

1) 0-cell as a base

2) Glue $C(a)$ (2-ball) to the base point

3) $C(b), C(c)$ (3-balls) gives a 3-sphere

4) $C(d), C(e)$ (4-balls) We get a 4-sphere S^4!

We know the space for $q=4$:

$$\text{space}_{q=0} \vee \text{space}_{q=2} \vee S^4_{q=4} \vee \text{space}_{q=6}$$

One still needs to calculate for other quantum gradings.