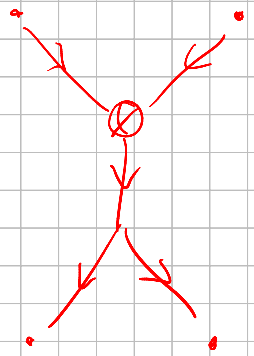
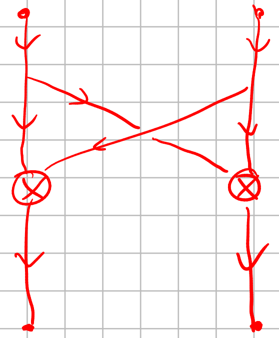
Hopf algebras, equivariant Lagrangian Floer homology and cornered instanton theory





Guillem Cazassus, Oxford

Joint with Paul Kirk, Artem Kotelskiy, Mike Miller, Wai-Kit Yeung

(Work in progress...)

• Equivariant Lagrangian Floer homology

$(M,\omega) \supset L_0, L_1$ (+ assumptions) $\rightsquigarrow HF(L_0,L_1)$

symplectic manifold

pair of Lagrangians

- compact + monotone
- exact + convex at ∞

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Pb: What if M and/or L_0, L_1 are ... singular?

Ex: $M = \tilde{M} /\!/ G = \mu^{-1}(0)/G \qquad G \circlearrowright \tilde{M} \xrightarrow{\mu} \mathfrak{g}^*$

$L_i = \tilde{L}_i / G$ (ex: Atiyah-Floer conjecture, $M = \mathcal{M}(\Sigma)$ ...)

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$L_i = \tilde{L}_i / G$ (ex: Atiyah-Floer conjecture, $M = \mathcal{M}(\Sigma)$)

$\Rightarrow$ try to substitute $HF(M; L_0, L_1)$ by

$HF_G(\tilde{M}; \tilde{L}_0, \tilde{L}_1)$

"equivariant Lagrangian Floer homology"

Approaches for defining $HF_G(M; L_0, L_1)$:

× Symplectic vortex equation:
Cieliebak-Gaio-Salamon, Mundet i Riera,
Frauenfelder, Tian-Xu, Woodward ...

× ∞-categories: Hendricks-Lipshitz-Sarkar

× Morse-Bott: Austin-Braam, Fukaya-Daemi

× Borel construction: $EG = \varinjlim_{N} EG_N$, T^*EG_N

Viterbo, Bourgeois-Oancea, Kim-Lau-Zheng,
C. (Work in progress...)

Another approach:

$$\underset{\substack{\uparrow \\ \text{Compact} \\ \text{Lie group}}}{G} \circlearrowright \underset{\substack{\nwarrow \\ \text{compact} \\ \text{smooth} \\ \text{manifold}}}{X}$$

$$H^G_*(X) = H_*(X \times_G EG)$$

Another approach :

$G \circlearrowright X$

Compact Lie group

compact smooth manifold

$$H^G_*(X) = H_*(X \times_G EG)$$

$$= H_*\left(C_*(X) \otimes_{C_*(G)} C_*(EG) \right)$$

Th (Gugenheim-May)

Another approach:

$G \circlearrowright X$

compact Lie group

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Goal: mimic this formula: "$HF_G(L_0,L_1) := H_*\left(CF(L_0,L_1) \otimes_{C_*(G)} C_*(EG)\right)$"

Another approach:

$$G \circlearrowright X$$

G: compact Lie group

X: compact smooth manifold

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Goal: mimic this formula: "$HF_G(L_0, L_1) := H_*\left(CF(L_0, L_1) \underset{C_*(G)}{\otimes} C_*(EG) \right)$"

$\Rightarrow$ need a G-action at the chain level.

Th: [C., Kirk, Kotelskiy, Miller, Yeung] Work over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

Let:
- G compact Lie group, $g : G \to \mathbb{R}$ Morse fct°
- (M, ω): symplectic, with Hamiltonian action $G \circlearrowright M \xrightarrow{\mu} \mathfrak{g}^*$

  moment map.
- $L_0, L_1 \subset M$ pair of Lagrangians

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be such that: • $e \in G$ is a local minimum for g

• (M, L_0, L_1) satisfy either:

Weinstein correspondence $\subset T^*G \times M^- \times M$

- × M, L_0, L_1 exact
- × M convex at ∞
- × L_0, L_1 compact
- × $\mu : M \to \mathfrak{g}^*$ proper

or

- × M, L_0, L_1 compact
- × $M, L_0, L_1, \Lambda_G(M)$ positively monotone
- × minimal Maslov number of $L_0, L_1, \Lambda_G(M)$ is proportional to an integer $N \geq 3$.

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Weinstein correspondence $\subset T^*G \times M^- \times M$

Then
× The Morse complex $CM_*(G,g)$ is an A_∞-algebra, e is a strict unit: $\mu^1(e) = 0$, $\mu^2(x,e) = \mu^2(e,x) = x$, $\mu^k(\dots, e, \dots) = 0$, $k \geq 3$.

× The Floer complex $CF(L_0, L_1)$ is an A_∞-module over $CM_*(G,g)$

$\rightarrow$ Equivariant Lagrangian Floer homologies

(use Mike Miller's thesis appendix)

$A = CM_*(G, g)$, $\bar{A} = A/\langle e \rangle$

→ Equivariant Lagrangian Floer homologies

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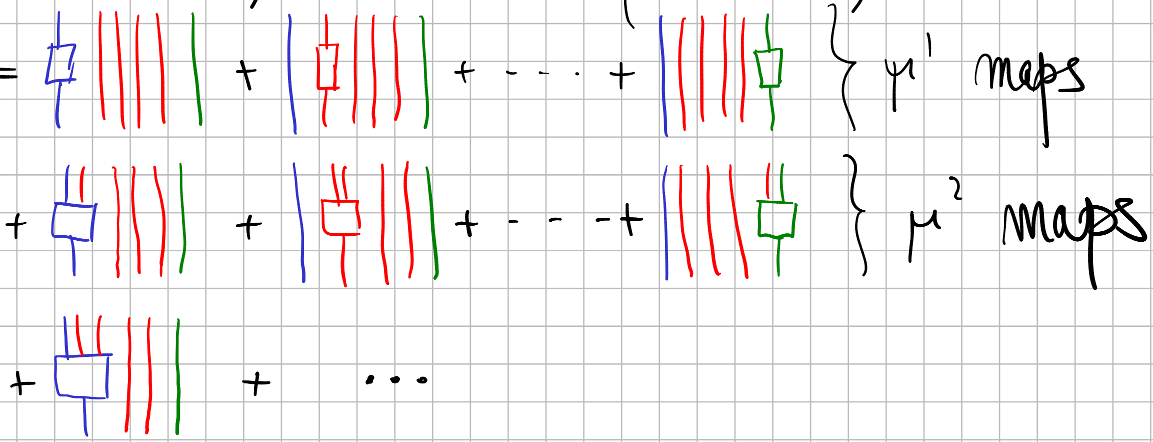
$A = CM_*(G,g)$, $\bar{A} = A/_{\langle e\rangle}$

strictly unital A_∞-algebra

left/right A_∞-modules

Bar construction : $B(M,A,N) = M \otimes \left(\bigoplus \bar{A}^{\otimes k}\right) \otimes N$

with differential $\partial =$



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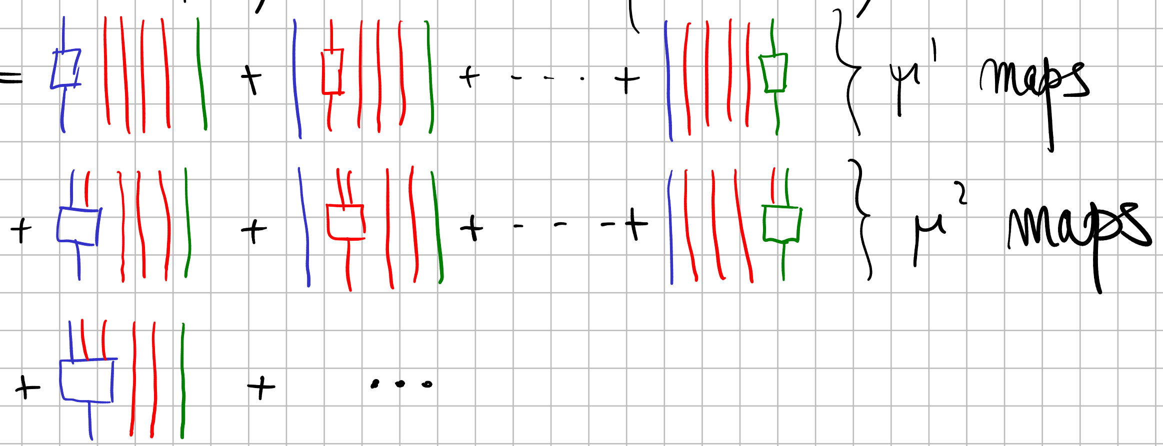

- $A = CM_*(G,g)$
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co-Bar construction: $cB(N,A,M) = \mathrm{Hom}_A(B(N,A,N), M)$

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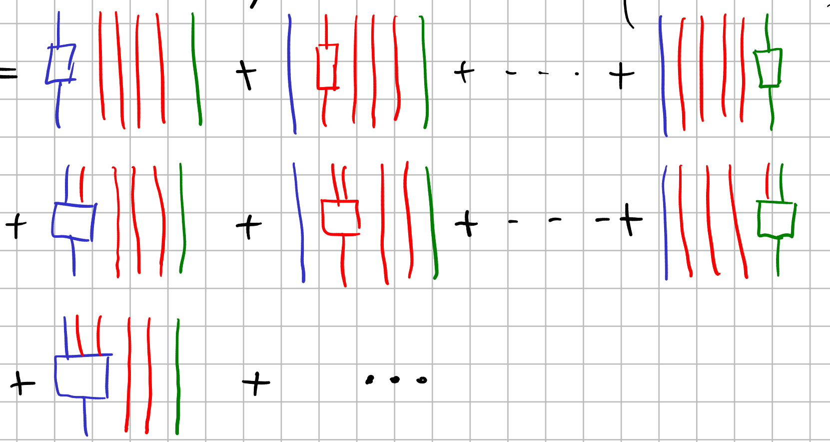

} μ^1 maps

} μ^2 maps

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Borel complex : $C_A^+(M) = B(M,A,N)$

co-Borel complex : $C_A^-(M) = cB(N,A,M)$

Tate complex : $C_A^\infty(M) = \mathrm{Cone}\left(\underline{N_M} : B(M,A,D_A) \longrightarrow C_A^-(M)\right)$

$= cB(N,A,A)$ "dualizing complex"

⇝ Equivariant Lagrangian Floer homologies

(use Mike Miller's thesis appendix)

strictly unital A_∞-algebra

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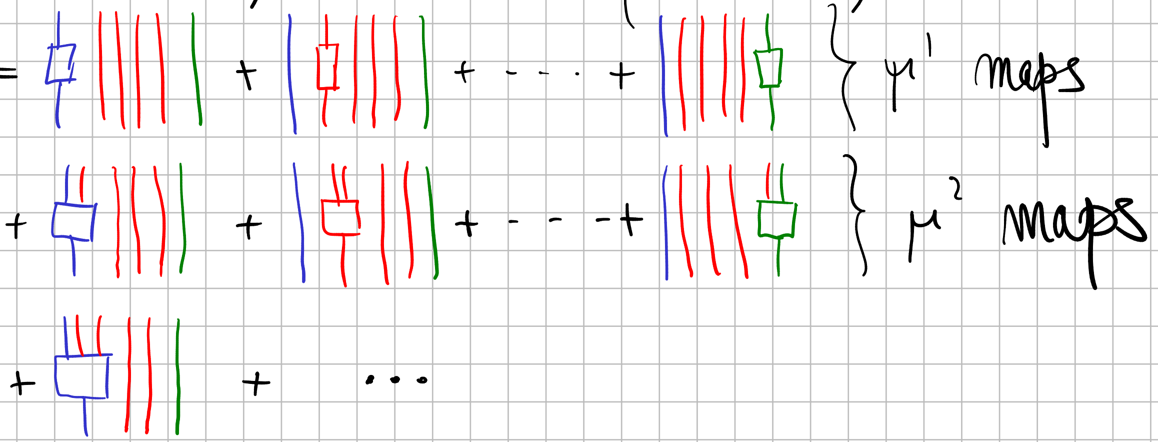

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⇝ Get homology groups $HF_G^+(L_0,L_1)$, $HF_G^-(L_0,L_1)$ and $HF_G^\infty(L_0,L_1)$.

