WEAKER COUSINS OF RAMSEY'S THEOREM OVER A WEAK BASE THEORY

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Abstract. The paper is devoted to a reverse-mathematical study of some well-known consequences of Ramsey's theorem for pairs, focused on the chain-antichain principle CAC, the ascending-descending sequence principle ADS, and the Cohesive Ramsey Theorem for pairs CRT^2_2. We study these principles over the base theory RCA^*_0, which is weaker than the usual base theory RCA_0 considered in reverse mathematics in that it allows only Δ^0_1-induction as opposed to Σ^0_1-induction. In RCA^*_0, it may happen that an unbounded subset of $\mathbb{N}$ is not in bijective correspondence with $\mathbb{N}$. Accordingly, Ramsey-theoretic principles split into at least two variants, "normal" and "long", depending on the sense in which the set witnessing the principle is required to be infinite.

We prove that the normal versions of our principles, like that of Ramsey's theorem for pairs and two colours, are equivalent to their relativizations to proper Σ^0_1-definable cuts. Because of this, they are all Π^0_3- but not Π^1_1-conservative over RCA^*_0, and, in any model of $\mathsf{RCA}^*_0 + \neg\mathsf{RCA}_0$, if they are true then they are computably true relative to some set. The long versions exhibit one of two behaviours: they either imply RCA_0 over RCA^*_0 or are Π^0_3-conservative over RCA^*_0. The conservation results are obtained using a variant of the so-called grouping principle.

We also show that the cohesion principle COH, a strengthening of CRT^2_2, is never computably true in a model of RCA^*_0 and, as a consequence, does not follow from RT^2_2 over RCA^*_0.

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1. Introduction

The logical strength of Ramsey-theoretic principles has been one of the most important research topics in reverse mathematics for over two decades. Statements from Ramsey theory are an appealing subject for logical analysis, because they are often not equivalent to any of the usual set existence principles encountered in second-order arithmetic, and they form a complex web of implications and nonimplications (see [8] for an introduction to the area). Moreover, characterizing the first-order consequences of Ramsey-theoretic statements is frequently an interesting and demanding task.

As is the custom in reverse mathematics, the strength of such statements is usually investigated over the base theory RCA_0, a fragment of second-order arithmetic that includes the Δ^0_1-comprehension axiom and the mathematical induction scheme for Σ^0_1-definable properties. A weaker alternative to RCA_0, introduced in [19] and known as RCA^*_0, allows induction only for Δ^0_1 properties. Working in a weak base theory makes it possible to track nontrivial uses of induction and to make some fine distinctions

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that disappear over RCA_0, but it also comes with additional technical and conceptual challenges.

An issue of particular relevance to Ramsey-theoretic principles is that many of them assert the existence of an *infinite* set $Y \subseteq \mathbb{N}$ that relates in a certain way to a given colouring of tuples. RCA_0^* is weak enough that the precise definition of what it means to be an infinite subset of $\mathbb{N}$ becomes important. Usually, one only requires that Y be unbounded in $\mathbb{N}$, and this gives rise to what we call "normal" versions of the principles. However, over RCA_0^* being unbounded is strictly weaker than being the range of a strictly increasing map with domain $\mathbb{N}$. "Long" versions of principles can be obtained by requiring Y to have the latter property.

The strength of Ramsey's Theorem over RCA_0^* was investigated in [23] and [11]. The upshot of that work is that in all nontrivial cases, the normal version of Ramsey's Theorem for a fixed length of tuples and number of colours is partially conservative but not Π^1_1-conservative over RCA_0^*. On the other hand, the long version of Ramsey's Theorem is strong enough to imply RCA_0.

In this paper, we ask the question whether the same general pattern also holds for other Ramsey-theoretic principles, in particular the various natural weakenings of Ramsey's Theorem for pairs that are commonly studied in reverse mathematics. Many of our results could be stated in relatively general way, but for illustrative purposes, we find it useful to concentrate on a small number of specific principles. We mostly consider two statements about linear orders, namely the chain-antichain principle CAC and the ascending-descending sequence principle ADS, as well as the cohesive version of Ramsey's theorem for pairs and two colours CRT^2_2. (The definitions are recalled in Section 2.1.) The statements CAC, ADS, and CRT^2_2 are not only combinatorially natural, but also reasonably well-understood in the traditional reverse-mathematical setting: over RCA_0 they form a strict linear order in terms of implication, and each of them is known to be fully conservative over a classical fragment of first-order arithmetic.

We show that normal versions of our principles, just like those of RT^n_k, belong to a class of statements that we call "pseudo–second-order". The behaviour of any such statement in a model of $\mathsf{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$ is governed by the proper Σ^0_1-definable cuts of the model. As a consequence, normal versions of CAC, ADS, and CRT^2_2 are Π^0_3- but not Π^1_1-conservative over RCA_0, and they have the curious feature that whenever they are true in a structure satisfying $\mathsf{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$, they are actually computably true in that structure relative to a set parameter witnessing the failure of $\mathrm{I}\Sigma^0_1$. We also show that CAC and ADS are significantly weaker than RT^2_2 in a technical sense related to closure properties of cuts. The strength of CRT^2_2 in this sense is left open, as is the question whether ADS or CAC imply CRT^2_2 over RCA_0^*.

We then show that long versions of Ramsey-theoretic principles tend to behave in one of two ways. Some, like CAC, imply RCA_0 by an easy argument dating back to [23]. Others, like CRT^2_2, are equivalent to normal versions of the corresponding principles in RCA_0^* or in its extension by Weak König's Lemma. As a result, these principles remain Π^0_3-conservative over RCA_0. In the case of ADS, both behaviours are possible depending on how exactly the principle is formalized.

We also study the cohesion principle COH, a well-known strengthening of CRT^2_2 that does not fit neatly into the classification into normal and long principles. It follows immediately from our results on CRT^2_2 that COH is not Π^1_1-conservative over RCA_0^*, which answers a question of Belanger [2]. Our main result about COH as such is that in contrast to many other statements we consider, it can never be computably true, even in a model of $\mathsf{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$. As a consequence, COH is not implied by CRT^2_2, ADS, or even RT^2_2 provably in RCA_0^*.

The remainder of this paper is structured as follows. In Section 2, we discuss the necessary definitions and background, including precise formulations of the normal and long versions of our principles. We study the normal versions in Section 3, the long versions in Section 4, and COH in Section 5.

2. Preliminaries

We assume that the reader has some familiarity with the language of second-order arithmetic and with the most common fragments of second-order arithmetic like RCA_0 and WKL_0, as described in [18] or [8]. We also assume familiarity with the usual induction and collection (or bounding) schemes encountered in first- and second-order arithmetic. Background in first-order arithmetic that is not covered in [8] will be discussed below.

The symbol ω denotes the set of standard natural numbers, while $\mathbb{N}$ denotes the set of natural numbers as formalized within an arithmetic theory. In other words, if $(M, \mathcal{X})$ is a model of some fragment of second-order arithmetic, then $\mathbb{N}^{(M,\mathcal{X})}$ is simply the first-order universe M. The symbol $\leq$ denotes the usual order on $\mathbb{N}$.

We write Δ^0_n, Σ^0_n, Π^0_n to denote the usual formula classes defined in terms of first-order quantifier alternations, but allowing second-order free variables. On the other hand, notation without the superscript 0, like Δ_n, Σ_n, Π_n, represents analogously defined classes of purely first-order, or "lightface", formulas, that do not contain any second-order variables at all. If we want to specify the second-order parameters appearing in a Σ^0_n formula, we use notation like $\Sigma_n(A)$. We extend these conventions to naming theories. If Γ is a class of formulas, then $\forall\Gamma$ denotes the class of universal closures of formulas from Γ. Note, for example, that $\forall\Sigma^0_n$ and $\forall\Pi^0_{n+1}$ are the same class.

The theory RCA^*_0, originally defined in [19], is obtained from RCA_0 by replacing the $\mathrm{I}\Sigma^0_1$ axiom with the weaker axiom of Δ^0_0-induction (by Δ^0_1-comprehension, this immediately implies induction for all Δ^0_1-definable properties) and adding a Π_2 axiom exp that explicitly guarantees the totality of exponentiation. The theory WKL^*_0 is obtained from WKL_0 in an analogous way. RCA^*_0 proves the collection scheme $\mathrm{B}\Sigma^0_1$, and the first-order consequences of RCA^*_0 and of WKL^*_0 are axiomatized by $\mathrm{B}\Sigma_1 + \exp$

When we consider a model $(M, \mathcal{X}) \vDash \mathsf{RCA}^*_0$ (or work in RCA^*_0 without reference to a specific model), a *set* is an element of the second-order universe $\mathcal{X}$. In contrast, a *Σ^0_n-definable* set or simply *Σ^0_n-set* is any subset of the first-order universe M that is definable in $(M, \mathcal{X})$ by a Σ^0_n formula (and likewise for Σ_n-sets, Π_n-sets etc.) A *Δ^0_n-definable set* or *Δ^0_n-set* is a Σ^0_n-set that is simultaneously a Π^0_n-set. Since in general the models we study only satisfy Δ^0_1-comprehension, Δ^0_n-sets for $n \geq 2$ and Σ^0_n-sets for $n \geq 1$ will not always be sets. We write Δ_1-Def(M) for the collection of the Δ_1-definable subsets of M and Δ^0_1-Def(M, A) for the collection of $\Delta_1(A)$-definable subsets, where $A \subseteq M$. If $(M, A) \vDash \mathrm{B}\Sigma_1(A) + \exp$, then $(M, \Delta^0_1\text{-Def}(M, A))$ is a model of RCA^*_0.

Already $\mathrm{I}\Delta_0 + \exp$ is strong enough to support a well-behaved universal Σ_1 formula $\mathrm{Sat}_1(x, y)$. We can define the Σ_1-set $0'$ as $\{e : \mathrm{Sat}_1(e, e)\}$.

A *cut* I in a model of arithmetic M is a downwards-closed subset of M which is also closed under successor. M is then an *end-extension* of I, and it is common to write $I \subseteq_e M$, or $I \subsetneq_e M$ if I is a proper cut. The cut I is a Σ^0_1-cut exactly if it is Σ^0_1-definable.

