Selected topics in graph theory

08.10.2018 — embedded graphs, part 1

Problem 1. Let $\mathcal{G} = \{K_5, K_6, K_7, K_8, K_{3,3}, \text{Petersen}\}$ and $\mathcal{S} = \{\text{torus}, \text{projective plane}\}$. For every $(G, \Sigma) \in \mathcal{G} \times \mathcal{S}$, decide if G can be embedded in Σ.

Problem 2. Draw an K_4 on a projective plane such that every face is a 4-face.

Problem 3. Show that:

1. Adding a twisted handle to the surface can be simulated with adding two crosscaps (and thus lead to isomorphic surfaces).

2. Adding a handle to projective plane can be simulated with adding two crosscaps.

Problem 4. Let G be a graph, let $\prod$ be a combinatorial orientable embedding of G, and let C be a $\prod$-separating cycle in G. Let G_L be the union of C-bridges that are left to C and let G_R be the union of C-bridges that are right to C. Prove that the Euler genus of $\prod$ equals the sum of Euler genera of the induced embeddings of $G_L + C$ and $G_R + C$. Deduce that the Euler genus of a disjoint union of $G_L + C$ and $G_R + C$ is smaller by exactly two than the Euler genus of G.

Problem 5. Let G be a graph, let $\prod$ be a combinatorial orientable embedding of G, and let C be a $\prod$-nonseparating $\prod$-twosided cycle in G. Consider a graph G' with an embedding $\prod'$ constructed from G and $\prod$ by duplicating the edges and vertices on the cycle C into two copies C_L and C_R and connecting C_L to the edges on the left side of C in $\prod$ and C_R to the edges on the right side of C in $\prod$. Prove that the Euler genus of $\prod'$ is exactly smaller by 2 than the Euler genus of $\prod$.

Problem 6. Let G, $\prod$, and C be as in the previous exercise, but assume that C is $\prod$-onesided. Try to make a definition of cutting along C that would be analogous to the previous operation of duplicating C. Prove that this operation decreases the Euler genus by exactly one.

Problem 7. Show a sequence of graphs $G_1, G_2, \ldots$ such that their Euler genera are bounded, but their orientable genera are unbounded.

Selected topics in graph theory

15.10.2018 — embedded graphs, part 2

combinatorial embeddings

A *combinatorial embedding* of a graph G is a pair (π, λ), where $\pi = (\pi_v)_{v \in V(G)}$, each π_v is a cyclic permutation of the edges incident with v, and $\lambda : E(G) \to \{-1, 1\}$. The permutation π_v defines the order of edge endpoitns around v and $\lambda(e) = -1$ if we switch left/right sides (go upside-down) while going along the edge e.

A *facial walk* is constructed as follows. We start with a tuple (v, e, a), where e is incident with v and $a \in \{-1, 1\}$. The meaning of this tuple is that we start from the vertex v, going along the edge e, and $a = 1$ if we go "normally" and $a = -1$ if we go upside down. In one step, we switch from a tuple (v, e, a) to (v', e', a') where v' is the second endpoint of e, $a' = a \cdot \lambda(e)$, and e' is the next edge after e in $\pi_{v'}$ if $a' = 1$ and the previous edge if $a' = -1$. We stop when we encounter the same tuple again. We discussed that each edge is contained in exactly two facial walks if we identify two walks that go around the same face, but in different directions (if $e = uv$ then starting with the tuples $(v, e, 1)$ and $(u, e, -1)$ lead to the same walk, but reversed).

An embedding $\Pi = (\pi, \lambda)$ of G is *orientable* if for every closed walk W in G we have $\prod_{e \in W} \lambda(e) = 1$. Given an embedding $\Pi = (\pi, \lambda)$ of G, a *swap* at a vertex v consists of reversing π_v and changing the sign of $\lambda(e)$ for every edge e incident with v.