Th: [C., Kirk, Kotelskiy, Miller, Yeung] Morse theoretic analog

$$\underset{\text{compact Lie group}}{G} \curvearrowright \underset{\text{smooth compact mfd}}{X}$$

$$\left.\begin{aligned} g &: G \longrightarrow \mathbb{R} \\ f &: X \longrightarrow \mathbb{R} \end{aligned}\right\} \text{Morse functions}$$

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compact Lie group

smooth compact mfd

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1. $CM_*(G,g)$: A_∞- algebra
2. $CM_*(X,f)$: A_∞ - module / $CM_*(G,g)$

such that get algebra & module structure on homology induced by:

$$m_G : G \times G \longrightarrow G \qquad\qquad (m_G)_* : H_*(G) \otimes H_*(G) \to H_*(G)$$

$$\rightsquigarrow$$

$$m_X : G \times X \longrightarrow X \qquad\qquad (m_X)_* : H_*(G) \otimes H_*(X) \to H_*(X)$$

Pushforwards on Morse homology (→ Kronheimer-Mrowka's book, sec 2.8)

$$\begin{tikzcd} & \mathbb{R} & & \mathbb{R} \\ \Phi : M \arrow[ur, "f"] \arrow[rr] & & N \arrow[ur, "g"] & \end{tikzcd}$$

smooth map

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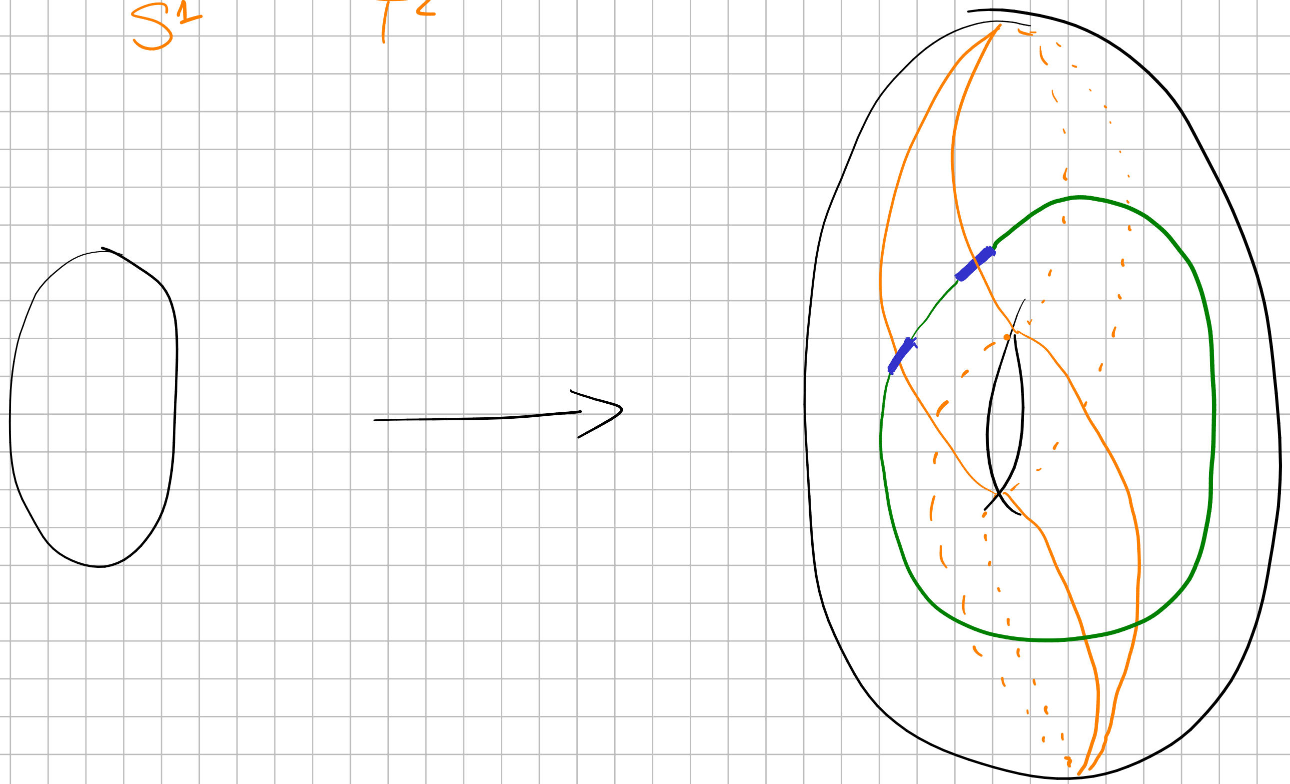
$$\begin{tikzcd} & \mathbb{R} & & \mathbb{R} \\ \Phi : \underbrace{M}_{S^1} \arrow[ur, "f"] \arrow[rr] & & \underbrace{N}_{T^2} \arrow[ur, "g"] \end{tikzcd}$$ smooth map



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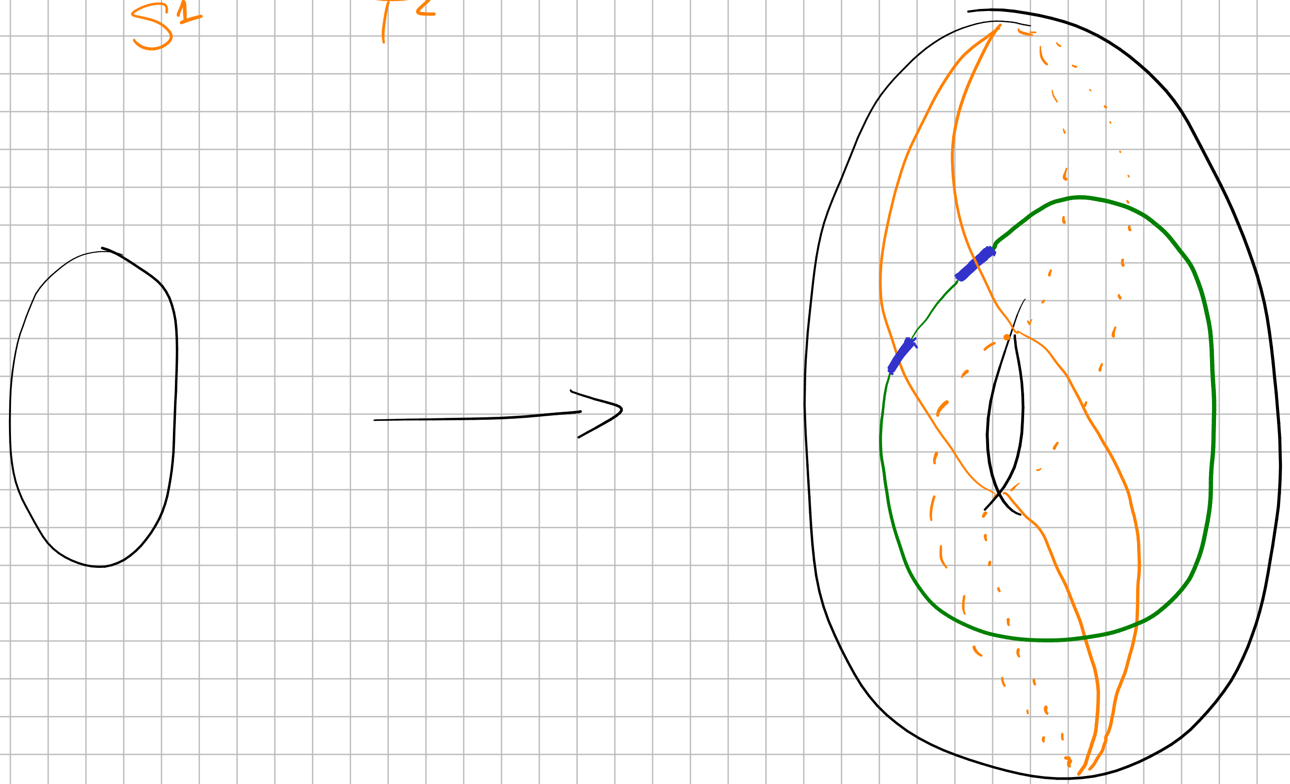
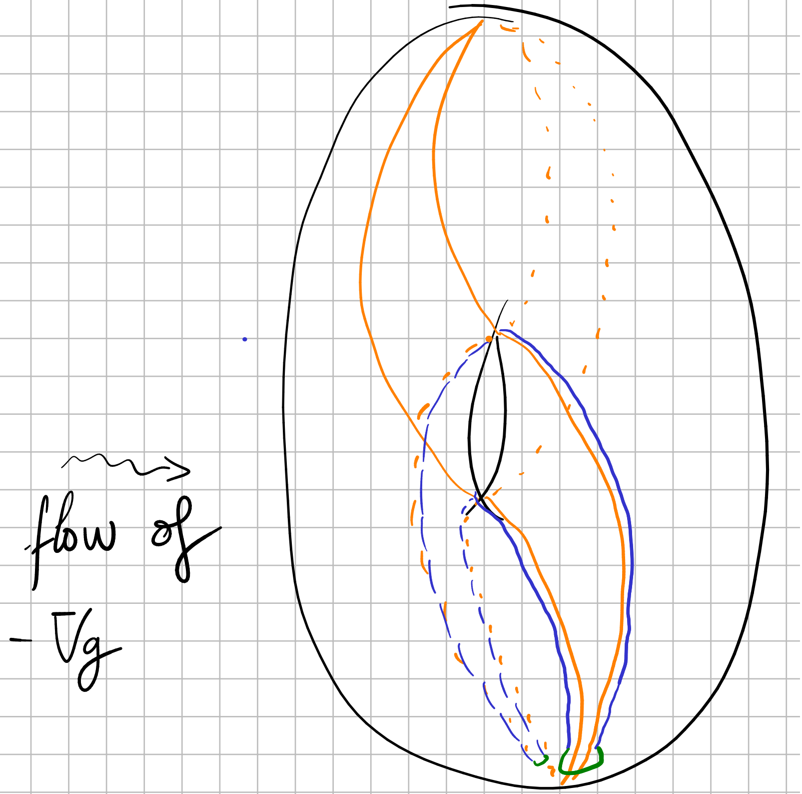
$$\Phi : \underbrace{\overset{f \nearrow \mathbb{R}}{M}}_{S^1} \longrightarrow \underbrace{\overset{g \nearrow \mathbb{R}}{N}}_{T^2} \quad \text{smooth map}$$





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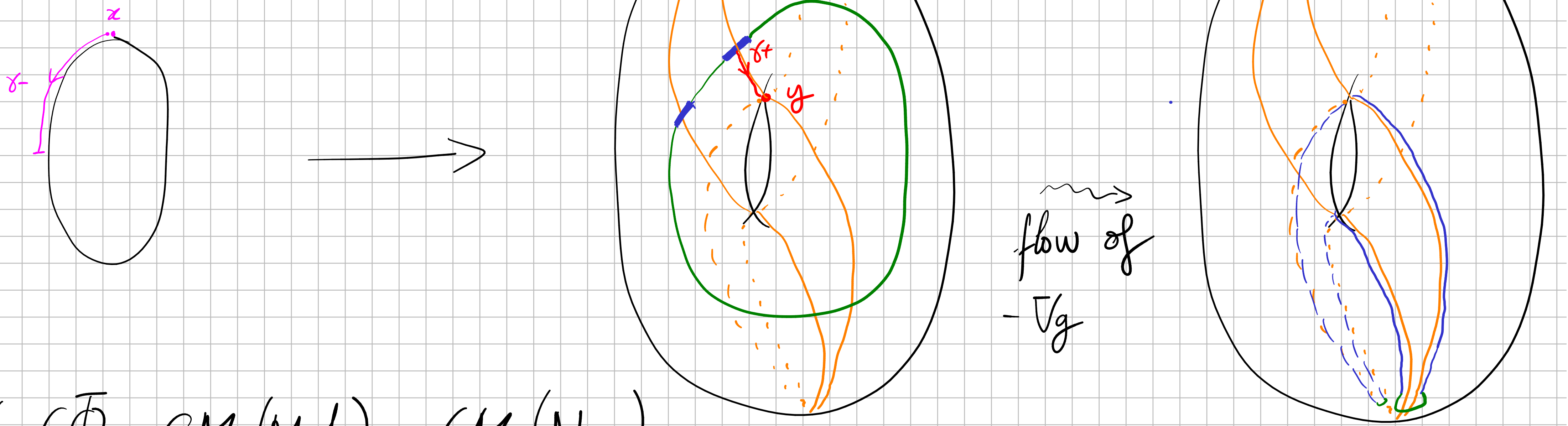
$\Phi : \underbrace{M}_{S^1} \longrightarrow \underbrace{N}_{T^2}$ smooth map



↳ $C\Phi_* : CM_*(M,f) \to CM_*(N,g)$

$$C\Phi_*(x) = \sum_y{}' \# \left\{ \overset{\bullet}{x} \longrightarrow \Phi \longrightarrow \overset{\bullet}{y} \right\} \cdot y$$

$\gamma_- : \mathbb{R}_{\leq 0} \longrightarrow M$

$\gamma_+ : \mathbb{R}_{\geq 0} \longrightarrow N$

$\Phi(\gamma_-(0)) = \gamma_+(0)$

Pushforwards of multiplication maps

* $m_G :$
$$\begin{tikzcd} G \times G \arrow[r] \arrow[dr] & G \arrow[dr, "g"] & \\ & \mathbb{R} & \mathbb{R} \end{tikzcd}$$