A set A is *unbounded* if for every $x \in \mathbb{N}$ there exists $y \in A$ with $y \geq x$. We write $A \subseteq_{\mathrm{cf}} \mathbb{N}$ to indicate that A is unbounded, and more generally $A \subseteq_{\mathrm{cf}} B$ to indicate that A is an unbounded subset of B. The set A *has cardinality* $\mathbb{N}$ if it contains an n-element finite subset for each $n \in \mathbb{N}$, or equivalently if it can be enumerated in increasing order as $\{a_n : n \in \mathbb{N}\}$. Provably in RCA^*_0, a set of cardinality $\mathbb{N}$ is unbounded. However, it was shown in [20, Lemma 3.2] that the statement "every unbounded set has cardinality $\mathbb{N}$" implies RCA_0 over RCA_0^*. In other words, $\mathsf{RCA}_0^* + \neg \mathrm{I}\Sigma_1^0$ proves the existence of an unbounded set that does not have cardinality $\mathbb{N}$. Each such set can be enumerated in increasing order as $A = \{a_i \mid i \in I\}$ for some proper Σ_1^0-cut I. Conversely, given a Σ_1^0 cut I, we can use Δ_1^0-comprehension to form the set $\{\langle w_0, \ldots, w_i\rangle : i \in I\}$, where each w_j is the smallest element witnessing that $j \in I$. Thus, we have:

Proposition 2.1. *Let $(M, \mathcal{X}) \vDash \mathsf{RCA}_0^*$. For each Σ_1^0-cut I there exists a set $A \in \mathcal{X}$ with $A \subseteq_{\mathrm{cf}} M$ that can be enumerated in increasing order as $A = \{a_i \mid i \in I\}$.*

If $A = \{a_i \mid i \in I\}$ for some Σ_1^0-cut I, we sometimes write a_{-1} for -1.

A bounded subset of a model $M \vDash \mathrm{I}\Delta_0 + \exp$ is *coded* in M if it has the form $(s)_{\mathrm{Ack}} = \{x \in M \mid M \vDash x \in_{\mathrm{Ack}} s\}$ for some $s \in M$, where $x \in_{\mathrm{Ack}} s$ denotes the usual Δ_0 formula expressing that the x^{th} digit in the binary expansion of s is 1. For a cut $I \subsetneq_e M$ we let $\mathrm{Cod}(M/I) = \{I \cap (s)_{\mathrm{Ack}} \mid s \in M\}$ stand for the collection of subsets of I which are coded in M. Note that $\mathrm{Cod}(M/I)$ can be viewed as a second-order structure on I. If I is closed under exponentiation, then $(I, \mathrm{Cod}(M/I)) \vDash \mathsf{WKL}_0^*$ [19, Theorem 4.8].

The following lemma states an important special case of a more general result about coding in models of $\mathrm{B}\Sigma_n^0 + \exp$.

Lemma 2.2 (Chong-Mourad [4])**.** *Let $(M, \mathcal{X}) \vDash \mathsf{RCA}_0^*$ and let I be a proper Σ_1^0-cut in $(M, \mathcal{X})$. If $X \subseteq I$ is such that both X and $I \setminus X$ are Σ_1^0-definable, then $X \in \mathrm{Cod}(M/I)$.*

The iterated exponential function is defined inductively as follows: $\exp_0(y) = 1$, $\exp_{x+1}(y) = y^{\exp_x(y)}$. The axiom supexp, provable in RCA_0 but not in RCA_0^*, states that the iterated exponential function is total, i.e. $\exp_x(y)$ exists for every x and y.

Proposition 2.3. *For each countable $(M, \mathcal{X}) \vDash \mathsf{WKL}_0$ there exists $K \supsetneq_e M$ such that $K \vDash \mathrm{B}\Sigma_1 + \exp$, M is a Σ_1-cut of K, and $\mathrm{Cod}(K/M) = \mathcal{X}$.*

Proof. Suppose that $(M, \mathcal{X})$ is a countable model of WKL_0. By [21], there exists a structure $L \supsetneq_e M$ such that $(L, \Delta_1\text{-Def}(L)) \vDash \mathsf{RCA}_0$ and $\mathrm{Cod}(L/M) = \mathcal{X}$. Fix some $a \in L \setminus M$. Note that since L satisfies $\mathrm{I}\Sigma_1^0$ and therefore supexp, the value $\exp_b(a)$ exists in L for each $b \in L$. Define $K \subseteq L$ so that $K = \sup(\{\exp_m(a) \mid m \in M\})$. Then $K \vDash \mathrm{B}\Sigma_1 + \exp$ and M is a Σ_1-cut in K since $m \in M$ if and only if $K \vDash \exists y\, (y = \exp_m(a))$. Furthermore, $\mathrm{Cod}(K/M) = \mathrm{Cod}(L/M) = \mathcal{X}$. □

2.1. **Normal and long versions of principles.** Many Ramsey-theoretic statements take the form $\forall X \subseteq \mathbb{N}\, (\alpha(X) \rightarrow \exists Y\, (Y \text{ is infinite} \wedge \beta(X, Y)))$, where α and β are arithmetical. In this context X and Y are often called, respectively, "instance" and "solution" of the statement. In RCA_0, "Y is infinite" is usually formalized as "Y is unbounded". However, "Y is infinite" could also be taken to mean "Y has cardinality $\mathbb{N}$", and, as explained above, the two concepts are not equivalent in RCA_0^*. Accordingly, over RCA_0^* typical Ramsey-theoretic principles will have at least two versions: one that we will take as the default and call the *normal* one, in which we only require the solution Y to be infinite in the sense of being unbounded; and a *long* version, in which we require Y to have cardinality $\mathbb{N}$. (The word "long" is intended to emphasize that Y has to be enumerated using $\mathbb{N}$ as opposed to a shorter cut.) When using standard abbreviations for various principles, we will distinguish the long versions from the normal ones by using the prefix ℓ-.

The distinction between the two versions of Ramseyan statements was first made in the context of Ramsey's Theorem itself by Yokoyama [23]. For any $n, k \in \omega$, let RT_k^n be the normal version of Ramsey's Theorem for n-tuples and k colours, "For every

$c\colon [\mathbb{N}]^n \to k$ there exists an unbounded set $H \subseteq \mathbb{N}$ such that $c \restriction [H]^n$ is constant", and let ℓ-RT^n_k be the long version, which requires H to have cardinality $\mathbb{N}$ (this is denoted by RT^{n+}_k in [23]). It was shown in [23] that ℓ-RT^2_2 implies $\mathrm{I}\Sigma^0_1$ over RCA^*_0, while RCA^*_0 extended by RT^n_k is Π_2-conservative over $\mathrm{I}\Delta_0 + \exp$. The study of RT^n_k over RCA^*_0 was taken quite a bit further in [11]. Results obtained in that paper include the $\forall\Pi^0_3$-conservativity of $\mathsf{RCA}^*_0 + \mathsf{RT}^n_k$ over RCA^*_0 for each n, k, a complete axiomatization of $\mathsf{RCA}^*_0 + \mathsf{RT}^n_2$ for each $n \geq 3$, and a complete axiomatization of $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2 + \neg\mathrm{I}\Sigma^0_1$.

The emphasis in the present paper is on principles about ordered sets, CAC and ADS, and on the Cohesive Ramsey Theorem CRT^2_2. Let us, therefore, give precise formulations of the normal and long versions for each of these principles in turn.

The chain-antichain principle CAC says that every partial order defined on $\mathbb{N}$ contains either an infinite chain or an infinite antichain. Over RCA^*_0, this gives rise to the following principles.

CAC : *For every partial order* $(\mathbb{N}, \preceq)$ *there exists an unbounded set* $S \subseteq \mathbb{N}$ *which is either a chain or an antichain in* $\preceq$.

ℓ-CAC: *For every partial order* $(\mathbb{N}, \preceq)$ *there exists a set* $S \subseteq \mathbb{N}$ *of cardinality* $\mathbb{N}$ *which is either a chain or an antichain in* $\preceq$.

It could be argued that a more natural formulation of CAC would require the existence of an unbounded chain or antichain in any partial order on an unbounded set, not necessarily on all of $\mathbb{N}$. However, we will prove in Lemma 3.2 that this is equivalent to the version given above and that an analogous equivalence also holds for the normal versions of other principles we study.

The ascending-descending sequence principle ADS says that every linear order on $\mathbb{N}$ contains either an unbounded increasing sequence or an unbounded decreasing sequence. There is a delicate issue here, as there can be more than one way of stating the requirement that the solution to ADS has to satisfy. In the literature (see e.g. [8,9]) an ascending sequence is usually taken to mean either (i) an infinite set $S \subseteq \mathbb{N}$ on which the ordering $\preceq$ agrees with the natural number ordering $\leq$ or (ii) a sequence $(s_i)_{i\in\mathbb{N}}$ properly understood (that is, a map with domain $\mathbb{N}$) such that $s_0 \prec s_1 \prec s_2 \prec \ldots$ but there is no requirement on how the s_i are ordered by $\leq$. One could refer to these as set and sequence solutions to ADS, respectively. (Set and sequence solutions corresponding to descending sequences are defined analogously.) Over RCA_0, versions of ADS formulated in terms of set and sequence solutions are equivalent: a set solution obviously computes a sequence solution, but given a sequence solution $(s_i)_{i\in\mathbb{N}}$ we can also obtain a set solution by taking the set of those numbers s_j that are $\leq$-greater than all s_i for $i < j$.

Over RCA^*_0, such a thinning out argument works for the normal version of ADS: if we are given a sequence solution $(s_i)_{i\in I}$ with $s_0 \prec s_1 \prec \ldots$ for some cut I, then the set S of those s_j for $j \in I$ such that $s_j > s_i$ for all $i < j$ can be obtained by Δ^0_1-comprehension and is unbounded provably in RCA^*_0. Thus, S is a set solution to ADS. However, if $(s_i)_{i\in\mathbb{N}}$ is a sequence solution to the long version of ADS, then without $\mathrm{I}\Sigma^0_1$ it may happen that the set S obtained in this way is no longer of cardinality $\mathbb{N}$; in other words, S might not be a set solution to the long version of ADS. This leads us formulate the following three variants of ADS:

ADS : *For every linear order* $(\mathbb{N}, \preceq)$ *there exists an unbounded set* $S \subseteq \mathbb{N}$ *such that either for all* $x, y \in S$ *it holds that* $x \leq y$ *iff* $x \preceq y$ *or for all* $x, y \in S$ *it holds that* $x \leq y$ *iff* $x \succeq y$.

ℓ-$\mathsf{ADS}^{\mathrm{set}}$**:** *For every linear order* $(\mathbb{N}, \preceq)$ *there exists a set* $S \subseteq \mathbb{N}$ *of cardinality* $\mathbb{N}$ *such that either for all* $x, y \in S$ *it holds that* $x \leq y$ *iff* $x \preceq y$ *or for all* $x, y \in S$ *it holds that* $x \leq y$ *iff* $x \succeq y$.

