Problems

1. Let $A = A^T > 0$ such that its eigenvalues are in $[10, 100]$. We consider the following method for solving $Ax^* = b$:

$$z_n = x_n - \frac{1}{100}(Ax_n - b)$$
$$x_{n+1} = z_n - \frac{1}{10}(Az_n - b)$$

   - Is this method convergent for any x_0?
   - Find the smallest a such that

$$\|x_{20} - x^*\|_2 \leq a\|x_0 - x^*\|_2$$

   for any x_0.

2. We consider the following method for solving $A^T Ax^* = A^T b$, with a nonsingular A:

$$x_{n+1} = x_n + \tau_n r_n$$

   where $r_n = A^T \tilde{r}_n$ for $\tilde{r}_n = b - Ax_n$ and τ_n is such that

$$\min_{\tau} \|x_n + \tau r_n - x^*\|_2 = \|x_n + \tau_n r_n - x^*\|_2.$$

   - find the formula for τ_n and show how to compute τ_n without knowing x^*.
   - Is this method convergent for any x_0?
   - Let assume that the largest eigenvalue of $A^T A$ is $M = \lambda_{max}(A^T A)$ and the smallest eigenvalue of $A^T A$ is $m = \lambda_{min}(A^T A)$. Find the smallest a such that

$$\|x_n - x^*\|_2 \leq a\|x_{n-1} - x^*\|_2$$

   for any x_0.

3. Let $A = A^T > 0$ $N \times N$ such that its eigenvalues are $\lambda_k = k$ for $k = 1, \ldots, N$. Let the eigenvector for λ_k be denoted by v_k. We consider the following method for solving $Ax^* = b$:

$$x_{n+1} = x_n - \frac{1}{10}(Ax_n - b)$$

   - Is this method convergent for any x_0 for $N = 10^4$?

- Assuming that $x_0 - x^* = \sum_{k=1}^{10} a_k v_k$ Find the smallest a such that

$$\|x_n - x^*\|_2 \leq a\|x_{n-1} - x^*\|_2$$

  for any x_0 and $N = 10^4$.

4. Let $A_\alpha = (a_{ij})$ $N \times N$ be defined by:

$$a_{ij} = \begin{cases} \alpha & i \neq j \\ 1 & i = j \end{cases}$$

   Find all values of α for which:

   - $A = A^T > 0$
   - The Jacobi method for solving $A_\alpha x^* = b$ is convergent for any initial iteration x_0.