$(x,y) \longmapsto g(x)+g(y)$

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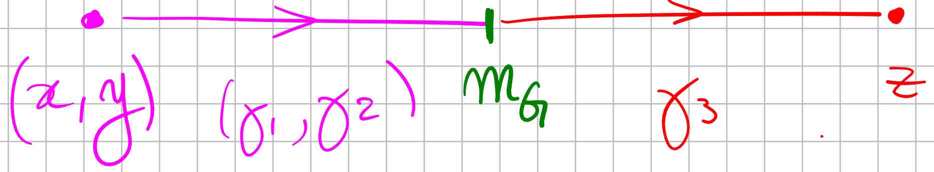
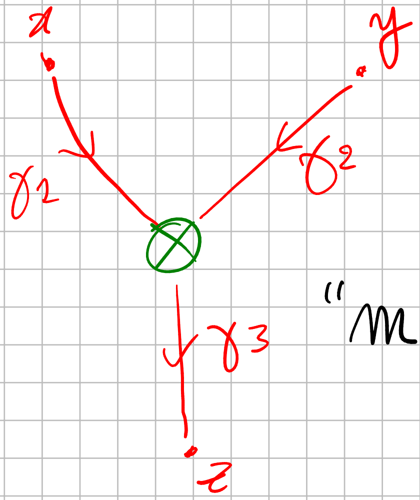


" grafted line "

$\Longleftrightarrow$



"multiplicative Y"

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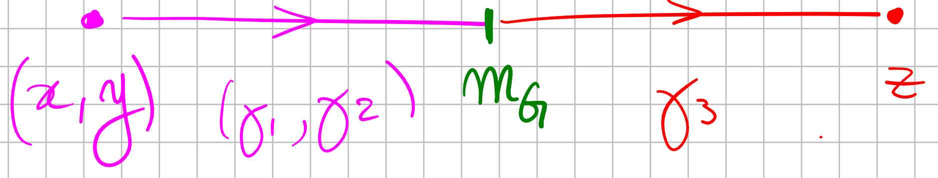

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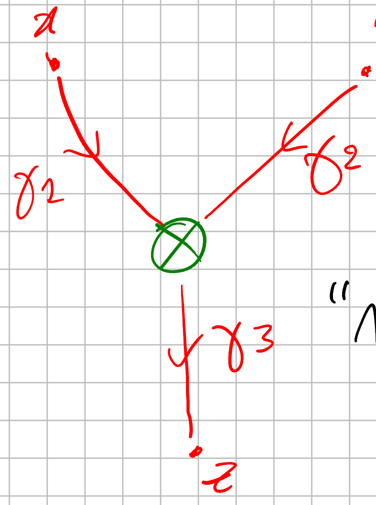

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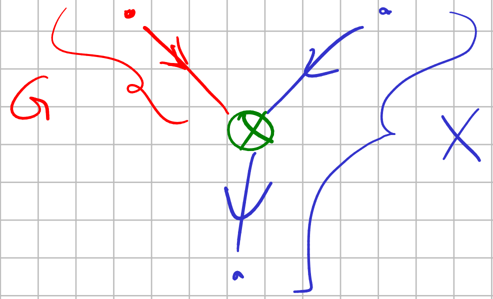

Pushforwards of multiplication maps

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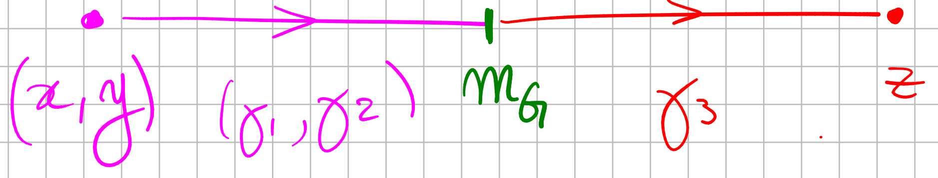

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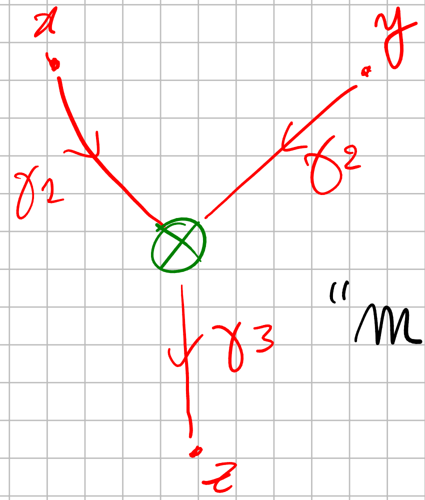

"multiplicative Y"

→ Get chain-level products:

$$\begin{cases} \mu_G : CM(G,g) \otimes CM(G,g) \longrightarrow CM(G,g) \\ \mu_X : CM(G,g) \otimes CM(X,f) \longrightarrow CM(X,f) \end{cases}$$

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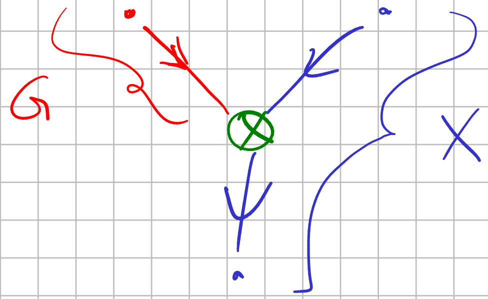

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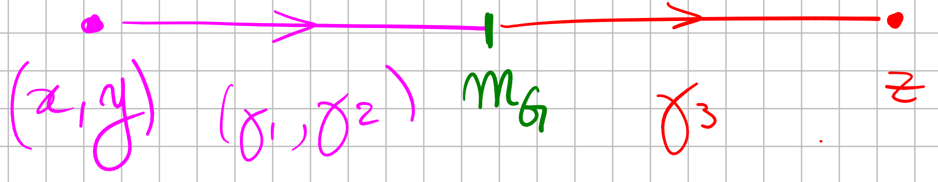

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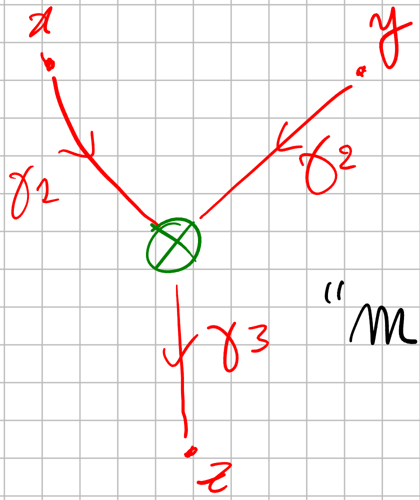

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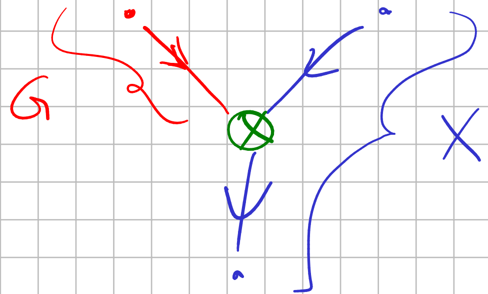

Q: Associative?

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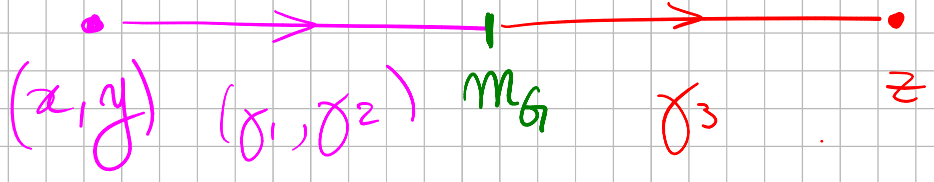

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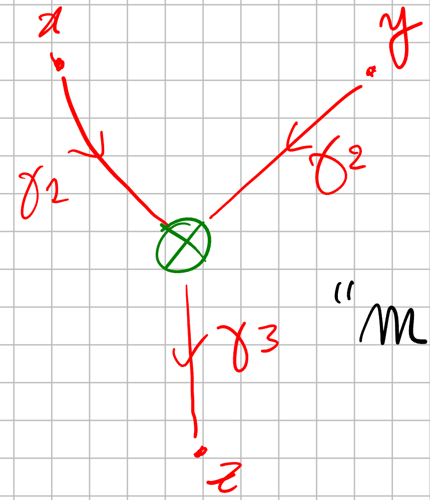

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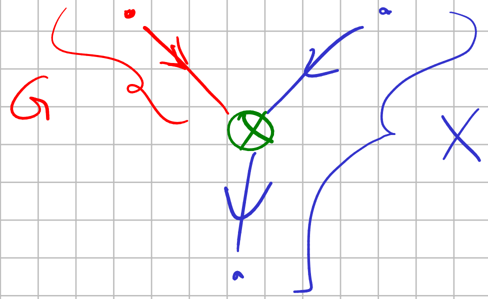

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Q: Associative?

A: Yes, up to homotopy...

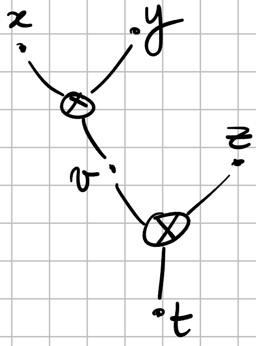
Associativity $x, y, z, t \in \text{Cirt}(G, g)$

$$\langle \mu_G(\mu_G(x,y),z) - \mu_G(x,\mu_G(y,z)), t \rangle$$

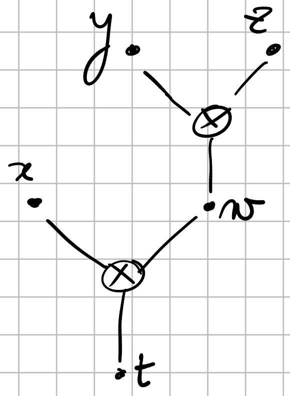
Associativity $x, y, z, t \in \mathrm{Crit}(G, g)$

$$\langle \mu_G(\mu_G(x,y),z) - \mu_G(x,\mu_G(y,z)), t \rangle$$

$$= \# \bigcup_{v} \Big\{$$



$$\Big\} - \# \bigcup_{w} \Big\{$$



$$\Big\}$$

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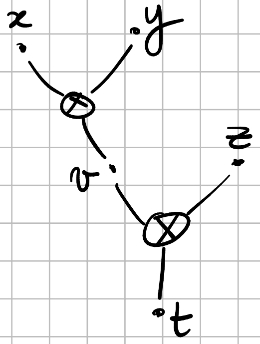

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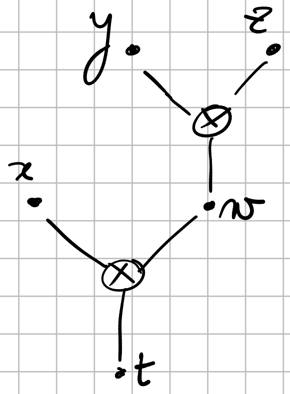


$\Big\}$

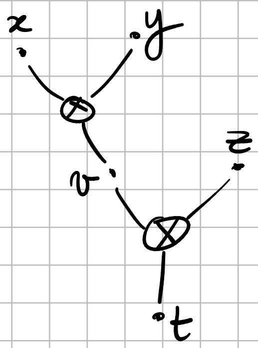
$\subset \partial \overline{A}$

$\subset \partial \overline{B}$

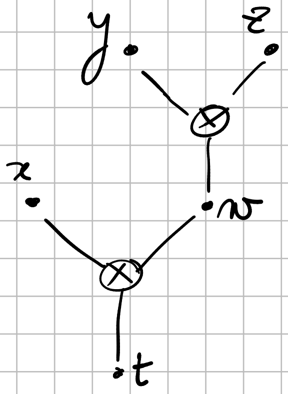
Associativity $x, y, z, t \in \mathrm{Crit}(G,g)$

$$\langle \mu_G(\mu_G(x,y),z) - \mu_G(x,\mu_G(y,z)), t \rangle$$

$= \# \bigcup_v \Big\{$



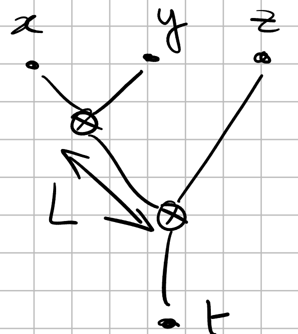
$\Big\} - \# \bigcup_w \Big\{$



$\Big\}$

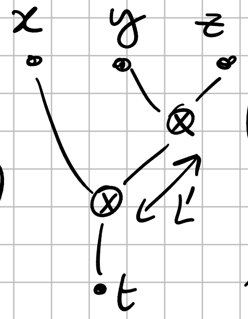
$\subset \partial \overline{A}$ $\qquad$ $\subset \partial \overline{B}$