ℓ-$\mathsf{ADS}^{\mathrm{seq}}$**:** *For every linear order* $(\mathbb{N}, \preceq)$ *there exists a sequence* $(s_i)_{i\in\mathbb{N}}$ *which is either strictly* $\preceq$*-increasing or strictly* $\preceq$*-decreasing.*

Notice that ℓ-$\mathsf{ADS}^{\mathrm{set}}$ clearly implies ℓ-$\mathsf{ADS}^{\mathrm{seq}}$. On the other hand, it will follow from Theorem 4.2 and Corollary 4.10 that the converse implication does not hold over RCA_0^*.

The final principle we focus on is the Cohesive Ramsey Theorem CRT^2_2. This says that for every 2-colouring c of pairs of natural numbers, there is an infinite set S on which c is *stable*, that is, for each $x \in S$, either $c(x, y) = 0$ for all sufficiently large $y \in S$ or $c(x, y) = 1$ for all sufficiently large $y \in S$. Thus, we define the following principles.

CRT^2_2 **:** *For every* $c\colon [\mathbb{N}]^2 \to 2$ *there exists an unbounded set* $S \subseteq \mathbb{N}$ *such that for each* $x \in S$ *there exists* $y \in S$ *such that* $c(x, z) = c(x, y)$ *holds for all* $z \in S$ *with* $z \geq y$.

ℓ-CRT^2_2**:** *For every* $c\colon [\mathbb{N}]^2 \to 2$ *there exists a set* $S \subseteq \mathbb{N}$ *of cardinality* $\mathbb{N}$ *such that for each* $x \in S$ *there exists* $y \in S$ *such that* $c(x, z) = c(x, y)$ *holds for all* $z \in S$ *with* $z \geq y$.

We also recall some principles that are not the main focus of this work but will be mentioned in one or more contexts.

Stable Ramsey's Theorem SRT^2_2 is RT^2_2 restricted to colourings c that are stable on $\mathbb{N}$.

A colouring $c\colon [A]^2 \to n$ is *transitive* if $c(x, y) = c(y, z) = i$ implies $c(x, z) = i$ for all $i < n$ and all $x < y < z$ elements of A. The colouring c is *semitransitive* if the above implication holds for all $i < n$ except at most one. The Erdös-Moser principle EM says that for any $c\colon [\mathbb{N}]^2 \to 2$, there is an infinite set $A \subseteq \mathbb{N}$ on which c is transitive.

Over RCA_0^*, both SRT^2_2 and EM have normal and long versions, which are defined in the natural way. RCA_0^* is able to prove the well-known equivalences of RT^2_2 with $\mathsf{SRT}^2_2 \land \mathsf{CRT}^2_2$ and with $\mathsf{EM} \land \mathsf{ADS}$.

The cohesive principle COH is recalled and studied in Section 5.

3. Normal principles

Hirschfeldt and Shore [9] proved that the sequence of implications $\mathsf{RT}^2_2 \to \mathsf{CAC} \to \mathsf{ADS} \to \mathsf{CRT}^2_2$ holds over RCA_0. Moreover, they showed that the first and third implication do not in general reverse over RCA_0. The strictness of the implication from CAC to ADS was shown in [15].

It is easy to check that the proofs of the implications from RT^2_2 to CAC and CRT^2_2, and of the one from CAC to ADS, do not require $\mathrm{I}\Sigma^0_1$. We can thus state the following lemma (see 4.1 for its "long" counterpart).

Lemma 3.1. *Over* RCA_0^**, the following sequences of implications hold:*

$$\mathsf{RT}^2_2 \to \mathsf{CAC} \to \mathsf{ADS},$$
$$\mathsf{RT}^2_2 \to \mathsf{CRT}^2_2.$$

None of the implications can be provably reversed in RCA_0^*.

Two issues left open by the lemma are whether ADS or at least CAC implies CRT^2_2 over RCA_0^*, and whether the implications above are still strict over $\mathsf{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$. It will be shown in Theorem 3.11 that RT^2_2, CAC, ADS, and CRT^2_2 do in fact remain pairwise distinct over $\mathsf{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$, and moreover, that they have pairwise distinct sets of arithmetical consequences. Interestingly, this is related to the fact that the principles are known to be distinct over WKL_0.

On the other hand, we were not able to determine whether RCA^*_0 proves $\mathsf{CAC} \to \mathsf{CRT}^2_2$. This question may be related to the problem whether CRT^2_2 is weaker than RT^2_2 in a specific technical sense discussed in Section 3.3.

3.1. **Basic observations.** In this subsection, we verify that some well-known and useful properties of the Ramsey-theoretic principles we consider still hold over RCA^*_0. First, we show that no generality is lost by restricting the principles to instances defined on all of $\mathbb{N}$ rather than on a more general infinite set.

Lemma 3.2. *Over RCA^*_0, each of RT^n_k, CAC, ADS, CRT^2_2 is equivalent to its generalization to orderings/colourings defined on an arbitrary unbounded subset of $\mathbb{N}$.*

Proof. For RT^n_k, this is implicit in [11]. The proofs are similar for all principles; we sketch them for ADS and CRT^2_2.

Working in RCA^*_0, assume ADS and let $(A, \preceq)$ be a linear order, where $A \subseteq_{\mathrm{cf}} \mathbb{N}$. Thus, $A = \{a_i \mid i \in I\}$, for some Σ^0_1-cut I in $\mathbb{N}$.

Define a linear order $\preceq'$ on $\mathbb{N}$ by

$$x \preceq' y \Leftrightarrow \exists i, j \in I\, (x \in (a_{i-1}, a_i] \wedge y \in (a_{j-1}, a_j] \wedge ((i \neq j \wedge a_i \prec a_j) \vee (i = j \wedge x \leq y)))$$

That is, elements are $\preceq'$-ordered according to the the $\preceq$-ordering between the nearest elements of A above them, if that makes sense, and according to the usual natural number ordering otherwise. Since $\preceq'$ is $\Delta_1(A, \preceq)$-definable, it exists as a set. Let $S' \subseteq_{\mathrm{cf}} \mathbb{N}$ be a strictly increasing or strictly decreasing sequence in $\preceq'$. Using $\Delta_1(S', A)$-comprehension, define $S \subseteq A$ by:

$$a \in S \Leftrightarrow a \in A \wedge \exists x \leq a\, (x \in S' \wedge [x, a) \cap A = \emptyset).$$

It is easy to check that S is unbounded and it is either a strictly increasing or a strictly decreasing sequence in $\preceq$.

For CRT^2_2, given $c\colon [A]^2 \to 2$, use Δ^0_1-comprehension to define $c'\colon [\mathbb{N}]^2 \to 2$ by:

$$c'(x, y) = \begin{cases} c(a_i, a_j) & \text{if } \exists i, j \in I\ (i \neq j \wedge x \in (a_{i-1}, a_i] \wedge y \in (a_{j-1}, a_j]), \\ 0 & \text{otherwise.} \end{cases}$$

If $S' \subseteq_{\mathrm{cf}} \mathbb{N}$ is such that c' is stable on S', then it is easy to define analogously as above $S \subseteq_{\mathrm{cf}} A$ on which c is stable by $\Delta_1(S', A)$-comprehension. □

We now check that in RCA^*_0, it is still true that ADS and CAC can be viewed as the restrictions of RT^2_2 to transitive and semitransitive colourings, respectively.

Proposition 3.3. *Over RCA^*_0, CAC and ADS are equivalent to RT^2_2 restricted to semitransitive 2-colourings and to transitive 2-colourings, respectively.*

Proof. This is just a verification that the arguments of [9] go through in RCA^*_0.

The implication from CAC to RT^2_2 for semitransitive 2-colourings is unproblematic. In the other direction, CAC follows easily from RT^2_3 for semitransitive 3-colourings, which is in turn derived from RT^2_2 for semitransitive 2-colourings. In the reduction from 3-colourings to 2-colourings, at one point we have to obtain an unbounded homogeneous set for a semitransitive 2-colouring defined on an unbounded subset of $\mathbb{N}$ rather than on $\mathbb{N}$. This is dealt with like in the proof of Lemma 3.2.

The implication from RT^2_2 for transitive 2-colourings to ADS is immediate. The other direction is [9, Theorem 5.3], which requires a comment. Given a transitive colouring $c\colon [\mathbb{N}]^2 \to 2$, we build a linear order $\preceq$ by inserting numbers $0, 1, \ldots$ into it one-by-one. When $\preceq$ is already defined on $\{0, \ldots, n-1\}$, we insert n into the order directly above the $\preceq$-largest $k < n$ such that $c(k, n) = 0$; if there is no such k, we place n at the bottom of $\preceq$. Then, we can check by induction on n that the ordering $\preceq$ agrees with c on $\{0, \ldots, n\}$ in the sense that for $i < j \leq n$, we have $i \prec j$ iff $c(i, j) = 0$. In [9], $\mathrm{I}\Sigma^0_1$ is invoked for this purpose, but it will be clear from the above description that the induction formula is actually bounded. The induction step uses the transitivity of c. □

3.2. **Between models and cuts.** In [11], it is shown that RT^n_k displays interesting behaviour in models of $\mathsf{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$: if I is a proper Σ^0_1-cut in a model $(M, \mathcal{X})$, then RT^n_k holds in the entire model $(M, \mathcal{X})$ if and only if it holds on the cut, that is in the structure $(I, \mathrm{Cod}(M/I))$. This equivalence provides important information about the first-order consequences of RT^n_k over RCA^*_0. It is apparent from the proof of the equivalence that it is not highly specific to RT^n_k and should hold for many other Ramsey-theoretic statements.

In Theorem 3.5 below, we identify a relatively broad syntactic class of sentences that all share the property of being equivalent to their own relativizations to Σ^0_1-cuts. We then verify that Ramsey-theoretic statements such as RT^n_k, CAC, ADS, and CRT^2_2 are equivalent to sentences from that class. It follows, for instance, that all these statements fail to be Π^1_1-conservative over RCA^*_0, and that they differ in their arithmetical consequences.

Definition 3.4. The $\mathcal{L}_2$-sentence χ belongs to the class of sentences pSO if there exists a sentence γ of second-order logic in a language $(\leq, R_1, \ldots, R_k)$, where $k \in \omega$ and each R_i is a relation symbol of arity $m_i \in \omega$, such that χ expresses:

> for any relations $R_1, \ldots, R_k$ on $\mathbb{N}$ and for each $D \subseteq_{\mathrm{cf}} \mathbb{N}$,
> there exists $H \subseteq_{\mathrm{cf}} D$ such that $(H, \leq, R_1, \ldots, R_k) \vDash \gamma$.