Problem 8. Prove that an embedding is orientable if and only if it can be changed by a sequence of swaps into an embedding with $\lambda(e) = 1$ for every edge e.

finding shortest Π-something cycles

For a graph G and a subgraph H, an *H-bridge* is a connected component C of $G - V(H)$ together with all edges connecting C with $V(H)$. A single edge connecting two vertices of H, but not in $E(H)$, is also an H-bridge.

A cycle C in a graph G with embedding Π is Π-*twosided* if the $\prod_{e \in C} \lambda(e) = 1$ and Π-*onesided* otherwise. In a Π-twosided cycle C we can distinguish two sides and partition the endpoints of the edges $G - E(C)$ incident with C into left and right of C. A Π-twosided cycle C is Π-*nonseparating* if there exists a C-bridge containing both an edge enpoint left of C and an edge endpoint right of C. Otherwise, a Π-twosided cycle C is Π-*separating*.

For a Π-separating cycle C, we can partition the C-bridges into left and right of C. Let G_L be the union of the left bridges and G_R be the union of the right bridges. C is Π-*noncontractible* if the Euler genera of the induced embeddings of $G_L \cup C$ and $G_R \cup C$ are positive, and Π-*contractible* otherwise.

Problem 9. A family $\mathcal{C}$ of cycles in G has the 3-*cycle property* if the following holds: For every triple of paths P_1, P_2, P_3 with common endpoints u, v but otherwise edge- and vertex-disjoint, it does not hold that exactly one of the cycles $P_1 \cup P_2$, $P_2 \cup P_3$, and $P_3 \cup P_1$ is in $\mathcal{C}$. Assume that you are given access to an oracle that, given a cycle C, in one computational step decides if $C \in \mathcal{C}$. Prove that with such an oracle one can in polynomial time find a shortest cycle belonging to $\mathcal{C}$.

Problem 10. Using the previous exercise, prove that one can in polynomial time find the shortest odd cycle in a graph.

Problem 11. Prove that, given a graph G and an embedding Π, the following families have the 3-cycle property:

1. Π-onesided cycles;
2. Π-nonseparating cycles;
3. Π-noncontractible cycles.

homotopic cycles

Let G be a graph and Π be its embedding. Two vertex-disjoint cycles C_1, C_2 are Π-*homotopic* if, after cutting along C_1 and C_2, there is one component which is planar and contains exactly one copy of C_1 and one copy of C_2.

Problem 12. Prove that if C_1 and C_2 are Π-homotopic and C_2 and C_3 are Π-homotopic, and C_1 and C_3 are vertex-disjoint, then C_1 and C_3 are Π-homotopic.

Problem 13. Let $C_1, \ldots, C_k$ be a sequence of Π-noncontractible, pairwise vertex-disjoint, and pairwise Π-nonhomotopic cycles. Prove that $k = \mathcal{O}(g)$ where g is the Euler genus of Π.

facewidth

Facewidth of an embedding Π of G is minimum number of facial walks whose union contains a Π-noncontractible cycle.

Problem 14. Prove that facewidth of a given embedding can be computed in polynomial time by reducing it to computation of edgewidth of another embedding.

Selected topics in graph theory

22.10.2018 — graphs minors and well-quasi-order

Recall that the partial order $(X, \leqslant)$ is a wqo, if in any infinite sequence $x_1, x_2, \ldots$ there is a pair of indexes $i < j$ such that that $x_i \leqslant x_j$.

Problem 15. Which of the following are wqo?

1. $\mathbb{N}^2$ with relation $(a, b) \leqslant (c, d) \Leftrightarrow (a \leqslant c \wedge b \leqslant d)$.
2. $\mathbb{N}^2$ with lexicographical order.
3. words over $\{a, b\}$ with lexicographical order.
4. $\mathbb{N}$ with $a \leqslant b$ if and only if $a \mid b$.
5. set of segments $X = \{[a, b] : a < b, a, b \in \mathbb{N}\}$, where $[a, b] \leqslant [c, d]$ if $b < c$ or $a = c$ and $b \leqslant d$.