$A = \bigcup_{L \geq 0} A_L \; , \; A_L = \Big\{$



$\Big\}$

$B = \bigcup_{L' \geq 0} B_{L'} \; , \; B_{L'} = \Big\{$

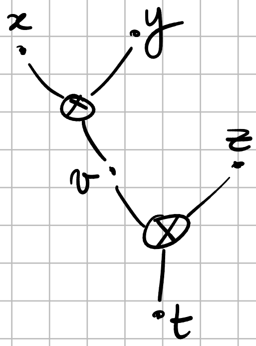


$\Big\}$

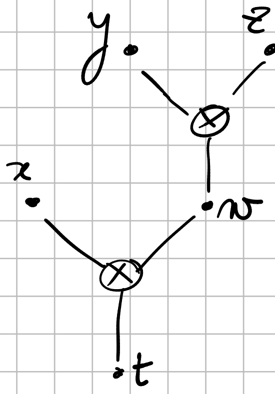
Associativity $x, y, z, t \in \mathrm{Crit}(G, g)$

$$\langle \mu_G(\mu_G(x,y),z) - \mu_G(x,\mu_G(y,z)), t \rangle$$

$$= \# \bigcup_v \Big\{$$



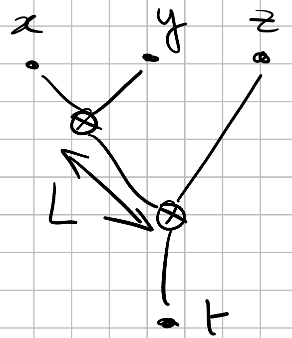
$$\Big\} - \# \bigcup_w \Big\{$$



$$\Big\}$$

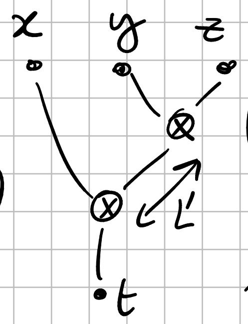
$\subset \partial\overline{\mathcal{A}}$ $\qquad$ $\subset \partial\overline{\mathcal{B}}$

$$\mathcal{A} = \bigcup_{L \geq 0} \mathcal{A}_L \;, \quad \mathcal{A}_L = \Big\{$$



$$\Big\}$$

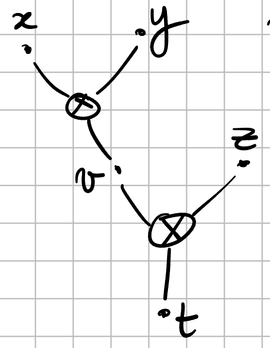
$$\mathcal{B} = \bigcup_{L' \geq 0} \mathcal{B}_{L'} \;, \quad \mathcal{B}_{L'} = \Big\{$$



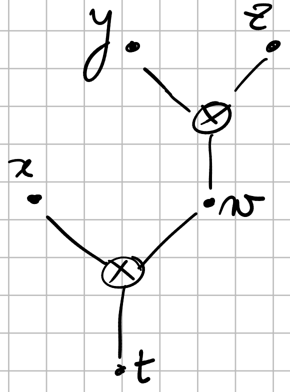
$$\Big\}$$

Let $\mathcal{M}(x, y, z; t) = \mathcal{A} \cup_{\mathcal{A}_0 = \mathcal{B}_0} \mathcal{B}$

$$\partial \overline{\mathcal{M}}(x, y, z; t) = \bigcup_{v} \Big\{$$

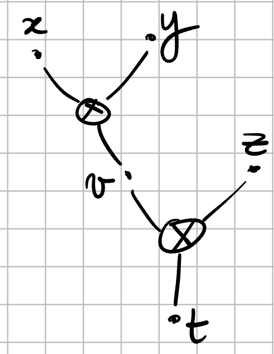


$$\Big\} \cup \bigcup_{w} \Big\{$$

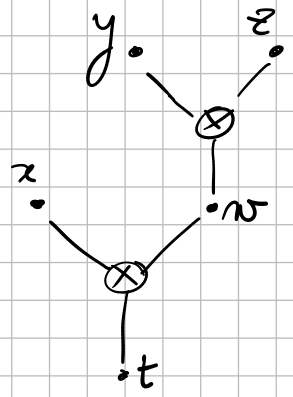


$$\Big\}$$

$\partial \overline{\mathcal{M}}(x,y,z;t) = \bigcup_{v} \{$

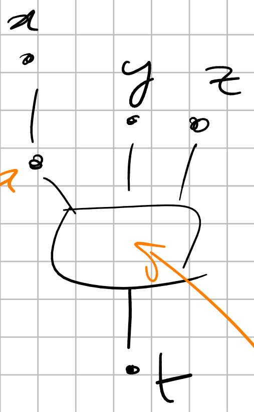


$\} \cup \bigcup_{w} \{$

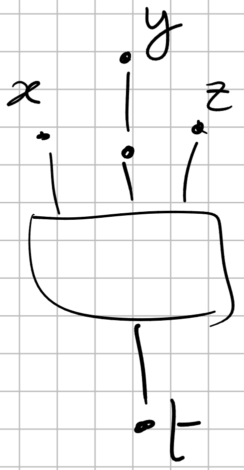


$\}$

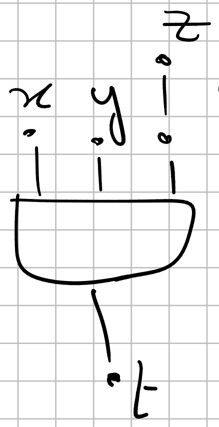
$\cup \{$



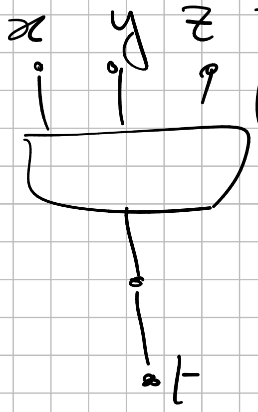
$\} \cup \{$



$\} \cup \{$

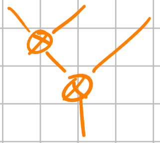


$\} \cup \{$

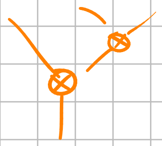


$\}$

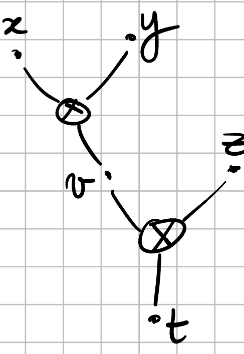
union over all possible breakings



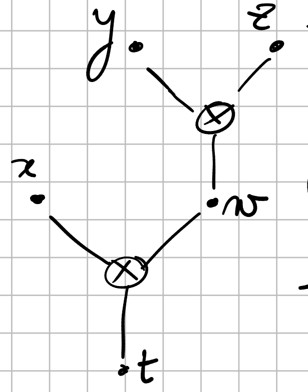
or



$$\partial \overline{\mathcal{M}}(x,y,z;t) = \bigcup_{v} \Big\{$$

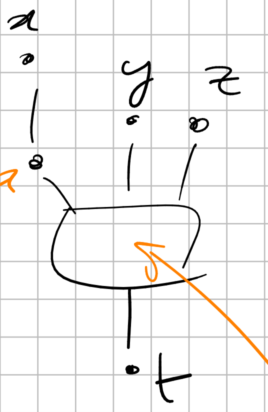


$$\Big\} \cup \bigcup_{w} \Big\{$$

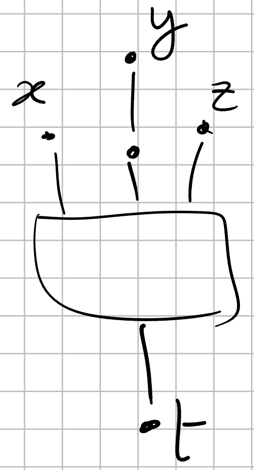


$$\Big\}$$

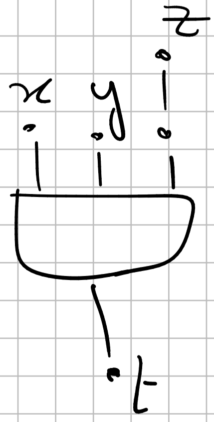
$$\cup \Big\{$$



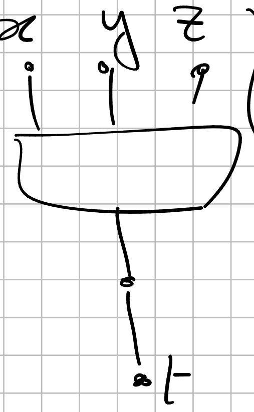
$$\Big\} \cup \Big\{$$



$$\Big\} \cup \Big\{$$



$$\Big\} \cup \Big\{$$



$$\Big\}$$

union over all possible breakings

$$\Rightarrow \quad \mu_G(\mu_G(\cdot,\cdot),\cdot) - \mu_G(\cdot,\mu_G(\cdot,\cdot)) = \mu^3 \circ (\mu' \otimes id \otimes id + id \otimes \mu' \otimes id + id \otimes id \otimes \mu')$$
$$+ \mu' \circ \mu^3$$

(third A_∞ - relation)

The A_∞-structure on $CM_*(G, \underline{g})$:

- μ^1: Morse differential.

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- μ^1 : Morse differential.
- Let $k \geq 2$, $T_k := \{(k+1)\text{-leafed metric rooted ribbon trees}\}$

↑ finite edges have a length

$x_1, \dots, x_k, y \in Crit(G,g)$

$\rightsquigarrow$ define $\mathcal{M}(x_1, \dots, x_k; y) = \{(\Gamma, \gamma) \mid \Gamma \in T_k,\ \gamma : \Gamma \to G,\ (*)\}$

$$\dim \mathcal{M}(x_1, \dots, x_k; y) = \underbrace{(k-2)}_{\dim T_k} + \underbrace{i(x_1) + \dots + i(x_k) - i(y)}_{\text{Morse indices}}$$

The A_∞-structure on $CM_*(G,g)$:

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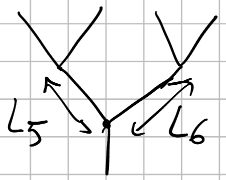
Define $\mu^k : CM_*(G,g)^{\otimes k} \longrightarrow CM_*(G,g)$

by $\mu^k(x_1, \dots, x_k) = \sum_y{}' \ \#\mathcal{M}(x_1, \dots, x_k; y) \cdot y$

when dim $= 0$,
$= 0$ otherwise

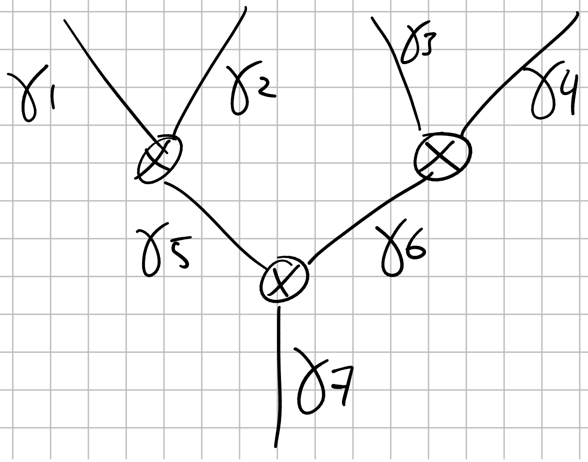
$$\mathcal{M}(x_1, \dots, x_k ; y) = \left\{ (\Gamma, \gamma) \,\middle|\, \Gamma \in \mathcal{T}_k \,,\; \gamma : \Gamma \to G,\; \underline{\underline{(*)}} \right\}$$

$$\mathcal{M}(x_1, \dots, x_k; y) = \left\{ (\Gamma, \gamma) \;\middle|\; \Gamma \in T_k \,,\; \gamma : \Gamma \to G,\; \underline{\underline{(*)}} \right\}$$

ex : $\Gamma =$ , $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7)$ with :

$\gamma_1, \gamma_2, \gamma_3, \gamma_4 : \mathbb{R}_{\leq 0} \longrightarrow G$ $\quad$ $\gamma_5 : [0, L_5] \to G$ $\quad$ $\gamma_7 : \mathbb{R}_{\geq 0} \longrightarrow G$

$\gamma_6 : [0, L_6] \to G$



$$\mathcal{M}(x_1, \dots, x_k; y) = \left\{ (\Gamma, \gamma) \;\middle|\; \Gamma \in T_k,\ \gamma : \Gamma \to G,\ \underline{\underline{(*)}} \right\}$$

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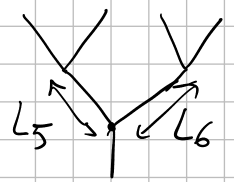


, $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7)$ with :