In the definition above, we slightly abuse notation by writing $(H, \leq, R_1, \ldots, R_k)$ instead of the more cumbersome $(H, \leq \cap H^2, R_1 \cap H^{m_1}, \ldots, R_k \cap H^{m_k})$. The fact that this structure satisfies γ is expressed by relativizing each first-order quantifier in γ to H and restricting each m-ary second-order quantifier to m-ary relations on H. Of course, when this is interpreted in a model of arithmetic $(M, \mathcal{X})$, "m-ary relations on H" are understood as elements of $\mathcal{X} \cap \mathcal{P}(H^m)$.

The abbreviation pSO stands for "pseudo–second-order": pSO sentences appear to use both first- and second-order quantification of $\mathcal{L}_2$, but they are relativized to arbitrarily small unbounded subsets of $\mathbb{N}$ in such a way that in cases where $\mathrm{I}\Sigma^0_1$ fails their behaviour is closer to that of first-order sentences; cf. Corollary 3.6.

Theorem 3.5. *If χ is a* pSO *sentence, then for every $(M, \mathcal{X}) \vDash \mathsf{RCA}^*_0$ and every proper Σ^0_1-cut I in $(M, \mathcal{X})$, it holds that $(M, \mathcal{X}) \vDash \chi$ if and only if $(I, \mathrm{Cod}(M/I)) \vDash \chi$.*

Proof. Let γ be a second-order sentence and for notational simplicity, assume that it contains only one unary relation symbol R in addition to $\leq$, and that all second-order quantifiers are unary. Let χ be a pSO sentence stating that for every set R and every unbounded set D there exists an unbounded subset $H \subseteq_{\mathrm{cf}} D$ such that $(H, \leq, R) \vDash \gamma$. Let $(M, \mathcal{X}) \vDash \mathsf{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$, and let $A \in \mathcal{X}$ be a cofinal subset of M enumerated by the cut I, $A = \{a_i \mid i \in I\}$, as in Proposition 2.1.

Suppose first that $(M,\mathcal{X}) \vDash \chi$. Let $R, D \in \mathrm{Cod}(M/I)$ be such that $D \subseteq_{\mathrm{cf}} I$. Define $R', D' \subseteq M$ by:

$$x \in R' \Leftrightarrow \exists i \in I\, (x = a_i \wedge i \in R),$$
$$x \in D' \Leftrightarrow \exists i \in I\, (x = a_i \wedge i \in D).$$

Since both R' and $M \setminus R'$ are Σ_1-definable in A and (the code for) R, we know that $R' \in \mathcal{X}$. Similarly, $D' \in \mathcal{X}$. Notice that $D' \subseteq_{\mathrm{cf}} M$, since $D \subseteq_{\mathrm{cf}} I$ and $A \subseteq_{\mathrm{cf}} M$.

By our assumption that $(M,\mathcal{X}) \vDash \chi$, there exists $H' \in \mathcal{X}$ such that $H' \subseteq_{\mathrm{cf}} D'$ and $(H', \leq, R') \vDash \gamma$. Let $H = \{i \in I \mid a_i \in H'\}$. Notice that both H and $I \setminus H$ are Σ_1-definable in H' and A, so $H \in \mathrm{Cod}(M/I)$ by Lemma 2.2. Moreover, $H \subseteq_{\mathrm{cf}} D$.

To show that $(H, \leq, R) \vDash \gamma$, we show that the map $H' \ni a_i \mapsto i \in H$ induces an isomorphism of the structures $(H', \leq, R'; \mathcal{X} \cap \mathcal{P}(H'))$ and $(H, \leq, R; \mathrm{Cod}(M/I) \cap \mathcal{P}(H'))$. The fact that this map is an isomorphism between $(H', \leq, R')$ and $(H, \leq, R)$ follows directly from the definitions. Thus, we only need to argue that this map also induces an isomorphism of the second-order structures $\mathcal{X} \cap \mathcal{P}(H')$ and $\mathrm{Cod}(M/I) \cap \mathcal{P}(H')$. If $X' \in \mathcal{X}$ is a subset of H', then $\{i \in I \mid a_i \in X'\}$ is in $\mathrm{Cod}(M/I)$ by Lemma 2.2. If $X', Y' \in \mathcal{X}$ are distinct subsets of H', then $\{i \in I \mid a_i \in X'\}$ and $\{i \in I \mid a_i \in Y'\}$ are clearly distinct. Finally, if $X \in \mathrm{Cod}(M/I)$ is a subset of H, then $X' = \{a_i \mid i \in X\}$ is in $\mathcal{X}$ by Δ^0_1-comprehension, and it is a subset of H'.

Now suppose that $(I, \mathrm{Cod}(M/I)) \vDash \chi$. Let $R, D \in \mathcal{X}$ be such that $D \subseteq_{\mathrm{cf}} M$. By replacing D with an appropriate unbounded subset if necessary, we may assume w.l.o.g. that $D \cap (a_{i-1}, a_i]$ has at most one element for each $i \in I$. We now transfer R, D to $R', D' \subseteq I$ defined as follows:

$$i \in R' \Leftrightarrow \exists x \in (a_{i-1}, a_i] \cap R,$$
$$i \in D' \Leftrightarrow \exists x \in (a_{i-1}, a_i] \cap D.$$

By Lemma 2.2, $R', D' \in \mathrm{Cod}(M/I)$. Notice that $D' \subseteq_{\mathrm{cf}} I$, given that $D \subseteq_{\mathrm{cf}} M$.

Since $(I, \mathrm{Cod}(M/I)) \vDash \chi$, there exists $H' \subseteq_{\mathrm{cf}} D'$ such that $(H', \leq, R') \vDash \gamma$. Define

$$H = \{x \in D \mid \exists i \in H'\, (x \in (a_{i-1}, a_i])\}.$$

Clearly $H \in \mathcal{X}$ and $H \subseteq_{\mathrm{cf}} D$. To show that $(H, \leq, R) \vDash \gamma$, it remains to prove that the structures $(H', \leq, R'; \mathrm{Cod}(M/I) \cap \mathcal{P}(H'))$ and $(H, \leq, R; \mathcal{X} \cap \mathcal{P}(H))$ are isomorphic. The isomorphism is induced by the map that takes $i \in H'$ to the unique element of $H \cap (a_{i-1}, a_i]$. The verification that this is indeed an isomorphism is similar to the one in the proof of the other direction. □

Corollary 3.6. *Let χ be a* pSO *sentence and let $(M,\mathcal{X}) \vDash \mathsf{RCA}^*_0$. If $A \in \mathcal{X}$ is such that $(M, A) \vDash \neg \mathrm{I}\Sigma_1(A)$, then $(M,\mathcal{X}) \vDash \chi$ if and only if $(M, \Delta^0_1\text{-}\mathrm{Def}(M, A)) \vDash \chi$.*

Proof. The right-hand side of the equivalence in Theorem 3.5 does not depend on $\mathcal{X}$ as long as a given proper cut I is Σ^0_1-definable in (M, A). □

Theorem 3.5 and Corollary 3.6 make it possible to prove a very simple criterion of Π^1_1-conservativity over RCA^*_0 for pSO sentences. We state the criterion in slightly greater generality, for boolean combinations of pSO sentences, so as to be able to conclude that some specific pSO sentences have distinct sets of first-order consequences over RCA^*_0.

Theorem 3.7. *Let ψ be a boolean combination of* pSO *sentences. Then the following are equivalent:*

(i) $\mathsf{RCA}^*_0 + \psi$ *is Π^1_1-conservative over* RCA^*_0,
(ii) $\mathsf{RCA}^*_0 + \neg \mathrm{I}\Sigma^0_1 \vdash \psi$,
(iii) $\mathsf{WKL}^*_0 \vdash \psi$.

Moreover, if $\mathsf{WKL}_0 \nvdash \psi$*, then* $\mathsf{RCA}_0^* + \psi$ *is not arithmetically conservative over* RCA_0^**.*

Proof. The implication (iii) $\to$ (i) is immediate from [19].

Assume that (i) holds. Note that by Corollary 3.6, $\mathsf{RCA}_0^* + \psi$ proves the Π^1_1 statement "for every A, if $\mathrm{I}\Sigma_1(A)$ fails, then ψ is true in the $\Delta_1(A)$-computable sets". Thus, by (i), this statement is provable in RCA_0^*. However, again by Corollary 3.6, in each model of $\mathsf{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$ this Π^1_1 statement is equivalent to ψ. This proves that (i) implies (ii).

Now assume that (iii) fails, and let $(M, \mathcal{X})$ be a countable model of $\mathsf{WKL}_0^* + \neg\psi$. If $(M, \mathcal{X}) \models \neg\mathrm{I}\Sigma^0_1$, then clearly $\mathsf{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1 \nvdash \psi$. Otherwise, $(M, \mathcal{X})$ is a model of WKL_0, so by Proposition 2.3 there exists a structure $(K, \Delta_1\text{-Def}(K)) \models \mathsf{RCA}_0^*$ in which M is a proper Σ^0_1-cut and $\mathrm{Cod}(K/M) = \mathcal{X}$. By Theorem 3.5, we get $(K, \Delta_1\text{-Def}(K)) \models \neg\psi$. This proves that (ii) implies (iii).

Note also that if we do have a countable model $(M, \mathcal{X})$ of $\mathsf{WKL}_0 + \neg\psi$, then the structure $(K, \Delta_1\text{-Def}(K))$ constructed as in the previous paragraph satisfies RCA_0^* but does not satisfy the first-order statement "$\neg\mathrm{I}\Sigma_1$ implies that the computable sets satisfy ψ". This proves that if $\mathsf{WKL}_0 \nvdash \psi$, then $\mathsf{RCA}_0^* + \psi$ is not arithmetically conservative over RCA_0^*. □

Remark 3.8. The assumption of the "moreover" part of Theorem 3.7 could be weakened to $\mathsf{WKL}_0^* + \mathrm{supexp} \nvdash \psi$, using essentially the same proof. Whether the assumption could be weakened simply to (iii) is related to the question whether every sufficiently saturated countable model of WKL_0^* is Σ_1-definable in an end-extension satisfying $\mathrm{B}\Sigma_1 + \exp$. Cf. [13, Section 5].

In [7], it is shown that *every* Π^1_2 sentence is Π^1_1-conservative over $\mathsf{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$ if and only if it is provable from $\mathsf{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1$. Note, however, that the criterion provided by Theorem 3.7 applies to conservativity over RCA_0^*, without $\neg\mathrm{I}\Sigma^0_1$ in the base theory.

We now show that the general facts about pSO sentences proved above apply in particular to the Ramsey-theoretic principles we study.