Problem 16. Consider a quasi-order $(A, \leqslant)$. For each $X \subseteq A$ we define the set $Forb(X) = \{a \in A : \forall x \in X, \neg(x \leqslant a)\}$. Show that if $(A, \leqslant)$ is a wqo then each subset $B \subseteq A$ closed under taking smaller elements, can be written as $B = Forb(X)$ for some finite X.

minors

The $k \times k$-*grid* is the graph with vertex set $V = \{(i, j), 1 \leqslant i, j \leqslant k\}$ and with an edge from (i, j) to $(i'j')$ when $|i - i'| + |j - j'| = 1$.

Problem 17. Is K_4 a minor of the 1000×1000-grid? Is K_5?.

Problem 18. The graph H consists of a cycle on six vertices $v_1v_2v_3v_4v_5v_6$ with additional edges v_1v_4, v_2v_5 v_3v_6. We take the graph G to be the 100×100-grid plus one diagonal per cell. Formally we add an edge from (i, j) to (i', j') when $(i - i')(j - j') = 1$. Is H a minor of G?

Problem 19. Show that for every planar graph H, there is a k such that H is a minor of the $k \times k$-grid.

Problem 20. Show that trees are not well-quasi-ordered by the subgraph relation.

Problem 21. Show that the set of connected simple graphs are not well-quasi-ordered under the contraction relation (we only allow the contraction operation, and disallow vertex and edge deletion).

Problem 22. Show that simple graphs are not well-quasi-ordered by the topological minor relation.

minimal bad sequences

A *series-parallel* graph is a triple (G, s, t) defined recursively as follows:

1. $(K_2 = (\{s, t\}, \{st\}), s, t)$ is a series parallel-graph.
2. Given two series-parallel graphs (G, s, t) and (G', s', t'), the graph obtained by identifying s with s' and t with t' is a series-parallel graph. (parallel composition)
3. Given two series-parallel graphs (G, s, t) and (G', s', t'), the graph obtained by identifying t with s' is a series-parallel graph. (series composition)

Problem 23. Prove that the set of series-parallel graphs is well-quasi-ordered by the minor relation.

Selected topics in graph theory

29.10.2018 — tree-width and brambles

Treewidth. Let G be a graph. A tree decomposition of G is a pair (T, β) where T is a tree and $\beta : V(T) \to 2^{V(G)}$ satisfies the following conditions:

(T1) $V(G) = \bigcup_{t \in T} \beta(t)$;

(T2) for every edge $uv \in E(G)$ there exists $t \in T$ such that $u, v \in \beta(t)$;

(T3) for every $v \in V(G)$ the set $\{t \mid v \in \beta(t)\}$ induces a connected subgraph of T.

The width of a decomposition is $\max_{t \in T} |\beta(t)| - 1$. Treewidth of a graph is the minimum possible width of its tree decomposition.

Path decompositions and pathwidth are analogous notions where T is required to be a path.

Problem 24. In graph G there is a robber and k cops. The robber has a very quick motorbike, and each cop has a helicopter. Between turns, each agent occupies a single vertex of the graph. Each turn consists of the following steps:

1. a subset of cops departs from their current positions and declares where they land;
2. the robber moves; he can move by an arbitrary distance, but cannot move through a vertex that is occupied by a cop (one that does not move this turn);
3. the cops that move land.

The cops win if at some moment a cop lands in a vertex with the robber. The robber wins if it is able to evade the cops for arbitrary long time. Show that the minimum number of cops needed to catch the robber is exactly one more than the treewidth of G.

Problem 25. Show that G is a forest if and only if its treewidth is at most 1.

Problem 26. Show a tree decomposition of width 2 of an n-vertex cycle.

Problem 27. What is the treewidth of the clique K_n?

Problem 28. A graph is *outerplanar* if it can be drawn in the plane such that each vertex is on the infinite face of the embedding. Prove that an outerplanar graph has treewidth at most 2.