$\gamma_1, \gamma_2, \gamma_3, \gamma_4 : \mathbb{R}_{\leq 0} \to G$

$\gamma_5 : [0, L_5] \to G$

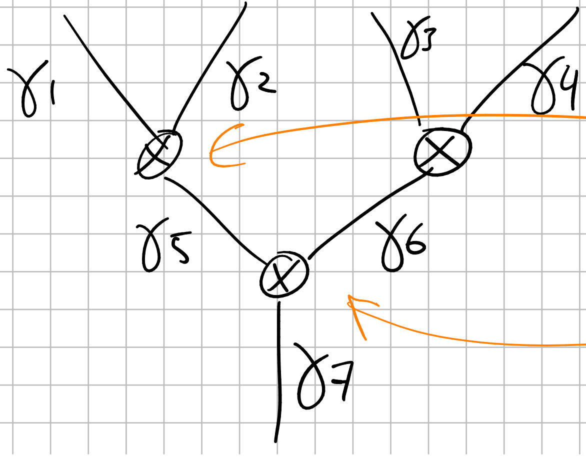
$\gamma_6 : [0, L_6] \to G$

$\gamma_7 : \mathbb{R}_{\geq 0} \to G$

$(*)$
- flow lines for a domain-dependent pseudo-gradient of g
- limits to $x_1, \dots x_k, y$ at the ends,
- satisfy multiplicative relations at vertices

ex: $\gamma_1(0)\gamma_2(0) = \gamma_5(0)$

$\gamma_5(L_5) \cdot \gamma_6(L_6) = \gamma_7(0)$



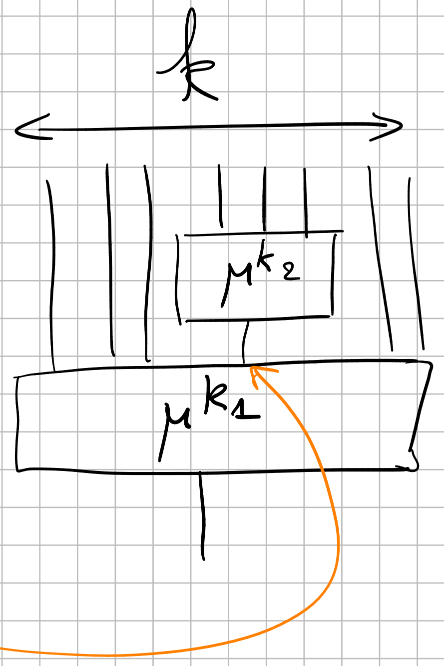
→ These $\mu^k : CM_*(G,g)^{\otimes k} \longrightarrow CM_*(G,g)$, defined by:

$$\mu^k(x_1,\dots,x_k) = \sum_y{}' \#\mathcal{M}(x_1,\dots,x_k;y)\cdot y$$

Satisfy the A_∞-relations:

- $\mu^1\circ\mu^1 = 0$ (μ^1 differential)
- $\mu^1\circ\mu^2 + \mu^2\circ(\mu^1\otimes id + id\otimes\mu^1) = 0$ (μ^2 chain map)
- ...

$$\forall k\geq 1,\ \sum_{\substack{k_1+k_2=k+1\\ 1\leq \ell\leq k_1}}{}'$$

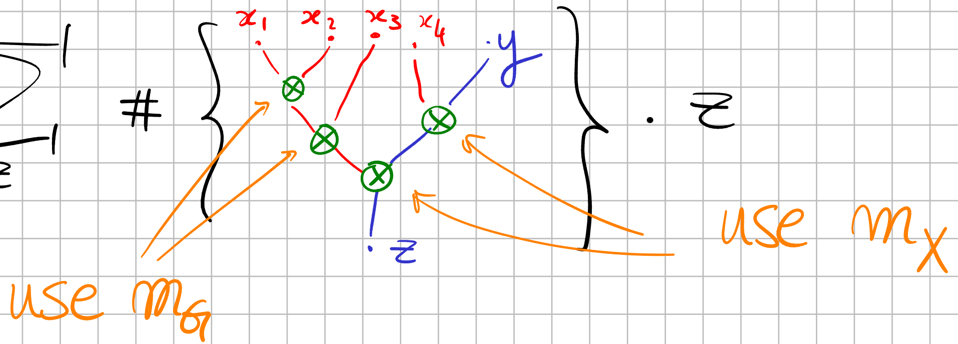


$= 0$

A_∞-module structure on $CM_*(X,f)$

Define $\mu_X^{k+1} : CM_*(G,g)^{\otimes k} \otimes CM_*(X,f) \to CM_*(X,f)$ analogously by:

$$\mu_X^{k+1}(x_1, \dots, x_k, y) = \sum_z{}' \# \{ \quad \} \cdot z$$



A_∞-module structure on $CM_*(X,f)$

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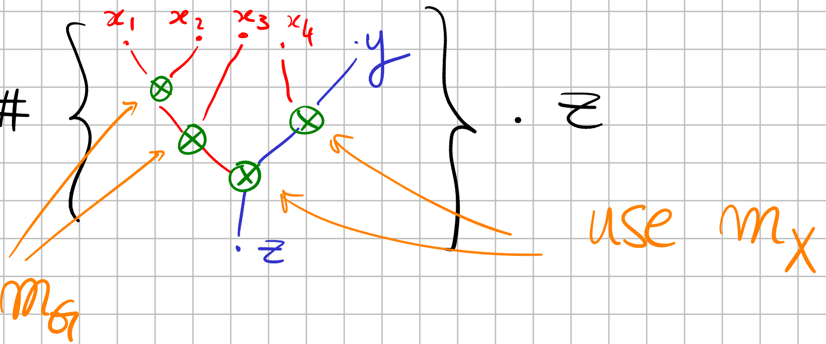
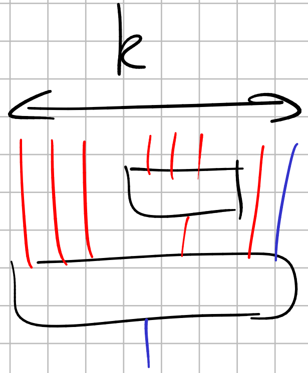


$$\cdot\, z$$

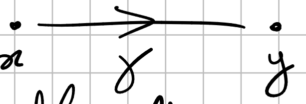
→ Satisfy the A_∞-relations for A_∞-modules:

$$\forall k \geqslant 1, \quad \sum{}'$$



$$= 0$$

From Morse to Floer

X	T^*X
$f : X \to \mathbb{R}$	$L_0 = 0_X$ zero-section $L_1 = \Gamma(df)$
$\mathrm{Crit}\, f$	$L_0 \cap L_1$
 flow line	 J-hol disc

From Morse to Floer

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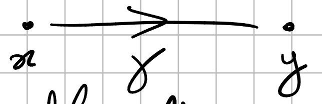
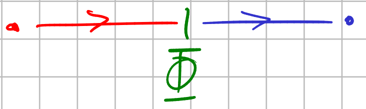
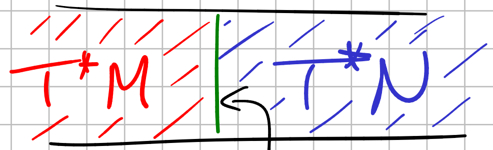
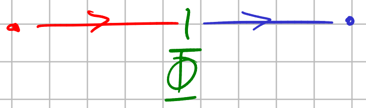
 flow line	 J-hol disc
 grafted line $\Phi : M \to N$	 quilt $N_{\Gamma(\Phi)} \subset (T^*M)^- \times (T^*N)$ conormal bundle (Lagrangian correspondence)

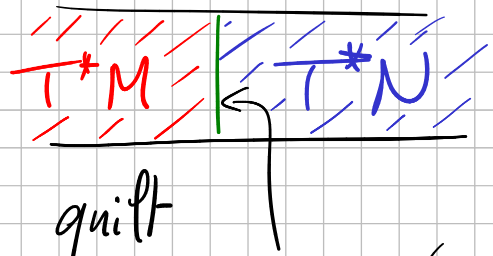
(Wehrheim – Woodward)

From Morse to Floer II



grafted line

$\Phi : M \to N$

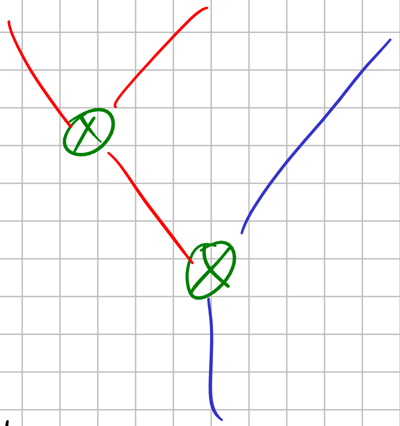


quilt

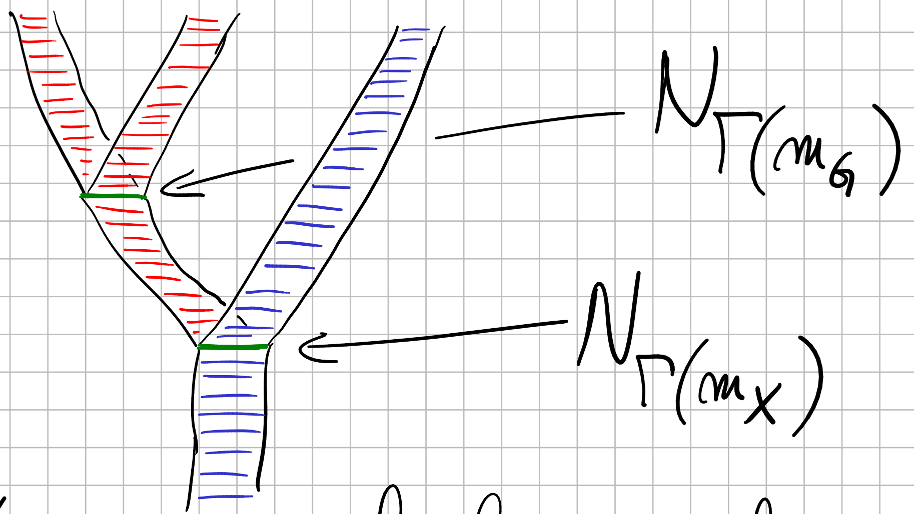
$N_{\Gamma(\Phi)} \subset (T^*M)^- \times (T^*N)$

conormal bundle

(Lagrangian correspondence)



multiplicative tree



"pseudo-holomorphic foam"

Rk: $N_{\Gamma(m_X)} = \Lambda_G(T^*X)$ ⟵ "Weinstein correspondence"

Def: (Weinstein correspondence)

$G \circlearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{g}^*$ Hamiltonian manifold

$$\Lambda_G(M) = \left\{ ((g,p), m, m') \;\middle|\; \begin{matrix} m' = g m \\ R^*_{g^{-1}} p = \mu(m) \end{matrix} \right\} \subset T^*G \times M^- \times M$$

Lagrangian submanifold

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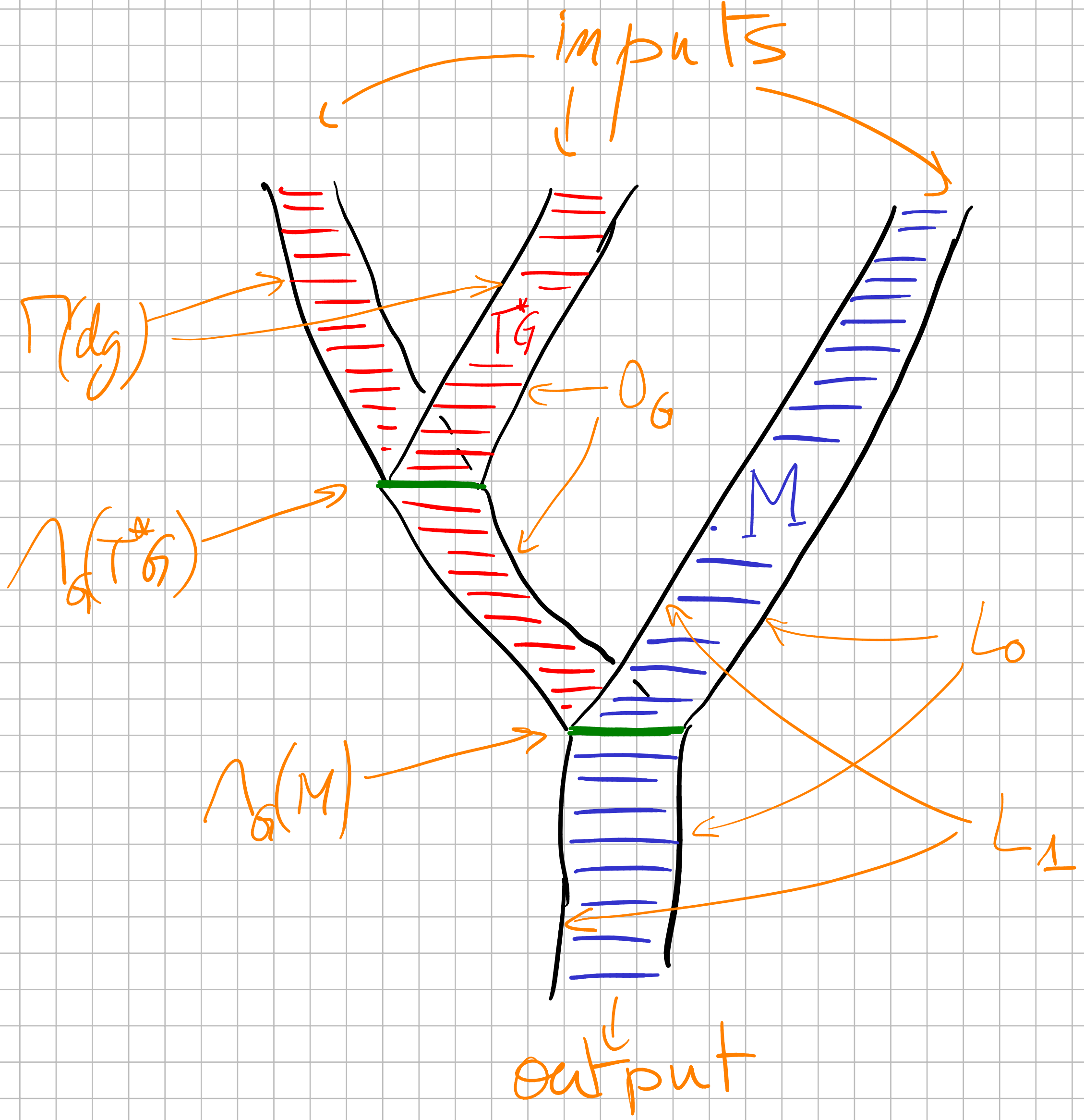
$\uparrow$ Lagrangian submanifold

$\Rightarrow$ Can transpose our construction to the Floer complex $CF(M; L_0, L_1)$.