Lemma 3.9. *Let* P *be one of the principles* RT^n_k*, for* $n, k \in \omega$*,* CAC*,* ADS*, and* CRT^2_2*. Then there exists a* pSO *sentence* χ *which is provably in* RCA_0^* *equivalent to* P*, both in the entire universe and on any proper* Σ^0_1*-cut.*

Proof. The proofs are similar for all the above principles P and rely on Lemma 3.2. We give a somewhat detailed argument for ADS and restrict ourselves to stating the appropriate χ for the other principles.

Let γ be the sentence

either R is not a linear order
or for every x, y it holds that $R(x, y)$ iff $x \leq y$
or for every x, y it holds that $R(x, y)$ iff $x \geq y$,

and let χ say that for every set R and every unbounded set D, there is $H \subseteq_{\mathrm{cf}} D$ such that $(H, \leq, R)$ satisfies γ. We claim that ADS is equivalent to χ provably in RCA_0^*. Clearly, if $\preceq$ is a linear order on $\mathbb{N}$, then χ applied with $D = \mathbb{N}$ and $R = \preceq$ implies the existence of a set H witnessing ADS for $\preceq$. Thus, χ implies ADS. In the other direction, given a relation R and an unbounded set D, either R is a linear order on D or not. In the latter case, $H = D$ witnesses χ. In the former, Lemma 3.2 lets us apply ADS to obtain either an ascending or a descending sequence in $R \cap D^2$, which witnesses χ. Thus, ADS implies χ.

The above argument also works in a structure of the form $(I, \mathrm{Cod}(M/I))$ for I a proper Σ^0_1-cut I in a model of RCA_0^*. To verify this one has to check that an analogue of Lemma 3.2 holds in $(I, \mathrm{Cod}(M/I))$, which is unproblematic.

For CAC, the corresponding pSO sentence χ says that for every set R and every unbounded set D there exists an unbounded $H \subseteq_{\mathrm{cf}} D$ such that $(H, \leq, R) \vDash \gamma$, where γ states that if R is a partial order, than it is a chain or antichain. For RT^n_k, the sentence γ states that if $R_1, \ldots, R_k$ form a colouring of unordered n-tuples, i.e. they are disjoint n-ary relations whose union is the set of all n-tuples that are strictly increasing with respect to $\leq$, then all but one of the relations R_j are in fact empty. For CRT^2_2, the appropriate γ says that the binary relation R is a stable colouring when restricted to the set of unordered pairs. □

Theorem 3.5, Corollary 3.6, and Lemma 3.9 immediately give:

Corollary 3.10. *Let* P *be one of:* RT^n_k*, for each* $n, k \in \omega$*,* CAC*,* ADS*, and* CRT^2_2*. Then for every* $(M, \mathcal{X}) \vDash \mathsf{RCA}^*_0$ *and each proper* Σ^0_1*-cut* I *of* M *it holds that* $(M, \mathcal{X}) \vDash \mathsf{P}$ *if and only if* $(I, \mathrm{Cod}(M/I)) \vDash \mathsf{P}$*. If* $A \in \mathcal{X}$ *is such that* $(M, A) \vDash \neg \mathrm{I}\Sigma_1(A)$*, then* $(M, \mathcal{X}) \vDash \mathsf{P}$ *if and only if* $(M, \Delta^0_1\text{-}\mathrm{Def}(M, A)) \vDash \mathsf{P}$*.*

For RT^n_k, the above result was shown in [11].

It follows from work of Towsner [22] that $\mathsf{WKL}_0 + \mathsf{CAC}$ does not prove RT^2_2 and $\mathsf{WKL}_0 + \mathsf{ADS}$ does not prove CAC. Therefore, none of the implications $\mathsf{RT}^2_2 \to \mathsf{CAC} \to \mathsf{ADS} \to \mathsf{CRT}^2_2 \to \top$ (where $\top$ is the constant True) available in RCA_0 can be reversed provably in WKL_0. It thus follows from Theorem 3.7 and Lemma 3.9 that all principles appearing in this sequence differ in strength over $\mathsf{RCA}^*_0 + \neg \mathrm{I}\Sigma^0_1$ and that they can even be distinguished by their first-order consequences over RCA^*_0.

Theorem 3.11. *Let* P *be one of the principles* $\mathsf{RT}^2_2, \mathsf{CAC}, \mathsf{ADS}, \mathsf{CRT}^2_2$*, and let* Q *be a principle to the right of* P *in this sequence or the constant* $\top$*. Then:*

(i) Q *does not imply* P *over* $\mathsf{RCA}^*_0 + \neg \mathrm{I}\Sigma^0_1$*,*
(ii) there is a first-order statement provable in $\mathsf{RCA}^*_0 + \mathsf{P}$ *but not in* $\mathsf{RCA}^*_0 + \mathsf{Q}$*,*
(iii) $\mathsf{RCA}^*_0 + \mathsf{P}$ *is not arithmetically conservative over* RCA^*_0*.*

Proof. By Lemma 3.9 we can treat $\mathsf{P} \to \mathsf{Q}$ as a boolean combination of pSO sentences. Since WKL_0 does not prove $\mathsf{Q} \to \mathsf{P}$, Theorem 3.7 gives (i) and additionally implies that there is an arithmetical sentence θ provable in $\mathsf{RCA}^*_0 + \mathsf{Q} \to \mathsf{P}$ but not in RCA^*_0. Then $\mathsf{RCA}^*_0 + \mathsf{P} \vdash \theta$ and $\mathsf{RCA}^*_0 + \neg\mathsf{Q} \vdash \theta$, so $\mathsf{RCA}^*_0 + \mathsf{Q} \nvdash \theta$, which proves (ii). Finally, (iii) is a special case of (ii). □

Together, Lemma 3.1 and Theorem 3.11 answer all questions about provability of implications between our principles in RCA^*_0 and $\mathsf{RCA}^*_0 + \neg \mathrm{I}\Sigma^0_1$ except the following.

Question 3.12. Does $\mathsf{RCA}^*_0 + \mathsf{ADS}$ or $\mathsf{RCA}^*_0 + \mathsf{CAC}$ prove CRT^2_2?

In the context of item (iii) of Theorem 3.11, note that CRT^2_2 is Π^1_1-conservative over RCA_0 [3], and while CAC and ADS are not Π^1_1-conservative over RCA_0 because they imply $\mathrm{B}\Sigma^0_2$, they are Π^1_1-conservative over $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$ [5].

We turn now to a more fine-grained analysis of conservativity issues. By [11], RT^n_k is $\forall\Pi^0_3$-conservative over RCA^*_0. *A fortiori*, all the weaker principles studied in this paper are also $\forall\Pi^0_3$-conservative over RCA^*_0. (We remark in passing that the techniques of [11] show that any pSO sentence that is true in some ω-model of WKL_0 is $\forall\Pi^0_3$-conservative over RCA^*_0.)

On the other hand, if P is one of RT^2_2, CAC, and ADS, then the statement "If $\mathrm{I}\Sigma_1$ fails, then any computable instance of P has a computable solution" is a Π_4 sentence of first-order arithmetic. So, essentially by Corollary 3.10, we get the following nonconservation result (proved in [11] for RT^2_2).

Corollary 3.13. *None of* RT^2_2*,* CAC*, and* ADS *is* Π_4*-conservative over* RCA^*_0*.*

Thus, we have tight bounds on the amount of conservativity of RT^2_2, CAC, and ADS over RCA^*_0. On the other hand, the sentence "If $\mathrm{I}\Sigma_1$ fails, then any computable instance of CRT^2_2 has a computable solution" is only Π_5. So, we get:

Corollary 3.14. CRT^2_2 *is* $\forall\Pi^0_3$*- but not* Π_5*-conservative over* RCA^*_0.

The following intriguing question remains open:

Question 3.15. Is $\mathsf{WKL}^*_0 + \mathsf{CRT}^2_2$ $\forall\Pi^0_4$-conservative over RCA^*_0?

3.3. **Closure properties.** To conclude our discussion of normal versions of combinatorial principles, we will show that over RCA^*_0 the principle CAC and all of its consequences are significantly weaker than RT^2_2 in a technical sense related to the closure properties of cuts.

Working in RCA^*_0, we write I^0_1 to denote the definable cut consisting of those numbers x such that each unbounded set $S \subseteq_{\mathrm{cf}} \mathbb{N}$ contains a finite subset of cardinality x. Note that by the correspondence between unbounded subsets of $\mathbb{N}$ and Σ^0_1-cuts stated in Proposition 2.1, I^0_1 is simply the intersection of all Σ^0_1-cuts. Thus, $\mathrm{I}^0_1 = \mathbb{N}$ exactly if $\mathrm{I}\Sigma^0_1$ holds.

It is easy to show in RCA^*_0 that I^0_1 is closed under multiplication: let S be an infinite set that does not contain a finite set of cardinality a^2 and compute a set S' by taking "every a-th element" of S. If S' is finite, then some infinite end-segment of S does not contain any finite subset of cardinality a. If S' is infinite, then S' itself is an infinite set with no subset of cardinality a, because otherwise we would find at least a^2 elements of S.

In [12, Section 3], it is shown that RT^2_2 implies a stronger closure property, namely that I^0_1 is closed under exponentiation. The proof of this result makes use of the well-known almost exponential lower bounds on finite Ramsey numbers for 2-colourings of pairs. The result has some interesting consequences, among them the fact that $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ has nonelementary proof speedup over RCA^*_0. (This was, in fact, the original motivation for studying connections between Ramsey-theoretic principles and closure properties of I^0_1.) Another consequence is that the theory $\mathsf{RCA}_0 + \mathsf{RT}^2_2 + \neg\mathrm{I}\Sigma_2$ does not prove that RT^2_2 holds in the family of Δ_2-definable sets [11]; this rules out a potential approach to separating the arithmetical consequences of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ from $\mathrm{B}\Sigma_2$.

Below, we show that ADS and CAC are weaker than RT^2_2 in this respect, as they do not imply the closure of I^0_1 under any superpolynomially growing function. Our argument will be a typical initial segment construction, resembling for instance the one in [14, Theorem 3.3], and it will once again make use of bounds on the finite version of the appropriate combinatorial principle. In this case, we will take advantage of the fact that, for $k \geq 2$, a partial order with $k^2 - k$ elements contains either a chain or an antichain of size at least k. Indeed, by Dilworth's theorem, if the largest antichain in a finite order has at most $k-1$ elements, then the order can be presented as the union of $k-1$ chains. If the order contains at least $k^2 - k$ elements, then one of those chains must have length at least k.

Theorem 3.16. *Let g be a Σ_1-definable function such that for every $k \in \omega$ there exists $n \in \omega$ such that* $\mathsf{RCA}^*_0 \vdash \forall x \geq n\, (g(x) \geq x^k)$*. Then neither* $\mathsf{WKL}^*_0 + \mathsf{CAC}$ *nor* $\mathsf{WKL}^*_0 + \mathsf{ADS}$ *proves that I^0_1 is closed under g.*

Proof. Of course, since CAC implies ADS over RCA^*_0, it is enough to show that $\mathsf{WKL}^*_0 + \mathsf{CAC}$ does not imply the closure of I^0_1 under any superpolynomially growing function g.