Problem 29. Show that if T is a tree, $\beta : V(T) \to 2^{V(G)}$ is a function such that $\bigcup_{t \in V(T)} \beta(t) = V(G)$ and for every $t_1t_2 \in E(T)$, if T_i is the connected component of $T - \{t_1t_2\}$ containing t_i, then $\beta(t_1) \cap \beta(t_2)$ is a separator between $\bigcup_{t \in T_1} \beta(t)$ and $\bigcup_{t \in T_2} \beta(t)$, then (T, β) is a tree decomposition of G.

Problem 30. Let $W \subseteq V(G)$ and let (T, β) be a tree decomposition of G. Show that either there exists $t \in V(T)$ with $W \subseteq \beta(t)$ or two vertices $w_1, w_2 \in W$ and an edge $t_1t_2 \in E(T)$ such that $\beta(t_1) \cap \beta(t_2)$ separates w_1 from w_2.

Problem 31. Show that the series–parallel graphs have treewidth at most 2.

Problem 32. Show that the treewidth of a $k \times k$ grid equals k or $k - 1$.

Problem 33. Show that if a graph G does not contain a k-vertex path as a subgraph, then the pathwidth of G is less than k.

Hint for Problem 33: Consider a DFS tree of the graph G.

Selected topics in graph theory

05.11.2018 — Brambles and Tangles

Problem 34. Prove that pathwidth of full ternary tree of depth h is at least h.

G is a chordal graph if it does not contain cycles longer than 3 as induced subgraphs. A graph G is an interval graph if there is a family of open intervals on a real line $(s_v)_{v \in V(G)}$ such that $uv \in E(G)$ if and only if $s_u \cap s_v \neq \emptyset$.

Problem 35. Show that G is chordal if and only if it admits a tree decomposition where every bag is a clique.

Problem 36. Show that G is interval if and only if it admits a path decomposition where every bag is a clique.

Problem 37. Let G be a graph containing an $r \times r$-grid as a subgraph. Show that G contains a tangle of order $k < r$.

When θ is a tangle and $(A, B) \in \theta$, we call A the small side of $\{A, B\}$ in θ.

Problem 38. Let G be a graph with a tangle θ of order k.

1. Justify the notion of small side by showing that if $(A, B) \in \theta$ and $\{A', B'\}$ is a separation of order $< k$ with $A' \subseteq A$ and $B' \supseteq B$, then $(A', B') \in \theta$.

2. Show that when $(A, B), (A', B') \in \theta$ also $(A \cup A', B \cap B') \in \theta$, as long as $\{A \cup A', B \cap B'\}$ is a separation of order $< k$.

3. Deduce that, for every set X of fewer than k vertices, exactly one of the components C of $G - X$ is big, in the sense that $(V(C - G), X \cup V(C)) \in \theta$.

k-block. A set of $U \subseteq V(G)$ of at least k vertices is k-inseparable in G if no two vertices from U can be separated in G by fewer than k other vertices. A maximal k-inseparable set of vertices is a $k - block$.

Problem 39. Show the following implications for a graph G:

1. G contains a k-block $\Rightarrow$ G has a bramble of order k.

2. G has a tangle of order $k \Rightarrow G$ has a bramble of order k.

3. G has bramble of order $3k \Rightarrow G$ has a tangle of order k

Selected topics in graph theory

19.11.2018 — planar grid minor theorem

grid minor theorem

Problem 40. Let G be a plane graph with face-vertex diameter d (i.e., for every two vertices v, u, there exists at most d faces of G such that the union of the edges on these faces form a subgraph with u and v in the same connected component). Show that the treewidth of G is $\mathcal{O}(d)$.

Problem 41. Let G be a planar graph and let $X \subseteq V(G)$ be a vertex cover of G. Show that the treewidth of G is $\mathcal{O}(\sqrt{|X|})$.