Define $\mu^{k+1} : \underbrace{CF(T^*G; 0_G, \Gamma(dg))}_{\simeq CM_*(G,g)}{}^{\otimes k} \otimes CF(M; L_0, L_1) \longrightarrow CF(M; L_0, L_1)$

by counting foams:



The other A_∞ - structure on $CM_*(X,f)$ (Fukaya)

$\Delta : X \longrightarrow X \times X$ $\rightarrow$ $\Delta_* : H_*(X) \longrightarrow H_*(X) \otimes H_*(X)$ coproduct

$x \mapsto (x,x)$ (~ cup product)

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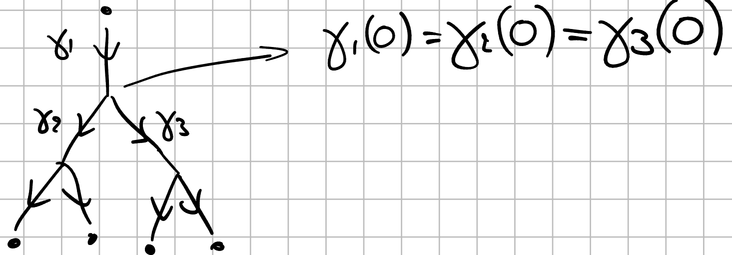
$x \mapsto (x,x)$ (~ cup product)

$\hookrightarrow$ Chain-level version : A_∞-coalgebra structure

$$\delta_k : CM_*(X) \to CM_*(X)^{\otimes k}$$

count



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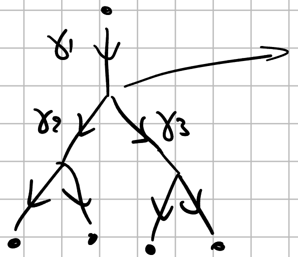
↳ Chain-level version: A_∞-coalgebra structure

$\delta_k : CM_*(X) \longrightarrow CM_*(X)^{\otimes k}$

count



$\gamma_1(0)=\gamma_2(0)=\gamma_3(0)$

↳ symplectic version: Fuk(M): A_∞-(co-)category ...

The other A_∞-structure on $CM_*(X,f)$ (Fukaya)

$\Delta : X \to X \times X$ → $\Delta_* : H_*(X) \longrightarrow H_*(X) \otimes H_*(X)$ coproduct

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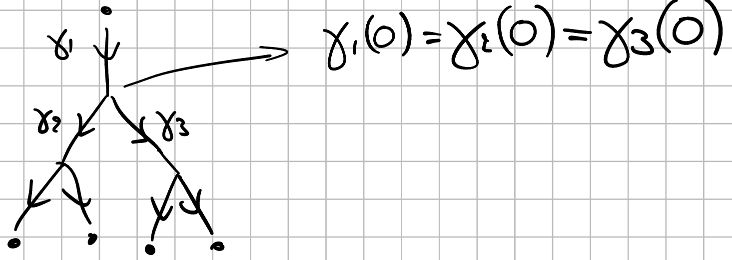
↳ Chain-level version : A_∞-coalgebra structure

$\delta_k : CM_*(X) \longrightarrow CM_*(X)^{\otimes k}$

count



$\gamma_1(0) = \gamma_2(0) = \gamma_3(0)$

↳ symplectic version: Fuk(M): A_∞- (co-)category ...

Rk: $\mu^1 = \delta_1$ same differentials

→ How are these two A_∞-structures related?

Hopf algebras, Hopf modules

$G \circlearrowright X \quad \rightsquigarrow \quad H_* G \circlearrowright H_* X$

group $\qquad$ mfd $\qquad\qquad$ Hopf algebra $\qquad$ Hopf module

Hopf algebras, Hopf modules

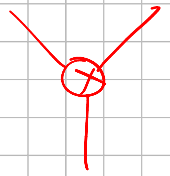
$G \circlearrowright X$ $\rightsquigarrow$ $H_*G \circlearrowright H_*X$

group mfd

Hopf algebra Hopf module

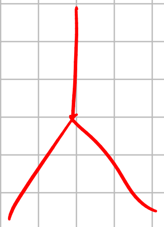
Def: Hopf algebra $(H, m, \Delta, \eta, \epsilon, S)$:

product



$$\begin{tikzcd} H \otimes H \arrow[d, "m"] \\ H \end{tikzcd}$$

coproduct

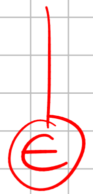


$$\begin{tikzcd} H \arrow[d, "\Delta"] \\ H \otimes H \end{tikzcd}$$

unit

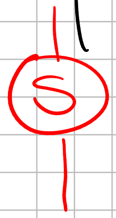
$$\begin{tikzcd} \mathbb{F}_2 \arrow[d, "\eta"] \\ H \end{tikzcd}$$

counit



$$\begin{tikzcd} H \arrow[d, "\epsilon"] \\ \mathbb{F}_2 \end{tikzcd}$$

antipode



$$\begin{tikzcd} H \arrow[d, "S"] \\ H \end{tikzcd}$$

(from $G \to G$
$g \mapsto g^{-1}$)

Hopf algebras, Hopf modules

$G \curvearrowright X$ ~> $H_* G \curvearrowright H_* X$

group mfd

Hopf algebra Hopf module

Def: Hopf algebra $(H, m, \Delta, \eta, \epsilon, S)$:

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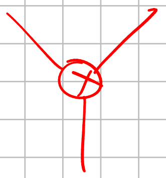
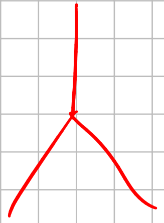


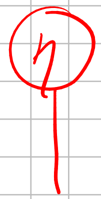
$$\begin{tikzcd} H\otimes H \arrow[d, "m"] \\ H \end{tikzcd}$$

coproduct



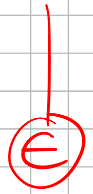
$$\begin{tikzcd} H \arrow[d, "\Delta"] \\ H\otimes H \end{tikzcd}$$

unit



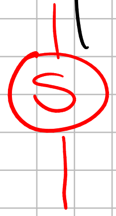
$$\begin{tikzcd} \mathbb{F}_2 \arrow[d, "\eta"] \\ H \end{tikzcd}$$

counit



$$\begin{tikzcd} H \arrow[d, "\epsilon"] \\ \mathbb{F}_2 \end{tikzcd}$$

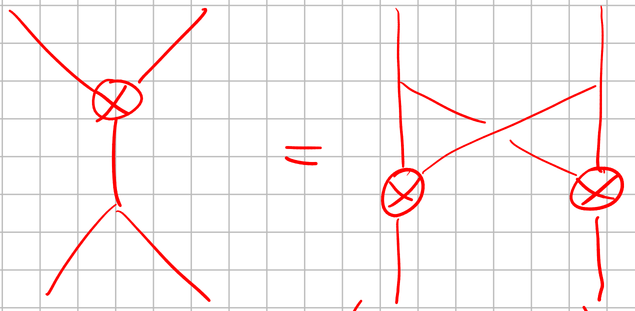
antipode



$$\begin{tikzcd} H \arrow[d, "S"] \\ H \end{tikzcd}$$

(from $G \to G$
$g \mapsto g^{-1}$)

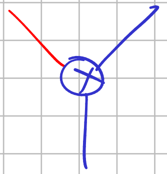
satisfy several relations, including:



$$\Delta \circ m = (m \otimes m) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta)$$

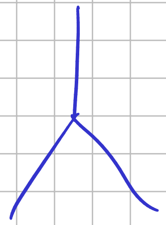
Def: Hopf module $H \circlearrowright (M, m_M, \Delta_M)$

product



$$\begin{tikzcd} H \otimes M \arrow[d, "m_M"] \\ M \end{tikzcd}$$

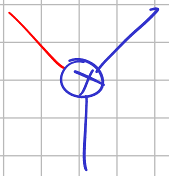
coproduct



$$\begin{tikzcd} M \arrow[d, "\Delta_M"] \\ M \otimes M \end{tikzcd}$$

Def: Hopf module $H \circlearrowright (M, m_{\underline{M}}, \Delta_M)$

product

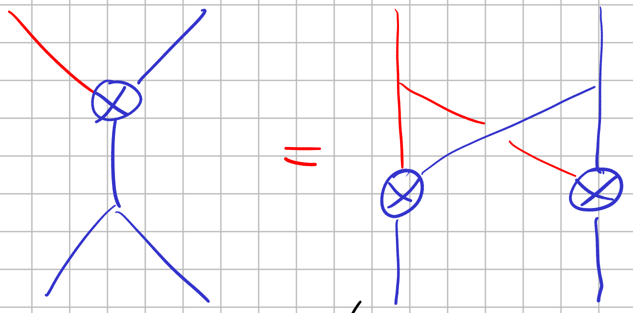


$$\begin{tikzcd} H \otimes M \arrow[d, "m_M"] \\ M \end{tikzcd}$$

coproduct

$$\begin{tikzcd} M \arrow[d, "\Delta_M"] \\ M \otimes M \end{tikzcd}$$

satisfy several relations, including:



$$\Delta_M \circ m_{\underline{M}} = (m_{\underline{M}} \otimes m_{\underline{M}}) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta_M)$$

Cornered instanton theory

- For Heegaard-Floer : Douglas-Lipshitz-Manolescu.

Cornered instanton theory

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Geometry (~ Moore-Tachikawa category)

Algebra

$$\begin{tikzcd}
\mathrm{Cob}_{1+1+1(+1)} \arrow[r] \arrow[rr, bend left, "\text{Cornered instanton theory}"] & \mathrm{Ham} \arrow[r] & \mathrm{Hopf}_\infty
\end{tikzcd}$$

Topology

Extended moduli spaces (Huebschmann-Jeffrey, Manolescu-Woodward, C.)

Cornered instanton theory

- For Heegaard-Floer: Douglas-Lipshitz-Manolescu.

Geometry (≈ Moore-Tachikawa category)

Algebra

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\mathrm{Cob}_{1+1+1(+1)} \arrow[r] \arrow[rr, bend left, "\text{Cornered instanton theory}"] & \mathrm{Ham} \arrow[r] & \mathrm{Hopf}_\infty \arrow[d, dashed, "H_*"] \\
& \mathrm{Lie}_{\mathbb{R}} \arrow[u, "T^*"] \arrow[r, dashed] & \mathrm{Hopf}
\end{tikzcd}$$

Homology

Topology

Toy model

Extended moduli spaces
(Huebschmann-Jeffrey,
Manolescu-Woodward, C.)