Let M be a countable nonstandard model of $\mathrm{I}\Delta_0 + \mathrm{supexp}$ and let $a \in M \setminus \omega$. We will construct a cut $I \subsetneq_e M$ in such a way that $(I, \mathrm{Cod}(M/I)) \vDash \mathsf{WKL}^*_0 + \mathsf{CAC}$ and

$\mathrm{I}^0_1(I, \mathrm{Cod}(M/I)) = \sup\{a^k \mid k \in \omega\}$. This will suffice to prove the theorem since it will hold that $a \in \mathrm{I}^0_1(I, \mathrm{Cod}(M/I))$ and for any superpolynomially growing function g, $g(a) \notin \mathrm{I}^0_1(I, \mathrm{Cod}(M/I))$.

Let $(S_n)_{n\in\omega}$ be an enumeration of all M-finite sets with cardinality below a^k for some $k \in \omega$, let $(c_n)_{n\in\omega}$ be an enumeration of all nonstandard elements of M and let $(\preceq_n)_{n\in\omega}$ be an enumeration of all M-coded partial orders with domain $[0, \exp_{a^a}(2)]$.

By induction on $n \in \omega$, we will construct a decreasing chain $F_0 \supseteq F_1 \supseteq F_2 \ldots$ of M-finite sets, maintaining the condition that for each n there is some $c \in M \setminus \omega$ such that $|F_n| \geq a^c$. Moreover, we will also make sure that for each $n \in \omega$, $|F_{3n}| \leq a^{c_n}$, that $[\min(F_{3n+1}), \max(F_{3n+1})] \cap S_n = \emptyset$, and that F_{3n+2} is either a chain or an antichain in the partial order $\preceq_n$.

We initialize the construction by setting $F_{-1} := \{1, 2, 4, 16, 2^{16}, \ldots, \exp_{a^a}(2)]$. In step $3n$, if $|F_{3n-1}| > a^{c_n}$, let $F_{3n} \subsetneq F_{3n-1}$ be such that $|F_{3n}| = a^{c_n}$ and $\min(F_{3n}) > \min(F_{3n-1})$. Otherwise, let $F_{3n} = F_{3n-1} \setminus \{\min(F_{3n-1})\}$.

In step $3n+1$, consider the set S_n. Let $k \in \omega$ be such that $|S_n| = a^k$. Assume w.l.o.g. (by taking a proper subset of F_{3n} if necessary) that F_{3n} has exactly a^c elements for some nonstandard $c \in M$, and let $(f_i)_{1\leq i\leq a^c}$ be the increasing enumeration of F_{3n}. Then F_{3n} can be split into a^{k+1} "intervals" as follows:

$$\{f_1, \ldots, f_{a^d}\} \cup \{f_{a^d+1}, \ldots, f_{2a^d}\} \cup \ldots \cup \{f_{(a^{k+1}-1)a^d+1}, \ldots, f_{a^{k+1}a^d}\}$$

where $d = c-k-1$. Since $a^{k+1} > a^k$, the pigeonhole principle implies that there is some $i_0 < a^{k+1}$ such that $[f_{i_0a^d+1}, f_{(i_0+1)a^d}] \cap S_n = \emptyset$. Let F_{3n+1} be the set $\{f_j \mid i_0a^d+1 \leq j \leq (i_0+1)a^d\}$. Notice that $|F_{3n+1}| \geq a^{c-k-1}$ and that $[\min(F_{3n+1}), \max(F_{3n+1})] \cap S_n = \emptyset$ as wanted.

In step $3n+2$, consider $\preceq_n \restriction F_{3n+1}$. By construction, $|F_{3n+2}| \geq a^c$ for some nonstandard c. Dilworth's theorem guarantees that there exists $C \subseteq F_{3n+2}$ such that $|C| \geq a^{c/2}$ and C is either a chain or an antichain in $\preceq_n$. Set $F_{3n+2} = C$.

Finally, let I be the initial segment $\sup\{\min(F_n) \mid n \in \omega\}$. We check that I satisfies the requirements of our construction.

Notice that $I \subsetneq_e M$, given that $\max(F_0) \in M \setminus I$. By the construction of step $3n$, I is a cut (that is, contains no greatest element) and for every $k \in \omega$, $F_k \cap I \subseteq_{\mathrm{cf}} I$. Moreover, $I \vDash \exp$ because $F_{-1} \cap I \subseteq_{\mathrm{cf}} I$, and if $x < y$ are two elements of F_{-1}, then $2^x \leq y$. Hence, $(I, \mathrm{Cod}(M/I)) \vDash \mathsf{WKL}^*_0$ by [19, Theorem 4.8].

If $\preceq$ is a partial order in $\mathrm{Cod}(M/I)$, then $\preceq = \preceq' \cap I$ for some M-finite set $\preceq'$. Note that there must exist $b \in M \setminus I$ such that $\preceq' \restriction [0, b]$ is a partial order; otherwise, I would be $\Delta_0(\preceq')$-definable as the set of $i \in M$ for which $\preceq' \restriction [0, i]$ is a poset. It is thus possible to extend $\preceq'$ to an order $\preceq''$ over M by making every element $\leq$-greater than b incomparable in $\preceq''$ with all other elements of M. The order $\preceq''$ is Δ_0-definable in M, so there exists $n \in \omega$ such that $\preceq'' \restriction [0, \exp_{a^a}(2)] = \preceq_n$. At step $3n+2$, we chose F_{3n+2} as a chain or antichain in $\preceq_n$. Thus, $F_{3n+2} \cap I \in \mathrm{Cod}(M/I)$ is either a chain or an antichain in $\preceq$, and moreover $F_{3n+2} \cap I \subseteq_{\mathrm{cf}} I$. So, $(I, \mathrm{Cod}(M/I)) \vDash \mathsf{CAC}$.

It remains to check that $\mathrm{I}^0_1(I, \mathrm{Cod}(M/I)) = \sup\{a^k \mid k \in \omega\}$. To prove the $\subseteq$ inclusion, let $c \in M$ be nonstandard and let $n \in \omega$ be such that $c = c_n$. Consider $F_{3n} \cap I \in \mathrm{Cod}(M/I)$, which is a cofinal subset of I. Since F_{3n} has at most a^c elements, then $F_{3n} \cap I$ has no finite subset of cardinality a^c. To prove the reverse inclusion, we have to show that for each $k \in \omega$ and each $U \in \mathrm{Cod}(M/I)$ such that $U \subseteq_{\mathrm{cf}} I$, there exists an M-finite set $V \subseteq U$ such that $|V| = a^k$. Let $U = U' \cap I$ for some M-finite set U'. Note that $|U'| \geq a^k$, because otherwise $|U'| = S_n$ for some $n \in \omega$ and the construction of step $3n+1$ guarantees that $[\min(F_{3n+1}), \max(F_{3n+1})] \cap U' = \emptyset$, so $U = U' \cap I \not\subseteq_{\mathrm{cf}} I$. So, let $V \subseteq U'$ be the set consisting of the first a^k elements of U'. Again, $V = S_n$

for some $n \in \omega$. The construction of step $3n+1$ guarantees that $V' \cap I \not\subseteq_{\mathrm{cf}} I$. This means that we must have $V \subseteq U$, because otherwise there would be a largest element of $V \cap U$, then an element $u \in U \setminus V$ above it, and then an element $v \in V \setminus U$ above u, contradicting the definition of V'. Therefore, V is an M-finite set of cardinality a^k contained in U. □

Remark 3.17. The technique used in the proof of Theorem 3.16 can also be used to show that $\mathsf{WKL}_0^* + \mathsf{RT}^n_k$ does not imply the closure of I^0_1 under any function of nonelementary growth rate. With a more careful choice of the initial model M, it can also be used to prove slight refinements of the theorem such as the $\forall\Pi^0_3$-conservativity of $\mathsf{WKL}_0^* + \mathsf{CAC} +$ "I^0_1 is not closed under any superpolynomially growing function" over RCA_0^*. We do not pursue this topic further in this paper.

Combining the techniques used above with the ones of [12, Section 3], one can show that over RCA_0^* the Erdös-Moser principle EM, or even a weakening that only requires the solution to be an unbounded set on which a given colouring is semitransitive, implies the closure of I^0_1 under exponentiation. This is because the lower bounds on general Ramsey numbers for pairs, along with the upper bounds on Ramsey numbers associated to orderings provided by Dilworth's theorem, imply that given $k \in \omega$, the smallest n such that any 2-colouring of pairs from $\{1, \ldots, n\}$ is semitransitive on a set of size k has size $2^{\Omega(\sqrt{k})}$.

On the other hand, it is quite unclear what closure properties of I^0_1, if any, are implied by CRT^2_2.

Question 3.18. Does CRT^2_2 imply that I^0_1 is closed under exp?

A positive answer to this question would give negative answers to Question 3.12 and Question 3.15. For the latter, notice that "I^0_1 is closed under exp"can be expressed by a $\forall\Pi^0_4$ sentence, and "I^0_1 in the computable sets is closed under exp" can even be expressed by a purely first-order Π_4 sentence.

One reason why it is not clear whether the techniques of Theorem 3.16 can be applied to CRT^2_2 is that this principle does not have an obvious "finite version" because of the relatively high quantifier complexity of its first-order part (what is a meaningful notion of "stable set" in the finite?). Answering Question 3.18 might require devising such a finite version of CRT^2_2 (or of SRT^2_2) and finding bounds on Ramsey numbers associated with it.

4. Long principles

We now focus our attention on the long versions of Ramsey-theoretic principles.

As in the case of normal versions, many implications with an easy proof in RCA_0 transfer to RCA_0^* with no particular difficulty. Additionally, as discussed in Section 2.1, it is straightforward to prove that ℓ-$\mathsf{ADS}^{\mathrm{set}}$ implies ℓ-$\mathsf{ADS}^{\mathrm{seq}}$. The following result summarizes the "easy" implications between our principles, as well as the non-implications known from RCA_0.

Lemma 4.1. *Over RCA_0^*, the following sequences of implications hold:*

$$\ell\text{-}\mathsf{RT}^2_2 \to \ell\text{-}\mathsf{CAC} \to \ell\text{-}\mathsf{ADS}^{\mathrm{set}} \to \ell\text{-}\mathsf{ADS}^{\mathrm{seq}},$$
$$\ell\text{-}\mathsf{RT}^2_2 \to \ell\text{-}\mathsf{CRT}^2_2.$$

None of the implications ℓ-$\mathsf{RT}^2_2 \to \ell$-$\mathsf{CAC} \to \ell$-$\mathsf{ADS}^{\mathrm{set}}$ and ℓ-$\mathsf{RT}^2_2 \to \ell$-CRT^2_2 can be provably reversed in RCA_0^*.

