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# Graph coloring, perfect graphs

## 1 Introduction to graph coloring

**Definition 1.** Let $G$ be a simple undirected graph and $k$ a positive integer. A function $c : V(G) \to \{1, \ldots, k\}$ is called a *k-coloring* of $G$ if $c(u) \neq c(v)$ for each $uv \in E(G)$.

**Definition 2.** A *chromatic number* $\chi(G)$ of a graph $G$ is the smallest integer $k$ such that $G$ admits a $k$-coloring.

**Example 3.** A chromatic number of a clique $K_n$ is $n$. Bipartite graphs can be defined as graphs which admit a 2-coloring.

Deciding whether a graph is $k$-colorable is difficult, both combinatorially and algorithmically. While 2-colorability is in $P$, 3-colorability is already $NP$-complete. Moreover, the problem of determining $\chi(G)$ is not approximable [3].

### 1.1 Greedy coloring and applications

Let us consider the following algorithm

**Algorithm 1:** Greedy coloring

Let $(v_1, \ldots, v_n)$ be an ordering of $V(G)$;
**for** $i := 1$ **to** $n$ **do**
  $c(v_i) := \min\{k \in \mathbb{Z}_+ : k \notin c\left(N(v_i) \cap \{v_1, \ldots, v_{i-1}\}\right)\}$;

Observe that the number of 'forbidden' colors for $v_i$ is bounded by $\deg v_i$. Consequently, the greedy algorithm uses at most $\Delta(G) + 1$ colors, where $\Delta(G) = \max_{v \in V(G)} \deg v$.

**Corollary 4.** *For every graph $G$ we have $\chi(G) \leq \Delta(G) + 1$.*

Let us improve this bound. Assume that $G$ is a connected graph and $T$ is its spanning tree rooted at $r$. Let us consider an ordering of $V(G)$ in which each vertex $v$ appears after its children in $T$. Now, for $v \neq r$ we have $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq \deg v - 1$, so $c(v_i) \leq \deg v_i$ for $v_i \neq r$. Unfortunately, the greedy may still need to use $\Delta(G) + 1$ colors if $\deg r = \Delta(G)$ and each child of $r$ happens to be colored using a different color. Nevertheless, we have a lot of freedom in choosing $T$ and it turns out that for each graph we can either find an appropriate tree $T$ or prove that no $\Delta(G)$-coloring exists.

**Theorem 5** (Brooks)**.** *Let $G$ be a connected graph. If $G \neq K_n$ and $G \neq C_{2n+1}$, i.e. $G$ is neither a clique nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

We omit the proof as it appears in the Discrete Mathematics course, see also [2].

**Definition 6.** A graph $G$ is *$k$-degenerate* if every subgraph of $G$ has a vertex of degree at most $k$. *Degeneracy* of $G$ is the smallest integer $k$, for which $G$ is $k$-degenerate.

Observe that a natural implementation of Algorithm 1 is to take a vertex of the minimum degree as $v_n$, recursively color $G[v_1, \ldots, v_{n-1}]$ and then color $v_n$ with the smallest available color, which is at most $1 + \deg v_n$. We conclude that a $k$-degenerate graph can be colored with $k+1$ colors.

### 1.2 Edge coloring

**Definition 7.** Let $G$ be a simple undirected graph and $k$ a positive integer. A function $c' : E(G) \to \{1, \ldots, k\}$ is called an *$k$-edge coloring* if $c(e_1) \neq c(e_2)$ for all incident edges $e_1, e_2 \in G$.

**Definition 8.** A *chromatic index* $\chi'(G)$ of $G$ is the smallest integer $k$ such that $G$ admits a $k$-edge coloring.

Clearly, for each vertex $v$ the edges incident to $v$ must be colored with pairwise different colors. This implies $\chi'(G) \geq \Delta(G)$. The upper bound is very close to the lower bound this time:

**Theorem 9** (Vizing)**.** *For each graph $G$ we have $\chi'(G) \leq \Delta(G) + 1$.*

Nevertheless, deciding whether $\chi'(G)$ is $\Delta(G)$ or $\Delta(G) + 1$ is $NP$-complete.