Cotangent bundles

$$\begin{tikzcd}
\mathrm{Cob}_{1+1+1(+1)} \arrow[r] \arrow[rr, bend left] & \mathrm{Ham} \arrow[r] & \mathrm{Hopf}_\infty \arrow[d, dashed, "H_*"] \\
 & \mathrm{Lie}_{\mathbb{R}} \arrow[u, "T^*"] \arrow[r, dashed] & \mathrm{Hopf}
\end{tikzcd}$$

cotangent bundles

conormal bundle $\subset (T^*X)^- \times T^*Y$

$$\begin{tikzcd}[column sep=large]
G \arrow[r, bend left=50, "X"{name=U}] \arrow[r, bend right=50, "Y"'{name=D}] & H \arrow[Rightarrow, from=U, to=D, "C"]
\end{tikzcd}
\quad \overset{T^*}{\longmapsto} \quad
\begin{tikzcd}[column sep=large]
G \arrow[r, bend left=50, "T^*X"{name=U}] \arrow[r, bend right=50, "T^*Y"'{name=D}] & H \arrow[Rightarrow, from=U, to=D, "N^*_C"]
\end{tikzcd}$$

- G, H : Lie groups
- $(G\times H) \circlearrowright X, Y$ smooth manifolds
- $C \subset X \times Y$ $(G\times H)$-invariant correspondence

- $(G\times H) \circlearrowright T^*X, T^*Y$ Hamiltonian manifolds
- $N^*_C \subset (T^*X)^- \times T^*Y$ $(G\times H)$-Lagrangian correspondence

composition of 1-morphisms:

$$\begin{tikzcd}
G \arrow[r, "X"] \arrow[rr, bend right, "(X\times Y)/H"'] & H \arrow[r, "Y"] & K
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "M"] \arrow[rr, bend right, "(M\times N)/\!/H"'] & H \arrow[r, "N"] & K
\end{tikzcd}$$

symplectic quotient

$$\begin{tikzcd}
\mathrm{Cob}_{1+1+1(+1)} \arrow[r] \arrow[rr, bend left] & \mathrm{Ham} \arrow[r] & \mathrm{Hopf}_\infty \arrow[d, dashed, "H_*"] \\
& \mathrm{Lie}_{\mathbb{R}} \arrow[u, hook, "T^*"'] \arrow[r, dashed] & \mathrm{Hopf}
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, bend left=50, "X"{name=U}] \arrow[r, bend right=50, "Y"'{name=D}] \arrow[Rightarrow, from=U, to=D, "C"] & H
\end{tikzcd}
\longmapsto
\begin{tikzcd}
H_*(G) \arrow[r, bend left=50, "H_*(X)"{name=U}] \arrow[r, bend right=50, "H_*(Y)"'{name=D}] \arrow[Rightarrow, from=U, to=D, "H_*(C)"] & H_*(H)
\end{tikzcd}$$

- G, H : Lie groups
- $(G \times H) \circlearrowright X, Y$ smooth manifolds
- $C \subset X \times Y$ $(G \times H)$-invariant correspondence

- $H_*(G), H_*(H)$ Hopf algebras
- $(H_*(G), H_*(H)) \circlearrowright H_*(X), H_*(Y)$ Hopf bimodules
- $H_*C : H_*(X) \to H_*(Y)$ morphism of Hopf bimodules.

$$\begin{tikzcd} \mathrm{Cob}_{1+1+1(+1)} \arrow[r] \arrow[rr, bend left] & \mathrm{Ham} \arrow[r] & \mathrm{Hopf}_\infty \arrow[d, dashed, "H_*"] \\ & \mathrm{Lie}_{\mathbb{R}} \arrow[u, "T^*"] \arrow[r, dashed] & \mathrm{Hopf} \end{tikzcd}$$

Ma'u - Wehrheim - Woodward A_∞ - functor

Fukaya categories

$$\begin{tikzcd} G \arrow[r, bend left=50, "M"{name=U}] \arrow[r, bend right=50, "N"'{name=D}] \arrow[Rightarrow, from=U, to=D, "L"] & H \end{tikzcd} \longmapsto \begin{tikzcd} CM_*(G) \arrow[r, bend left=50, "\mathrm{Fuk}(M)"{name=U}] \arrow[r, bend right=50, "\mathrm{Fuk}(N)"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\Phi_L"] & CM_*(H) \end{tikzcd}$$

- G, H : Lie groups
- $(G \times H) \circlearrowright M, N$: Hamiltonian manifolds
- $L \subset M^- \times N$ $(G\times H)$-Lagrangian correspondence

- $CM_*(G), CM_*(H)$: "Hopf_∞ algebras"
- $(CM_*(G), CM_*(H)) \circlearrowright \mathrm{Fuk}(M), \mathrm{Fuk}(N)$ "Hopf_∞ bimodules"
- $\Phi_L : \mathrm{Fuk}(M) \to \mathrm{Fuk}(N)$ "morphism of Hopf_∞ bimodules"

Hopf$_\infty$ - algebras (Saneblidze - Umble)

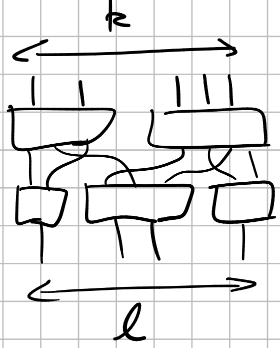
$\approx$ A_∞ - bialgebras + incorporate units, counits, antipodes

Hopf$_\infty$-algebras (Saneblidze - Umble)

$\approx$ A_∞-bialgebras + incorporate units, counits, antipodes

Def [Saneblidze-Umble] An A_∞-bialgebra $\left(H, \{P^k_\ell\}_{k,\ell \geq 1}\right)$ is a family of maps $P^k_\ell : H^{\otimes k} \to H^{\otimes \ell}$ satisfying a family of relations (R^k_ℓ) determined by the combinatorics of the "biassociahedra" $\{KK_{k,\ell}\}_{k,\ell\geq 1}$.

(R^k_ℓ) : " $\displaystyle\sum_{\partial(KK_{k,\ell})}$



$= 0$ "

Hopf$_\infty$-algebras (Saneblidze - Umble)

≈ A_∞-bialgebras + incorporate units, counits, antipodes

Def [Saneblidze-Umble] An A_∞-bialgebra $\left(H, \{P^k_\ell\}_{k,\ell \geqslant 1}\right)$ is a family of maps $P^k_\ell : H^{\otimes k} \to H^{\otimes \ell}$ satisfying a family of relations (R^k_ℓ) determined by the combinatorics of the "biassociahedra" $\{KK_{k,\ell}\}_{k,\ell \geqslant 1}$.

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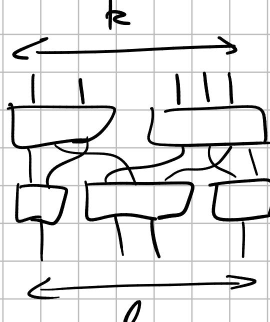
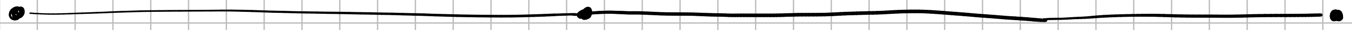
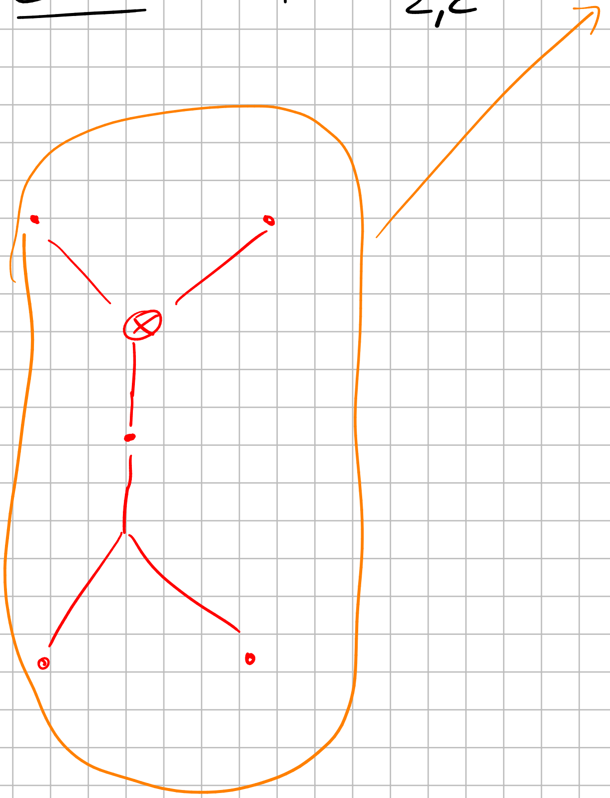


$= 0$ "

Rk: Contains
- A_∞-algebra $(H, \{P^k_1\}_k)$
- A_∞-coalgebra $(H, \{P^1_\ell\}_\ell)$

Ex: $KK_{2,2} =$





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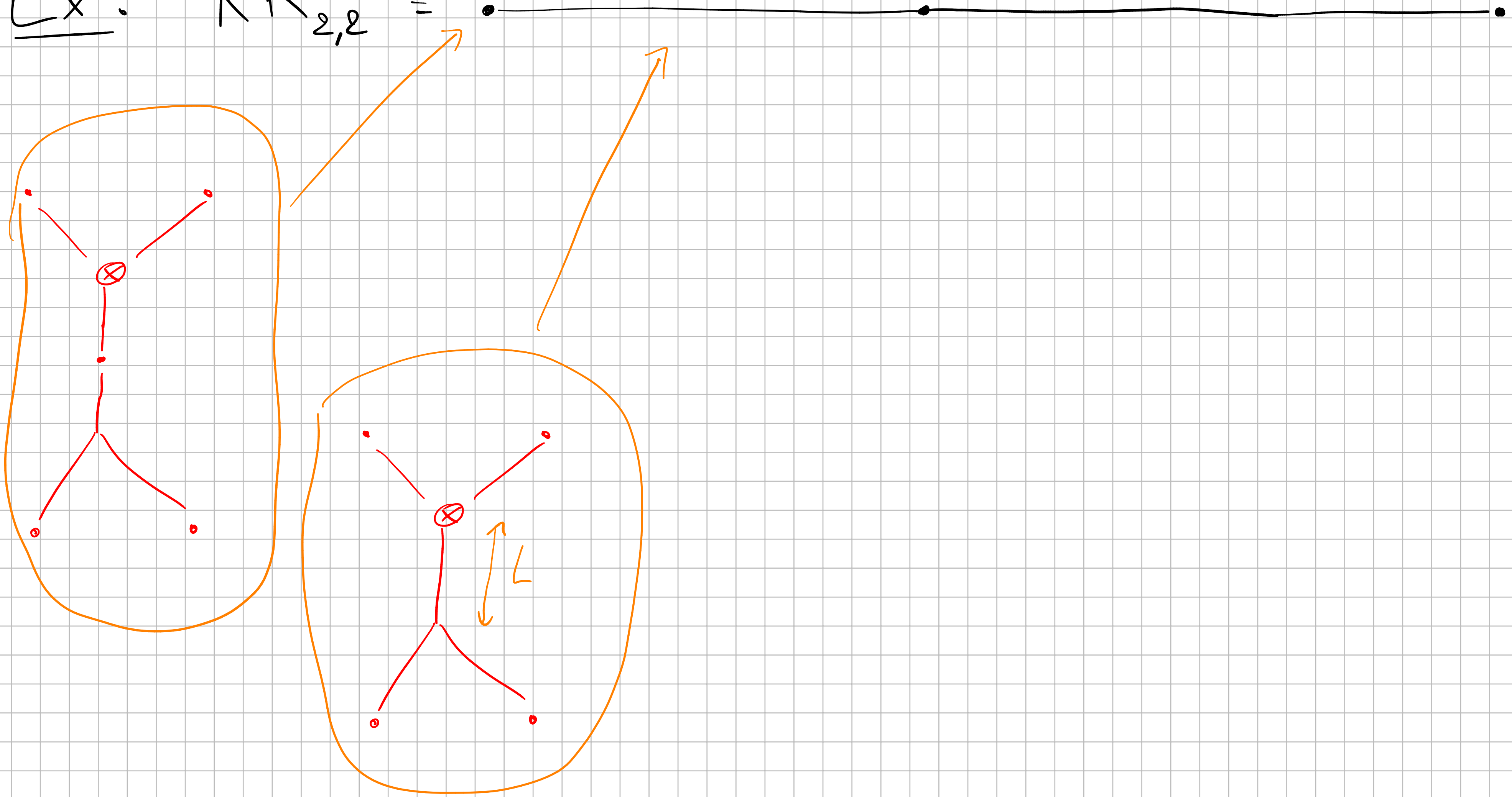

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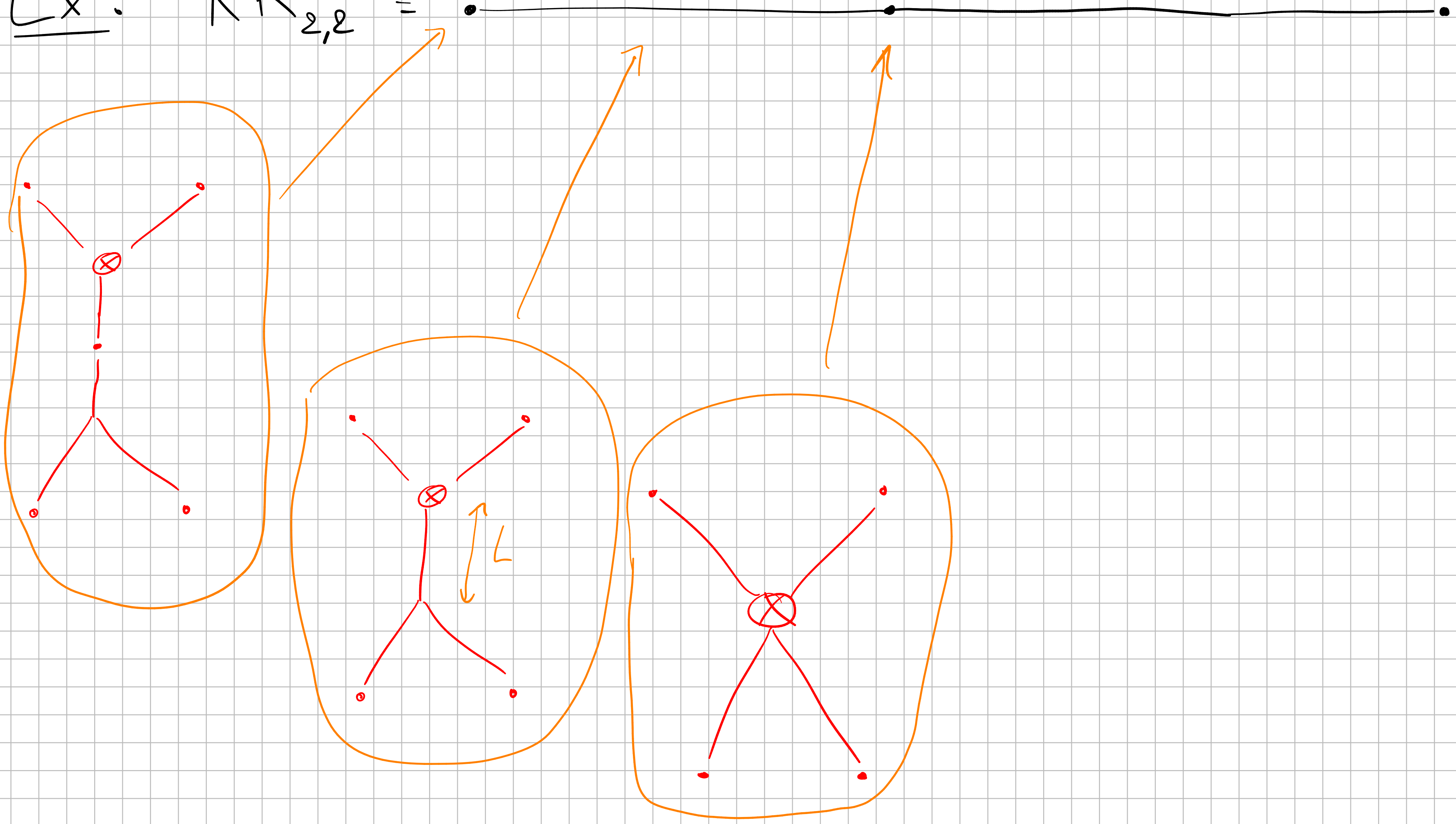