In the rest of this section, we describe some results obtained in an attempt to answer questions left open by Lemma 4.1. It will follow from these results (specifically from Theorem 4.2 and Theorem 4.8) that also the implication ℓ-$\mathsf{ADS}^{\mathrm{set}} \to \ell$-$\mathsf{ADS}^{\mathrm{seq}}$ cannot be provably reversed in RCA_0^*, and that ℓ-$\mathsf{ADS}^{\mathrm{set}}$ implies ℓ-CRT_2^2.

Perhaps more interestingly, it turns out that all of the long principles we consider behave in one of two contrasting ways. Some of them are like ℓ-RT_2^2, in that they are rather easily seen to imply Σ_1^0-induction. On the other hand, other long principles are partially conservative over RCA_0^*, which makes them closer to normal principles in a well-defined technical sense. We begin by discussing the former type of behaviour.

Theorem 4.2. *Over* RCA_0^*, *each of the principles* ℓ-RT_2^2, ℓ-CAC, ℓ-$\mathsf{ADS}^{\mathrm{set}}$ *implies* $\mathrm{I}\Sigma_1^0$.

Proof. The proof for ℓ-RT_2^2 was given by Yokoyama in [23], and it uses a transitive colouring, so essentially the same argument works for each of the principles listed above. We describe the argument for the weakest of these principles, namely ℓ-$\mathsf{ADS}^{\mathrm{set}}$.

Working in RCA_0^*, suppose that $\mathrm{I}\Sigma_1^0$ fails, and that an unbounded set A is enumerated in increasing order as $\{a_i \mid i \in I\}$ for I a proper Σ_1^0-cut. We define a linear order $\preceq$ on $\mathbb{N}$ in the following way:

$$x \preceq y \Leftrightarrow \begin{array}{l} \exists i \in I\,(x \in (a_{i-1}, a_i] \wedge y \in (a_{i-1}, a_i] \wedge x \geq y) \\ \vee\ \exists i, j \in I\,(i < j \wedge x \in (a_{i-1}, a_i] \wedge y \in (a_{j-1}, a_j]). \end{array}$$

That is, we invert the usual ordering $\leq$ on each interval $(a_{i-1}, a_i]$, but we compare elements from different intervals in the usual way. The order $\preceq$ is a set by $\Delta_1(A)$-comprehension.

If $S \subseteq \mathbb{N}$ is such that any two elements $x, y \in S$ satisfy $x \preceq y \leftrightarrow y \leq x$, then S has to be contained in an interval of the form $(a_{i-1}, a_i]$, so it is finite. On the other hand, if all $x, y \in S$ satisfy $x \preceq y \leftrightarrow x \leq y$, then S can contain at most one element from each $(a_{i-1}, a_i]$, so the cardinality of S is strictly less than $\mathbb{N}$. $\square$

Remark 4.3. Note that the ordering $\preceq$ used in the proof of Theorem 4.2 is *stable*, in the sense that for every x, there are only finitely many y such that $y \preceq x$. Thus, $\mathrm{I}\Sigma_1^0$ is implied already by what one could call "ℓ-$\mathsf{SADS}^{\mathrm{set}}$", the long, set-solution version of the stable ADS principle SADS from [9].

To show that the long versions of other principles are logically weak, we introduce an auxiliary statement, a version of the *grouping principle* GP_2^2 considered in [17]. The original grouping principle is a weakening of RT_2^2 stating that, for any 2-colouring of pairs and any notion of largeness of finite sets (suitably defined), there is an infinite sequence of large finite sets $G_0, G_1, \ldots$ (the *groups*) such that for each $i < j$ the colouring is constant on $G_i \times G_j$. We consider a weaker version tailored to RCA_0^*, in which the number of groups can be a proper cut, but the cardinality of individual groups should eventually exceed any finite number.

Definition 4.4. The *growing grouping principle* GGP_2^2 states that for every colouring $c\colon [\mathbb{N}]^2 \to 2$ there exists a sequence of finite sets $(G_i)_{i \in I}$ such that

(i) for every $i < j \in I$ and every $x \in G_i$, $y \in G_j$ it holds that $x < y$,
(ii) for every $i < j \in I$, the colouring $c \restriction (G_i \times G_j)$ is constant,
(iii) for every $i \in I$, $|G_i| \leq |G_{i+1}|$ and $\sup_{i \in I} |G_i| = \mathbb{N}$.

Note that $\mathsf{RCA}_0 + \mathsf{RT}_2^2 \vdash \mathsf{GGP}_2^2$. We prove a possibly surprising result on the behaviour of GGP_2^2 under $\neg\mathrm{I}\Sigma_1^0$.

Lemma 4.5. $\mathsf{WKL}_0^* + \neg\mathrm{I}\Sigma_1^0$ *implies* GGP_2^2. *Moreover,* GGP_2^2 *restricted to transitive colourings is provable in* $\mathsf{RCA}_0^* + \neg\mathrm{I}\Sigma_1^0$.

Remark 4.6. Lemma 4.5 implies in particular that GGP^2_2 is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$. In contrast, Yokoyama [private communication] has pointed out that GGP^2_2 is not arithmetically conservative over RCA_0. This can be seen as follows. It is shown in [17, Theorem 5.7 & Corollary 5.9] that RCA_0 extended by a statement $\mathsf{GP}(\mathsf{L}_\omega)$ intermediate between GGP^2_2 and GP^2_2 proves the principle known as 2-DNC and, as a consequence, an arithmetical statement $\mathrm{C}\Sigma_2$ unprovable in RCA_0. However, it is clear from the proof of [17, Theorem 5.7] that $\mathsf{RCA}_0 + \mathsf{GGP}^2_2$ is enough for the argument to go through.

Proof of Lemma 4.5. The proof uses the technique of building a grouping by thinning out a family of finite sets first "from below" and then "from above". This method was applied to construct large finite groupings in [14].

Work in $\mathsf{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$, and assume that $A = \{a_i \mid i \in I\}$ is an unbounded set, where I is a proper Σ^0_1-cut. By possibly thinning out A (which can only decrease I), we may also assume that for each $i \in I$, $a_0 > 2^i$ and $|(a_i, a_{i+1}]| \geq a_0 a_i 2^{a_i}$.

Let $c\colon [\mathbb{N}]^2 \to 2$. We want to obtain a sequence of sets $(G_i)_{i\in I}$ witnessing GGP^2_2 such that $G_i \subseteq (a_{i-1}, a_i]$ for each i. We proceed in two main stages.

(1) We stabilize the colour "from below". For each $i \in I$, build a finite sequence of finite sets $B^i_{-1} \supseteq B^i_0 \supseteq \ldots \supseteq B^i_{a_{i-1}}$ in the following way. Let $B^i_{-1} = (a_{i-1}, a_i]$, and for each $0 \leq x \leq a_{i-1}$ let $B^i_x = \{y \in B^i_{x-1} \mid c(x, y) = k\}$, where $k \in \{0, 1\}$ is such that $|\{y \in B^i_{x-1} \mid c(x, y) = k\}| \geq |\{y \in B^i_{x-1} \mid c(x, y) = 1 - k\}|$. We can choose for instance $k = 0$ if the two values are equal. Let $G'_i = B^i_{a_{i-1}}$.

At this point, for each $i \in I$ and each $x \leq a_{i-1}$ the colouring c is constant on $\{x\} \times G'_i$. Moreover, we have $G'_0 = [0, a_0]$ and $|G'_i| \geq a_0 a_i 2^{a_i - a_{i-1} - 1} \geq a_0 a_i$ for each $0 < i \in I$. Note that the sequence $(G'_i)_{i\in I}$ is $\Delta_1(A)$-definable.

(2) We stabilize the colour "from above". For each $i \in I$, we can construct an infinite sequence of finite sets $G'_i = D^i_i \supseteq D^i_{i+1} \supseteq D^i_{i+2} \supseteq \ldots$ indexed by $i \leq j \in I$, with a single step of the construction essentially like in stage (1). That is, given $j > i$, we let D^i_j be $|\{x \in D^i_{j-1} \mid c(x, \min G'_j) = k\}|$ for that k for which this set is larger. We only need to compare each $x \in D^i_{j-1}$ with one element of G'_j, because we have already arranged for c to be constant on $\{x\} \times G'_j$. Note that for each $i < j \in I$ the colouring c is constant on $D^i_j \times G'_j$. Also, each D^0_j is nonempty, while for $0 < i \leq j \in I$ we have $|D^i_j| \geq a_0 a_i 2^{-j} \geq 2a_i$.

Intuitively, we would want to define $G_i = \bigcap_{i \leq j \in I} D^i_j$, but then being a member of G_i might not be Δ^0_1-definable. However, if we fix $m \in I$ and consider only the sets $\bigcap^m_{j=i} D^i_j$ for $i \leq m$, we obtain a node of length $a_m + 1$ in the computable binary tree T defined as follows. A finite 0-1 sequence τ belongs to T if the largest m such that $\mathrm{lh}(\tau) > a_m$ satisfies (if we identify τ with the finite set it codes):

(i) $\tau \cap [0, a_m] \subseteq \bigcup^m_{i=0} G'_i$,
(ii) for every $i < j \leq m$, the colouring c is constant on $(G'_i \cap \tau) \times (G'_j \cap \tau)$,
(iii) $|\tau \cap G'_i| > a_i$ for every $i \leq m$.

The tree T is infinite because for arbitrary $m \in I$ there exists a node in T of length a_m, and the set $A = \{a_i \mid i \in I\}$ is unbounded. By WKL T has an infinite path G and we get the desired grouping $(G_i)_{i\in I}$ by taking $G_i = G \cap (a_{i-1}, a_i]$.

Now assume additionally that c is a transitive colouring. By the argument from the proof of Proposition 3.3, we can think of c as given by a linear ordering $\preceq$. The first stage of the construction, "from below", is exactly as before. In the "from above" stage, we will make a small change. If we built the sets D^i_j for c as in the previous construction, then, in terms of $\preceq$, we would look at the position of $\min G'_j$ in the $\preceq$-ordering relative to the elements of D^i_{j-1}. This would split D^i_{j-1} into a "top part" and a "bottom part" with respect to $\preceq$, and we would take whichever of these two parts were larger. Now, we will take the $\preceq$-bottom *half* of D^i_{j-1} if $\min G'_j$ lies above it, and the top $\preceq$-*half* if it does not. (Do this in a way that includes the $\preceq$-midpoint in case $|D^i_{j-1}|$ is odd, so that $|D^i_j|$ is exactly $\lceil |D^i_{j-1}|/2 \rceil$.)