## 2 Perfect graphs

We have already defined the chromatic number $\chi(G)$. Let us recall more graph invariants. By $\omega(G)$ we denote the clique number of $G$, i.e. the maximum size of a clique that is a subgraph of $G$, and by $\alpha(G)$ the independence number of $G$, i.e. the maximum size of a stable set in $G$. Note that $\alpha(G) = \omega(\bar{G})$ since cliques in $G$ correspond to stable sets in $\bar{G}$. These invariants provide lower bounds on the chromatic number:

**Fact 10.** *For each graph $G$ we have $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.*

*Proof.* For the first inequality note that if $K \subseteq V(G)$ induces a clique, then each vertex $v \in K$ is colored with a different color. For the second one observe that each color class forms a stable set, so $|V(G)|$ can be partitioned into $\chi(G)$ stable sets. □

The perfect graphs, which we shall define in a moment, are graphs for which the colorings behave in a controlled and structured way. Simultaneously this class is quite large and includes many important graph classes. This makes it one of the central classes in graph theory.

First, let us make an attempt to define *nice* graphs. Call a graph *nice* if $\chi(G) = \omega(G)$, i.e. a graph $G$ is nice if it has both a coloring and a clique which is a witness that the number of colors is optimal.

This may seem a natural definition, but there are some problems with it. For a graph $G$ define $\hat{G}$ as $G \dot{\cup} K_{\chi(G)}$, where $\dot{\cup}$ is a disjoint union. Clearly $\hat{G}$ is a nice graph, so an arbitrary graph can be made nice if we add a sufficiently large clique. Consequently, we cannot tell anything interesting about the structure of nice graphs. The definition of perfect graphs fixes that problem.

**Definition 11.** A graph $G$ is *perfect* if each induced subgraph $H$ of $G$ is nice, i.e. for each $X \subseteq V(G)$ the graph $H = G[X]$ satisfies $\chi(H) = \omega(H)$.

Observe that the perfect graphs are defined so that this class is *hereditary*. That is if $G$ is perfect then for any $v \in G$ the graph $G - v$ is still perfect and consequently each induced subgraph of a perfect graph is perfect. More generally, the fact that a graph class is closed under induced subgraphs or even under minors, makes it much more convenient to work with it. This is what the nice graphs missed.

As a hereditary class perfect graphs can be characterized by a family of minimal forbidden induced subgraphs. This family is known and relatively simple but infinite, which is possible as the induced subgraph order is not WQO. The following theorem has been conjectured by Berge in 1960's and the proof by Chudnovsky et al. [1], who develop involved structural theorems for perfect graphs, has been published in 2006.

**Theorem 12** (Strong Perfect Graph Theorem)**.** *A graph $G$ is perfect if and only if it does not contain an odd hole or an odd antihole, i.e. for any integer $k \geq 2$ neither $C_{2k+1}$ nor $\overline{C_{2k+1}}$ is an induced subgraph of $G$.*

Observe that the family of forbidden induced subgraphs is closed under complements, which implies the same property for the class of perfect graphs.

**Corollary 13** (Weak Perfect Graph Theorem)**.** *If a graph $G$ is perfect then $\bar{G}$ is perfect as well.*

In this lecture we shall give two alternative proves of the weak perfect graph theorem. Both are due to Lovász and date back to 1970's. The first one is purely combinatorial and proves the theorem directly. The other besides combinatorics uses linear algebra and gives an equivalent characterization of perfect graphs, which immediately implies the weak perfect graph theorem, but is also interesting on its own.

## 2.1 Combinatorial proof

First, let us develop a way of making perfect graphs by *cloning* vertices.

**Definition 14.** Let $G$ be a graph and let $v \in V(G)$. We say that $\hat{G}$ is obtained from $G$ by *cloning* $v$ or adding a *true twin* of $v$, if $\hat{G}$ is obtained from $G$ by adding a vertex $\hat{v}$ adjacent to $v$ so that $v$ and $\hat{v}$ have the same neighbours in $V(G) \setminus \{v\}$.

**Lemma 15.** *Let $G$ be a perfect graph and let $\hat{G}$ be obtained from $G$ by adding a true twin of $v \in V(G)$. Then $\hat{G}$ is perfect.*

*Proof.* We perform induction on the number of vertices. For $|V(G)| = 1$ clearly the lemma holds, which gives the inductive base. Therefore, let us perform an inductive step, i.e. take a perfect graph $G$ and assume that the lemma holds for all perfect graphs $G'$ with fewer vertices.

Formally, we need to prove that for each induced subgraph $H$ of $\hat{G}$ we have $\chi(H) = \omega(H)$. Observe that if $H \neq \hat{G}$ then either $H$ is isomorphic to an induced subgraph of $G$ (and automatically perfect) or $v, \hat{v} \in V(H)$ and $H$ is obtained from $H \setminus \{\hat{v}\}$ by cloning $v$ (and perfect by the inductive hypothesis). Thus, it suffices to show that $\chi(\hat{G}) = \omega(\hat{G})$. Fact 10 allows to prove the $\leq$ inequality only, i.e. it is enough to show that $\hat{G}$ can be colored with $\omega(\hat{G})$ colors. This type of reasoning is a standard beginning of proofs concerning perfect graphs; in subsequent proofs we omit these details.