Problem 42. Let G be a planar graph and let $X \subseteq V(G)$ and $\tau \in \mathbb{Z}_+$ be such that $G - X$ has treewidth at most τ. Show that the treewidth of G is $\mathcal{O}(\tau\sqrt{|X|})$.

Problem 43. Show that for every planar graph H there exists a constant $t(H)$ such that every graph G that does not have H as a minor has treewidth at most $t(H)$.

Problem 44. Show that for every integer k there exists an integer t such that the following holds: if a graph G does not contain k pairwise vertex-disjoint cycles then the treewidth of G is at most t. What is the best possible asymptotical dependency of t on k if we restrict the question to planar graphs?

algorithms on tree decompositions

Problem 45. Show that, given a tree decomposition of width t of a graph G, one can in polynomial time obtain a tree decomposition (T, β) of G of the same width with the following properties: T is a rooted tree and every $t \in V(T)$ is one of the four types:

(leaf) $|\beta(t)| = 1$ and t is a leaf of T;

(introduce) t has exactly one child s and $\beta(t) = \beta(s) \cup \{v\}$ for some $v \in \beta(t) \setminus \beta(s)$;

(forget) t has exactly one child s and $\beta(t) = \beta(s) \setminus \{v\}$ for some $v \in \beta(s) \setminus \beta(t)$;

(join) t has exactly two children s_1 and s_2 with $\beta(t) = \beta(s_1) = \beta(s_2)$.

Problem 46. Given an n-vertex graph G and its tree decomposition of width t, show that one can:

1. find minimum vertex cover and maximum independent set in G in time $2^{O(t)}n^{O(1)}$;
2. find minimum dominating set in G in time $2^{O(t)}n^{O(1)}$;
3. check if G is k-colorable in time $2^{O(t \log k)}n^{O(1)}$;
4. check if G is Hamiltonian in time $2^{O(t \log t)}n^{O(1)}$.

Problem 47. Show that, given a planar graph G and an integer k, one can check in time $2^{O(\sqrt{k})}n^{O(1)}$ whether:

1. has an independent set of size k;
2. has a simple path on k vertices as a subgraph.

03.12.2018 — concluding graph minors

Problem 48. For $n \geqslant 1$ let G_n be defined as follows. We start with G_n being an $n \times n$ grid and let $v_1, v_2, \ldots, v_n$ be vertices on one side of this grid, in this order. For every $1 \leqslant k \leqslant \lfloor n/2 \rfloor - 1$, we add an edge $v_{2k-1}v_{2k+2}$.

1. Prove that the Euler genus of G_n is $\Omega(n)$.
2. Prove if $X \subseteq V(G_n)$ such that $G - X$ is planar, then $|X| = \Omega(n)$.
3. Generalizing the previous two points, prove that there exists a constant $\delta > 0$ such that for every $n \geqslant 1$, if $X \subseteq V(G_n)$ and $G_n - X$ has Euler genus at most δn, then $|X| > \delta n$.
4. Show that G_n is 100-nearly-embeddable.

Problem 49. Show an example of an instance of Planar Disjoint Paths problem with k terminal pairs where the treewidth of the graph is $2^{\Omega(k)}$ and there exists a unique solution that passes through every vertex of the graph.

algorithmic applications

Problem 50. Let G be an undirected graph and consider an embedding of G in $\mathbb{R}^3$ (i.e., vertices are points, edges are differentiable curves, and edges are disjoint except for endpoints). We say that two vertex-disjoint cycles C_1 and C_2 in G are *linked* if their images in $\mathbb{R}^3$ cannot be separated with continuous maps of $\mathbb{R}^3$. A family of vertex-disjoint cycles is *linked* if every pair of cycles in the family is linked. Show that for every fixed $k \geqslant 1$, one can check in time $\mathcal{O}(|V(G)|^3)$ if a given graph G can be embedded in $\mathbb{R}^3$ such that there exists a family of k cycles in G that is linked.