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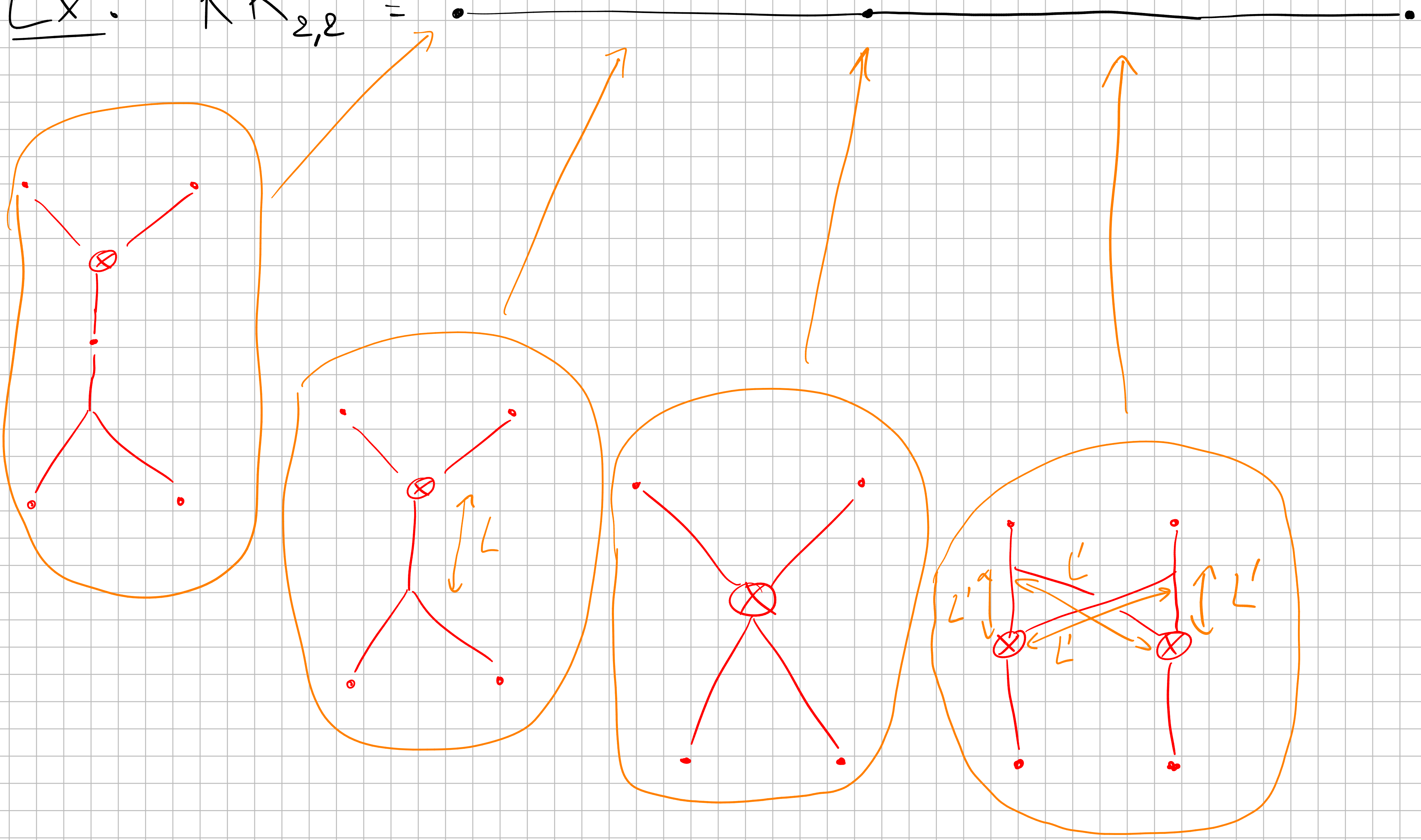

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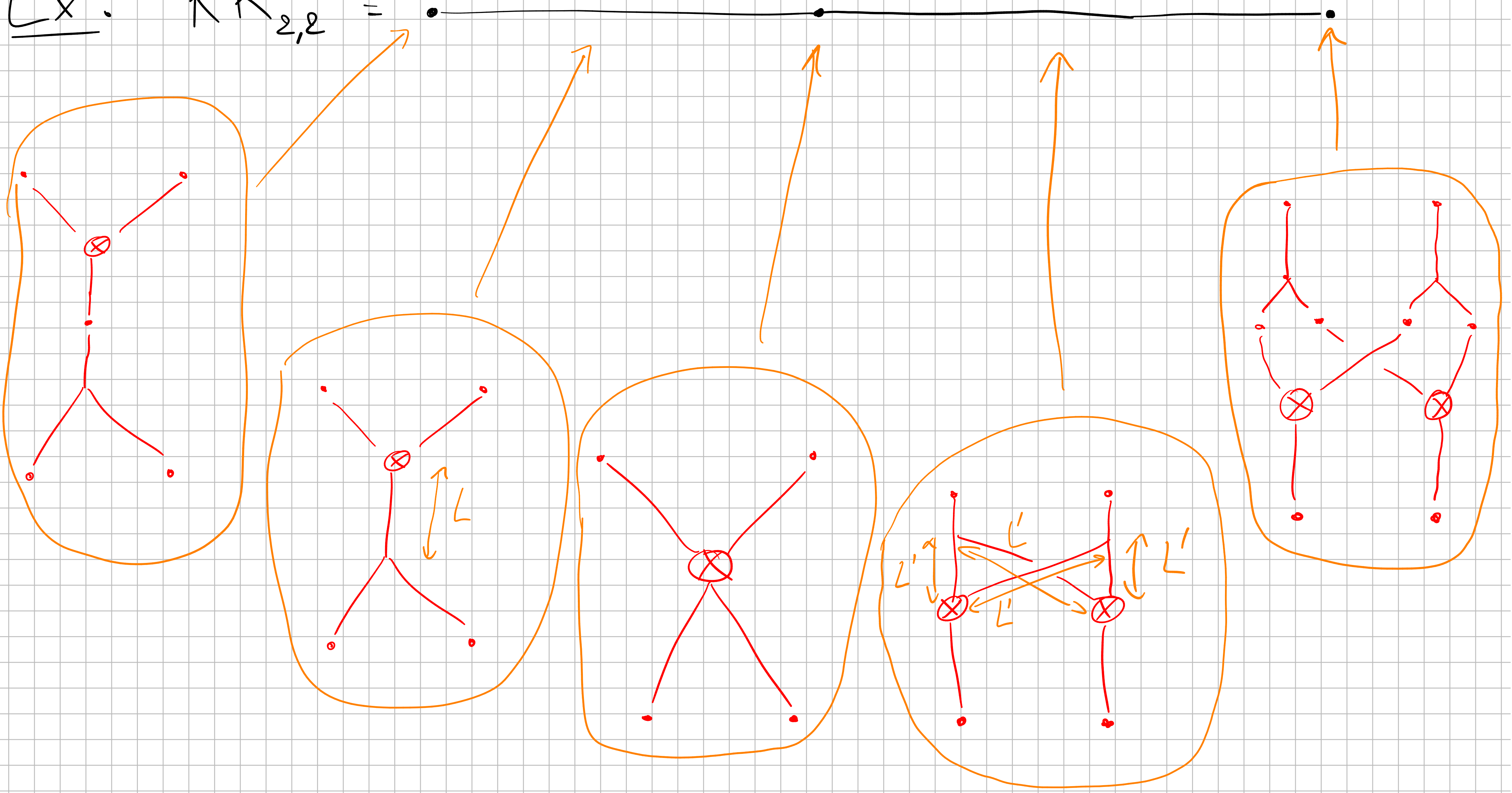

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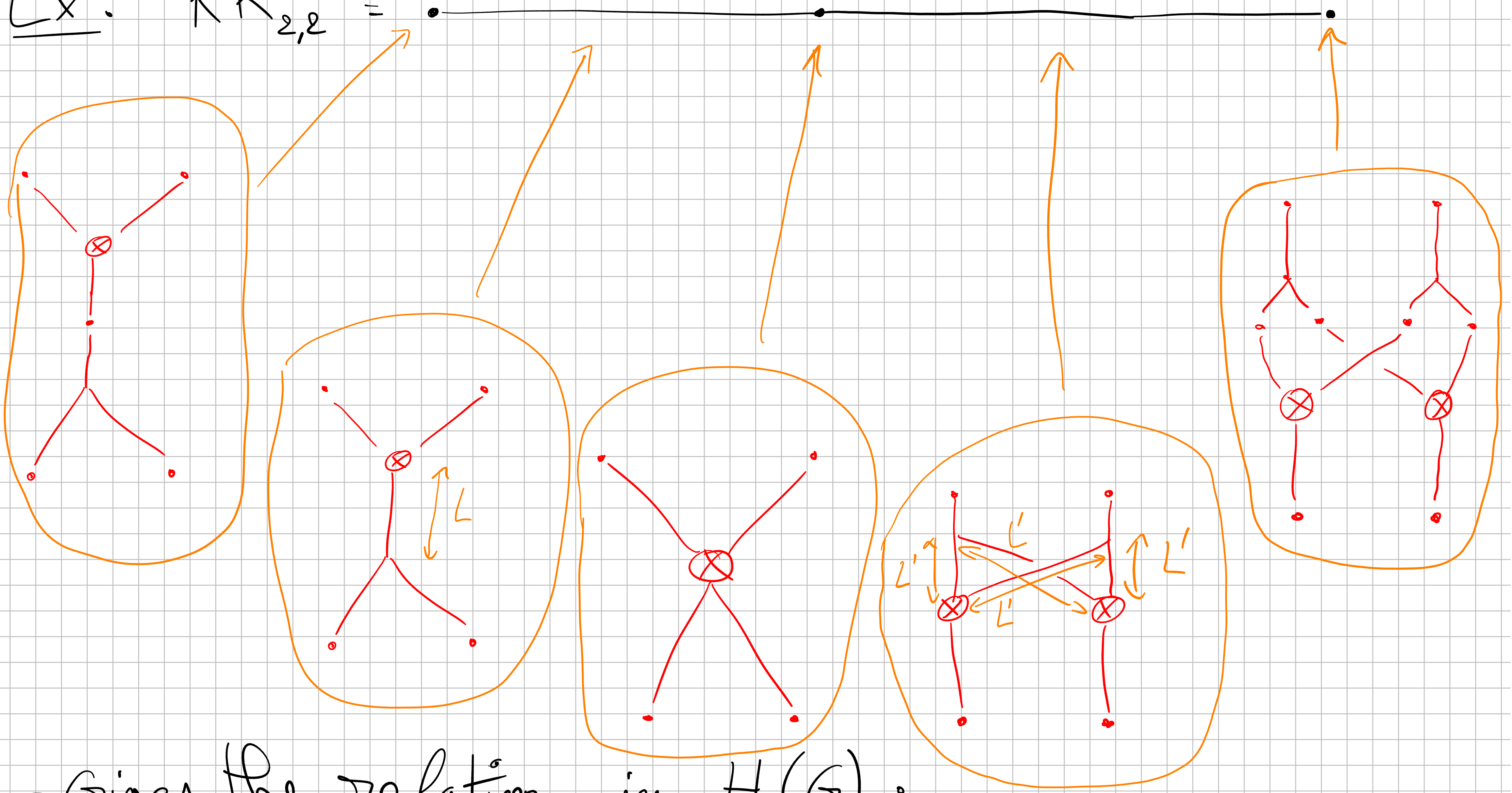

→ Gives the relation in $H_*(G)$:

$$\Delta \circ m = (m \otimes m) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta)$$

Foam version:

$KK_{2,2} =$