Note that $\omega(\hat{G}) \geq \omega(G)$ since $G$ is a subgraph of $\hat{G}$ and simultaneously $\omega(\hat{G}) \leq \omega(G) + 1$ since we have added a single vertex. This gives two cases.

If $\omega(\hat{G}) = \omega(G)+1$, then we can extend a coloring of $G$ with $\chi(G) = \omega(G)$ colors to a coloring of $\hat{G}$ with $\omega(G) + 1 = \omega(\hat{G})$ colors just by coloring $\hat{v}$ with the additional color. This simple argument shows $\chi(\hat{G}) = \omega(\hat{G})$.

The other case where $\omega(\hat{G}) = \omega(G)$ is more difficult. Note that $v$ did not belong to any maximum clique in $G$, since otherwise we would have a strictly larger clique in $\hat{G}$. Let us consider a coloring of $G$ with $\omega(G)$ colors, which defines color classes $A_1, \dots, A_{\omega(G)}$; wlog we may assume $v \in A_1$. Let us consider a graph $H = G \setminus (A_1 \setminus \{v\})$. Observe that $H - v$ can be colored with $\chi(G) - 1$ colors (with $A_2, \dots, A_{\omega(G)}$ as color classes), so $\omega(H - v) \leq \omega(G) - 1$. Moreover $\omega(H) \leq \omega(G) - 1$, since no clique of size of $\omega(G)$ contains $v$. As an induced subgraph of $G$, $H$ is a perfect graph, so it admits a coloring with $\omega(G) - 1$ colors. This gives an coloring of $\hat{G}$ with $\omega(G)$ colors, since $A_1$ is stable and $\hat{G} \setminus A_1$ is isomorphic to $H$. Consequently, $\chi(\hat{G}) = \omega(\hat{G})$.

□

Having proved the lemma, let us proceed with the proof of the weak perfect graph theorem.

*Proof.* Again, by an inductive argument it suffices to show that $\bar{G}$ can be colored with $\alpha(G)$ colors. Instead of coloring $\bar{G}$, it is more convenient to think of 'coloring' $G$, so that each color class is a clique.

Consider the following naive approach: take a clique $K$ in $G$. Color the vertices of $K$ with a single color and apply the inductive hypothesis to $G \setminus K$. Let $\mathcal{K}$ be the family of all cliques in $G$. This approach succeeds unless for each clique $K \in \mathcal{K}$ we have $\alpha(G) = \alpha(G \setminus K)$, i.e. for each $K \in \mathcal{K}$ there exists a stable set $A_K$ of size $\alpha(G)$ disjoint with $K$. We shall show that this is impossible.

Let us fix a single $A_K$ for each clique $K$ and for each $v \in V(G)$ define

$$\ell(v) = |\{K \in \mathcal{K} : v \in A_K\}|.$$

Let us define a graph $H$ which is obtained from $G$ by cloning each $v \in V(G)$ exactly $\ell(v)$ times. In particular, if $\ell(v) = 0$ then we remove $v$ and otherwise $\ell(v) - 1$ times we add a true twin of $v$. Note that the order of adding twins and removing vertices does not influence the result. By Lemma 15, $H$ is a perfect graph. Moreover, removing vertices does not increase the independence number and adding twins preserves it, so $\alpha(H) \leq \alpha(G)$. Observe that

$$|V(H)| = \sum_{x \in V(G)} \ell(v) = \sum_{K \in \mathcal{K}} A_K = \alpha(G)|\mathcal{K}|.$$

Consequently, by Fact 10,

$$\chi(H) \geq \frac{|V(H)|}{\alpha(H)} = \frac{\alpha(G)|\mathcal{K}|}{\alpha(H)} \geq \frac{\alpha(G)|\mathcal{K}|}{\alpha(G)} = |\mathcal{K}|$$

Now, note that all cliques in $H$ are obtained from cliques in $G$ by taking clones of vertices of the original cliques, and to obtain a maximal clique in $H$ we must take all clones. This means that

$$\omega(H) = \max_{K \in \mathcal{K}} \sum_{v \in K} l(v) = \max_{K \in \mathcal{K}} \sum_{K' \in \mathcal{K}} |K \cap A_{K'}|.$$

Clearly, a stable set and a clique have at most one common vertex while for $K = K'$ we know that $K \cap A_K = \emptyset$, i.e.

$$\omega(H) = \max_{K \in \mathcal{K}} \sum_{K' \in \mathcal{K}} |K \cap A_{K'}| \leq |\mathcal{K}| - 1 \leq \chi(H) - 1.$$

This is a contradiction since $H$ is perfect. □

## 2.2 Algebraic proof

We shall deduce the weak perfect graph theorem from the following characterization of perfect graphs.