Problem 51. Show that for every fixed $k \geqslant 1$ there exists an algorithm running in time $\mathcal{O}(|V(G)|^3)$ that checks, given a planar graph G, if there exists a planar embedding of G for which one can select at most k faces of the embedding such that all vertices of G lie on at least one of the selected faces.

Problem 52. Show that for every fixed $k \geqslant 1$ there exists an algorithm running in time $\mathcal{O}(|V(G)|^3)$ that checks, given a planar graph G, if there exists a supergraph G' of G that is also planar and has diameter at most k.

10.12.2018 — introduction to spectral graph theory

Problem 53. Let G be a d-regular multigraph. Show that the following conditions are equivalent:

1. G is disconnected,
2. $\lambda_2(G) = d$,
3. $h^E(G) = 0$.

Problem 54. Let G be a connected d-regular multigraph. Show that the following conditions are equivalent:

1. G is bipartite,
2. $\lambda_n(G) = -d$,
3. $h^V(G) = 0$.

Problem 55. Prove that G^2 is disconnected if and only if G is disconnected or bipartite.

Problem 56. Show that if G is a simple n-vertex graph, then $\sum_{i=1}^{n} \lambda_i(G) = 0$.

Problem 57. Compute h^E, h^V, and all eigenvalues λ_i of K_n, $K_{n,n}$, and C_n.

Problem 58. Show that the eigenvalues of the Petersen graph are 3, 5×1, and 4×-2.

Problem 59. Use the previous exercise to show that K_{10} cannot be decomposed into a union of three edge-disjoint Petersen graphs on the same vertex set.

Problem 60. Let G be a d-regular multigraph and let G' be constructed from G by adding a loop at every vertex. Prove that $\lambda_2(G') = 1 + \lambda_2(G)$ while $\lambda(G') = \max(1 + \lambda_2(G), -\lambda_n(G) - 1)$.

Problem 61. Let G be a d-regular connected multigraph. Show that $d - \lambda_2(G) = \Omega(d^{-1}n^{-2})$. Show that if additionally G has a loop at every vertex then $d - \lambda(G) = \Omega(d^{-1}n^{-2})$.

Problem 62. Let G be a d-regular multigraph on n vertices such that $d - \lambda(G) \geqslant \gamma$ for some $\gamma > 0$. Prove that there exists a constant c depending only on d and γ such that the diameter of G is at most $c \log n$.

Problem 63. Let G be a simple n-vertex d-regular graph with $d \geqslant 3$ and eigenvalues λ_i for $1 \leqslant i \leqslant n$. The line graph $L(G)$ of G is defined as $V(L(G)) = E(G)$ and two edges $e, f \in E(G)$ are adjacent in $L(G)$ if they are incident with a common vertex. Prove that the eigenvalues of $L(G)$ are $d - 2 + \lambda_i$ and additionally $(|E(G)| - |V(G)|) \times -2$.
Hint: consider a matrix B of dimension $|V(G)| \times |E(G)|$ that encodes vertex-edge incidencies. Look at BB^T and B^TB.

Selected topics in graph theory

17.12.2018 — spectral graph theory

constructions of expanders

Problem 64. For fixed integers d, n, let $\mathcal{G}_{d,n}$ be the family of d-regular vertex-labeled bipartite graphs with each side of the bipartition of size n. Show that there exists $\alpha = \Omega(1/\text{poly}(d))$ (independent of n) such that if we chose $G \in \mathcal{G}_{d,n}$ uniformly at random with bipartion sizes V_1 and V_2 then with high probability every vertex set $X \subseteq V_1$ of size at most αn has at least $(d-2)|X|$ neighbors.

Problem 65. Let q be a prime and let $\mathbb{F}_q$ be the field of size q. Let G_q be a graph with vertex set $\mathbb{F}_q^2$ and two vertices (a, b) and (c, d) are connected by an edge if and only if $ac = b + d$.

1. Show that (a, b) is adjacent to all vertices of on the line $ax - b$.
2. Compute adjacency matrix of G_q^2.
3. Compute eigenvalues of the adjacency matrix of G_q^2.
4. Show that $\lambda(G_q) \leqslant \sqrt{q}$.

reducing randomness

Problem 66. A randomized algorithm with two-sided error for a language L is an algorithm A that takes as input a word x and a sequence y of $n(x)$ random bits and outputs a single bit $A(x, y)$ with the property that the number of $n(x)$-bit sequences y with $[x \notin L] \Leftrightarrow A(x, y)$ is at most $2^{n(x)}/4$.

1. Consider boosting the error probability of an algorithm by conducting $2t + 1$ independent runs and taking the majority answer. What is the number of used random bits and what is the error probability?
2. Replace independent runs with random walks on an expander with $2^{n(x)}$ nodes. How many random bits do you need to get a similar error probability as in the previous point?

graph products

Let G and H be graphs. The *cartesian product* $G \square H$ is a graph with $V(G \square H) = V(G) \times V(H)$ and $(u, x)(v, y) \in E(G \square H)$ if and only if $u = v$ and $xy \in E(H)$ or $uv \in E(G)$ and $x = y$. The *tensor product* $G \times H$ is a graph with $V(G \square H) = V(G) \times V(H)$ and $(u, x)(v, y) \in E(G \square H)$ if and only if $uv \in E(G)$ and $xy \in E(H)$.

Problem 67. Let A, B, C be graphs with A and B being bipartite. Which of the following graphs are necessarily bipartite: $A \times B$, $A \square B$, $A \times C$, $A \square C$?

Problem 68. Let G and H be connected graphs. Is $G \square H$ necessarily connected? Is $G \times H$ necessarily connected?

Problem 69. Let G be a d_G-regular graph with n_G vertices and eigenvalues $\lambda_1^G \geqslant \lambda_2^G \geqslant \ldots \geqslant \lambda_{n_G}^G$ and let H be a d_H-regular graph with n_H vertices and eigenvalues $\lambda_1^H \geqslant \lambda_2^H \geqslant \ldots \geqslant \lambda_{n_H}^H$.

1. Show that the eigenvalues of $G \times H$ are $\lambda_i^G \lambda_j^H$ for $1 \leqslant i \leqslant n_G$ and $1 \leqslant j \leqslant n_H$.
2. Show that the eigenvalues of $G \square H$ are $\lambda_i^G + \lambda_j^H$ for $1 \leqslant i \leqslant n_G$ and $1 \leqslant j \leqslant n_H$.

3. Show that if G is simple then the eigenvalues of the complement of G are $(n - 1 - d_G)$ and $(-1 - \lambda_i^G)$ for $2 \leqslant i \leqslant n_G$.

Problem 70. Let G be a d-regular graph. Show that:

1. $G \square G$ is $2d$-regular;
2. $d - \lambda_2(G) = 2d - \lambda_2(G \square G)$;
3. $d - \lambda(G) \leqslant 2d - \lambda(G \square G)$;
4. $G \times G$ is d^2-regular;
5. $d(d - \lambda(G)) = d^2 - \lambda(G \times G)$.

Selected topics in graph theory

07.01.2019 — spectral graph theory

Problem 71. The graph G_n is defined as follows. The vertex set of G_n consists of all binary words of length n and two vertices are adjacent if and only if their words differ on exactly one position. Compute the eigenvalues of the adjacency matrix of G_n.

Problem 72. Let G be a d-regular multigraph and let $X \subseteq V(G)$ be a set of size at most $\alpha|V(G)|$ for some $0 < \alpha \leqslant 1/2$. Show that $|E(X, V(G) \setminus X)| \geqslant (1-\alpha)(d - \lambda_2(G))|X|$.

Problem 73. Show that a d-regular multigraph contains no independent set of size larger than $\lambda(G)/d \cdot |V(G)|$.