**Theorem 16** (Lovász)**.** *A graph $G$ is perfect if and only if $|V(H)| \leq \alpha(H)\omega(H)$ for each induced subgraph $H$ of $G$.*

Clearly $\alpha \cdot \omega$ is invariant under complements, so the weak perfect graph theorem is an immediate consequence.

*Proof.* Combining Fact 10 with the definition of perfect graphs we obtain the $\Rightarrow$ implication, so let us concentrate of proving the $\Leftarrow$ part.

As usually, we apply an inductive argument. Consequently we can assume that each induced subgraph $H$ of $G$ is perfect and it suffices to show that $G$ can be colored with $\omega(G)$ colors. For brevity we denote $\alpha = \alpha(G), \omega = \omega(G)$, $V = V(G)$ and $n = |V|$.

Again, we shall apply the naive approach and see when it fails. More precisely, we take a stable $A$, color $A$ with a single color and recursively color $H = G \setminus A$ with $\omega(G \setminus A)$ colors. We can choose $A$ arbitrarily, so the naive approach fails only if $\omega(G \setminus A) = \omega$ for each stable set $A$, i.e. for each $A$ there exists a clique $K_A$ of size $\omega$ disjoint with $A$.

Provided $n \leq \alpha\omega$, we shall again show that there is no 'space' in $G$ for these cliques and independent sets. This time, the 'space' is going to formalized as the rank of a particular matrix.

Let $A_0$ be an arbitrary independent set of size $\alpha$ in $G$ and $A_0 = \{a^1, \ldots, a^\alpha\}$. By inductive hypothesis each $G - a^i$ is perfect and consequently can be colored with $\omega$ colors; let the color classes be $A^i_1, \ldots, A^i_\omega$.

This way we get a collection $\mathcal{A} = \{A_0\} \cup \{A^i_j : i \in \{1, \ldots, \alpha\}, j \in \{1, \ldots, \omega\}\}$ of $\alpha\omega + 1$ independent sets.

**Claim 17.** Let $K$ be a clique of size $\omega$. Then $K$ with is disjoint with exactly one $A \in \mathcal{A}$.

*Proof.* Let us fix $a^i \in A_0$. If $a^i \notin K$, then then $K$ is a clique of size $\omega$ in $G - a^i$, so it hits each color class $A^i_j$. On the other hand, if $a^i \in K$, then $K - a^i$ is a clique of size $\omega - 1$ in $G - a^i$, so it hits all but one color class $A^i_j$.

Consequently, if $K \cap A_0 = \emptyset$, then $K$ is hits all cliques $A^i_j$. Otherwise $K \cap A_0 = \{a^i\}$ for some $i$ and $K$ hits all color classes $A^{i'}_j$ for $i' \neq i$ while it is disjoint with a single color class $A^i_j$. □

Let $\mathcal{A} = \{A_0, \ldots, A_{\alpha\omega}\}$ and let us fix cliques $K_0, \ldots, K_{\alpha\omega}$ of size $\omega$ so that $K_i$ is disjoint with $A_i$. Let us define matrices $X, Y$ of size $(\alpha\omega + 1)m \times n$, with the characteristic vector of $A_i$ as the $i$-th row of $X$ and the characteristic vector of $K_i$ as the $i$-th row of $Y$.

Let us consider a matrix $Z = X \cdot Y^T$ and its single entry $z_{ij}$. Observe $z_{ij} = \sum_{v \in V} [v \in A_i][v \in K_j] = |A_i \cap K_j|$ and this quantity is 0 for $i = j$. On the other hand the Claim implies that each column of $Z$ contains exactly one zero, while the other entries are ones. This means that

$$Z = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

In particular $Z$ is of full rank, i.e. $rk(Z) = \alpha\omega + 1$. On the other hand $rk(X), rk(Y) \leq n$, so $rk(Z) \leq \min(rk(X), rk(Y)) \leq n$. This implies $\alpha\omega + 1 \leq n$, a contradiction with $|V(G)| \leq \alpha(G)\omega(G)$. $\square$

## References


[1] Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas. The strong perfect graph theorem. *Annals of Mathematics*, 164:51–229, 2006.

[2] Reinhard Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.

[3] David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, 3(6):103–128, 2007.