Problem 74. Let H be a d-regular multigraph that is connected but not bipartite and let G be a $|V(H)|$-regular multigraph.

1. Show that in $G ⓩ H$, every two vertices $(v, w_1) \in V(G ⓩ H)$ and $(v, w_2) \in V(G ⓩ H)$ are in the same connected component.
2. Show that if we additionally assume that G is connected, then $G ⓩ H$ is connected as well.

Problem 75. Let G be a d-regular multigraph and let $F \subseteq E(G)$ be a set of edges not containing any loop. Let $(X_t)_{t \geqslant 0}$ be a standard random walk in G, where X_0 is defined to be a random endpoint of an edge from F chosen uniformly at random. Show that the probability that while stepping from X_i to X_{i+1} we use an edge from F is at most

$$\frac{|F|}{|E|} + \left(\frac{\lambda(G)}{d}\right)^i.$$

Problem 76. Let G be a d-regular multigraph and let $\delta < \frac{d-\lambda(G)}{6}$. Let F be a set of at most $\delta|V(G)|$ edges of G. Show that $G - F$ has a connected component with at least $\left(1 - \frac{2\delta}{d-\lambda(G)}\right)|V(G)|$ vertices.

Selected topics in graph theory

14.01.2019 — spectral graph theory: remainders

Problem 77. Show that a d-regular multigraph contains no independent set of size larger than $\lambda(G)/d \cdot |V(G)|$.

Problem 78. Let H be a d-regular multigraph that is connected but not bipartite and let G be a $|V(H)|$-regular multigraph.

1. Show that in $G \circledz H$, every two vertices $(v, w_1) \in V(G \circledz H)$ and $(v, w_2) \in V(G \circledz H)$ are in the same connected component.
2. Show that if we additionally assume that G is connected, then $G \circledz H$ is connected as well.

Problem 79. Let G be a d-regular multigraph and let $\delta < \frac{d-\lambda(G)}{6}$. Let F be a set of at most $\delta|V(G)|$ edges of G. Show that $G - F$ has a connected component with at least $\left(1 - \frac{2\delta}{d-\lambda(G)}\right)|V(G)|$ vertices.

Selected topics in graph theory

21.01.2019 — colouring, colouring planar graphs, discharging

greedy algorithm During the lecture we considered the greedy algorithm, which took some ordering of the vertices $v_1, v_2, \ldots, v_n$ and assigned to the vertex v_i the smallest possible color not used by $N(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}$.

Problem 80. Show that there is a vertex ordering such that the greedy algorithm will use only $\chi(G)$ colors.

Problem 81. Show a bipartite graph with $2n$ vertices and set its vertices such that the greedy algorithm will use n colors.

choosability

Problem 82. Give an example of a graph G such that $ch(G) > \chi(G)$.

Problem 83. Given an integer $k > 1$. Give an example of a bipartite graph G_k such that $ch(G_k) \geqslant k$.

Euler's formula and discharging

Problem 84. Given a graph G embedded on the torus, show that there is a vertex $v \in V(G)$ of degree $\leqslant 6$.

Problem 85. Show that if we can draw a 6-regular graph on the torus, it is triangularized after drawing.

Problem 86. Using the previous two tasks, try to prove that the graphs that you can draw on the torus are 6-colored. Has the argument with Kempe's chains worked?

edge colouring

Problem 87. Determine the edge chromatic number of K_n.

Problem 88. Prove that every bipartite graph satisfies $\chi'(G) = \Delta(G)$.

different strange

Problem 89. Let G be a triangulation of the plane. Show that the dual of G is 2-colorable if and only if every vertex of G has even degree.

Problem 90. Show that a plane triangulation is 3-colorable if and only if every vertex has even degree.

Problem 91. Consider a set of straight lines on the plane such that no 3 lines intersect in a single point. Let G be the natural graph induced by these lines (intersections are vertices, segments between intersections are edges). Show that this graph is 3-colorable.