By Lemma 2.2, there is an element $s > I$ coding the set of those pairs $\langle i, j\rangle$ with $i < j \in I$ for which D^i_j is the $\preceq$-top half of D^i_{j-1}. We can think of s as a subset of $[0, b] \times [0, b]$ for some $b < \log a_0$. We can use s to generalize the new definition of D^i_j to $i \in I$ and $j \in [i, b]$: D^i_j is the $\preceq$-top half of D^i_{j-1} if $\langle i, j\rangle \in s$, and the $\preceq$-bottom half otherwise. Let $G_i = \bigcap_{j=i}^{b} D^i_j$. It is easy to check that $(G_i)_{i\in I}$ is Δ^0_1-definable and that it witnesses GGP^2_2 for $\preceq$. □

Remark 4.7. Note that the reason why the proof of GGP^2_2 for transitive colourings does not obviously generalize to arbitrary ones is that in general, if $i \in I < j$, then it is not clear how to split a subset of $(a_{i-1}, a_i]$ into a "more red" and a "more blue" half with respect to a (nonexistent) element of G'_j. If the colouring is transitive and given by an ordering $\preceq$, then even though we cannot actually compare the elements of $(a_{i-1}, a_i]$ to a nonexistent element, we can say which ones form the top and bottom half.

Theorem 4.8. RCA^*_0 *proves* $\ell\text{-}\mathsf{ADS}^{\mathrm{seq}} \leftrightarrow \mathsf{ADS}$, *and* WKL^*_0 *proves* $\ell\text{-}\mathsf{CRT}^2_2 \leftrightarrow \mathsf{CRT}^2_2$.

Proof. Let us first consider the case of ADS. Clearly, $\ell\text{-}\mathsf{ADS}^{\mathrm{seq}}$ implies ADS, and the two principles are equivalent over RCA_0. So, we only need to prove $\ell\text{-}\mathsf{ADS}^{\mathrm{seq}}$ from ADS working in $\mathsf{RCA}^*_0 + \neg \mathrm{I}\Sigma^0_1$.

Let $(\mathbb{N}, \preceq)$ be an instance of $\ell\text{-}\mathsf{ADS}^{\mathrm{seq}}$. By Lemma 4.5, we can apply GGP^2_2 to the colouring given by $\preceq$, obtaining a sequence of finite sets $G_0 < G_1 < \ldots < G_i < \ldots$, where $i \in I$ for some Σ^0_1-cut I. By Lemma 3.2, we can apply ADS to the order $\preceq$ restricted to the set $A = \{\min(G_i) \mid i \in I\}$. Without loss of generality, assume that this gives us an unbounded set $S \subseteq A$ such that for any $x, y \in S$, $x \preceq y$ iff $x \geq y$. Assume $S = \{\min(G_{i_j}) \mid j \in J\}$ for some cut $J \subseteq I$. Now consider the descending sequence in $\preceq$ defined as follows: first list the elements of G_{i_0} in $\preceq$-descending order, then the elements of G_{i_1} in $\preceq$-descending order, and so on. This sequence can be obtained using $\Delta_1(S, \preceq)$-comprehension, and it has length $\mathbb{N}$, because $S \subseteq_{\mathrm{cf}} A \subseteq_{\mathrm{cf}} \mathbb{N}$, so $\sup_{j\in J} |G_{i_j}| = \sup_{i\in I} |G_i| = \mathbb{N}$.

A similar argument shows that $\mathsf{RCA}^*_0 + \mathsf{GGP}^2_2$ proves $\mathsf{CRT}^2_2 \to \ell\text{-}\mathsf{CRT}^2_2$. However, the instance to which we apply GGP^2_2 in that argument is not necessarily transitive, so Lemma 4.5 only implies $\mathsf{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1 \vdash \mathsf{CRT}^2_2 \to \ell\text{-}\mathsf{CRT}^2_2$ and thus $\mathsf{WKL}^*_0 \vdash \mathsf{CRT}^2_2 \leftrightarrow \ell\text{-}\mathsf{CRT}^2_2$. □

Remark 4.9. There is version of ADS, called ADC in [1], in which the solution is an infinite set S such that either each element of S has only finitely many predecessors or each element of S has only finitely many successors. This principle is known to be equivalent to ADS in RCA_0, but strictly weaker in terms of Weihrauch reducibility. It is not hard to verify using the techniques of Section 3 that the normal version of ADC is provably in RCA^*_0 equivalent to ADS. Moreover, a slight modification of the previous proof shows that the long version of ADC is also equivalent to ADS.

Theorem 4.8 allows us to show that $\ell\text{-}\mathsf{ADS}^{\mathrm{seq}}$ and $\ell\text{-}\mathsf{CRT}^2_2$ are weak principles in the sense that they are partially conservative over RCA^*_0.

Corollary 4.10. *Both* $\mathsf{WKL}^*_0 + \ell\text{-}\mathsf{ADS}^{\mathrm{seq}}$ *and* $\mathsf{WKL}^*_0 + \ell\text{-}\mathsf{CRT}^2_2$ *follow from* $\mathsf{WKL}^*_0 + \mathsf{RT}^2_2$. *As a consequence, these theories are* $\forall\Pi^0_3$*-conservative over* RCA^*_0 *and do not imply* $\mathrm{I}\Sigma^0_1$.

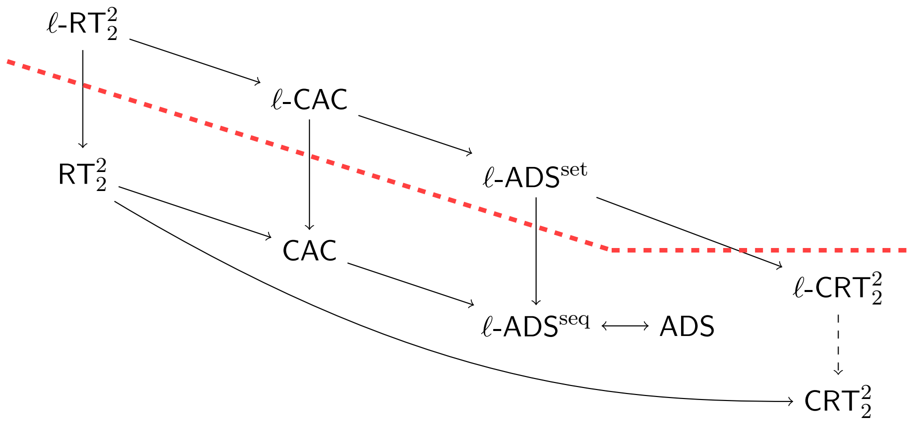

Figure 1. Summary of relations between the various versions of RT^2_2, CAC, ADS and CRT^2_2 over RCA^*_0. Solid arrows represent implications provable in RCA^*_0 that do not provably reverse in RCA^*_0. The dashed arrow represents an implication for which the reversal is open. Also the implications from CAC and ADS to CRT^2_2 and from any of $\mathsf{RT}^2_2, \mathsf{CAC}, \mathsf{ADS}$ to ℓ-CRT^2_2 are open. All indicated theories above the thick dashed line imply $\mathrm{I}\Sigma^0_1$, and all indicated theories below the line are $\forall\Pi^0_3$-conservative over RCA^*_0.

Proof. It is immediate from Theorem 4.8 and Lemma 3.1 that both $\mathsf{WKL}^*_0 + \ell$-$\mathsf{ADS}^{\mathrm{seq}}$ and $\mathsf{WKL}^*_0 + \ell$-CRT^2_2 follow from $\mathsf{WKL}^*_0 + \mathsf{RT}^2_2$.

To prove the $\forall\Pi^0_3$-conservativity of $\mathsf{WKL}^*_0 + \mathsf{RT}^2_2$ over RCA^*_0, note that the proof of $\forall\Pi^0_3$-conservativity of $\mathsf{RCA}^*_0 + \mathsf{RT}^n_k$ over RCA^*_0 in [11] in fact shows that any Σ^0_3 sentence consistent with RCA^*_0 is satisfied in some model of $\mathsf{RCA}^*_0 + \mathsf{RT}^n_k$ of the form $(I, \mathrm{Cod}(M/I))$ for I a proper Σ^0_1-cut in a model $M \vDash \mathrm{I}\Delta^0_1 + \exp$. By [19, Theorem 4.8], any such model $(I, \mathrm{Cod}(M/I))$ satisfies WKL^*_0 as well.

Of course, each theory that is at least Π_1-conservative over RCA^*_0 is consistent with $\neg\mathrm{Con}(\mathrm{I}\Delta_0)$ and thus cannot imply even $\mathrm{I}\Delta_0 + \mathrm{supexp}$, where supexp expresses the totality of the iterated exponential function. □

Our results from Sections 3 and 4 on the relationships between the normal and long versions of $\mathsf{RT}^2_2, \mathsf{CAC}, \mathsf{ADS}$, and CRT^2_2 are summarized in Figure 1. One phenomenon apparent from the figure is that all of the principles considered up to this point either imply $\mathrm{I}\Sigma^0_1$ or are $\forall\Pi^0_3$-conservative over RCA^*_0.

The main open problems related to normal versions of our principles concern CRT^2_2 and have already been stated in Section 3. Among the long principles, questions about those that imply $\mathrm{I}\Sigma^0_1$ move us back to the traditional realm of reverse mathematics over RCA_0. As for the weaker long principles, an important matter is to settle the status of GGP^2_2.

Question 4.11. Does $\mathsf{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ imply GGP^2_2? Is GGP^2_2 equivalent to WKL^*_0 over $\mathsf{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$?

A more specialized but related group of problems concerns ℓ-CRT^2_2.

Question 4.12. Is ℓ-CRT^2_2 equivalent to CRT^2_2 over RCA^*_0? Does it follow from $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$?

By the argument used to prove Theorem 4.8, if GGP^2_2 is provable in $\mathsf{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$, then both parts of Question 4.12 have a positive answer.

In the context of Question 4.11, we mention a potentially interesting connection between GGP^2_2 and the long version of the Erdös-Moser principle: over $\mathsf{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$, ℓ-EM is equivalent to $\mathsf{EM} \wedge \mathsf{GGP}^2_2$. This equivalence implies in particular that ℓ-EM does not prove $\mathrm{I}\Sigma^0_1$ and is in fact $\forall\Pi^0_3$-conservative over RCA^*_0. However, we do not know if the $\forall\Pi^0_3$-conservative long principles considered earlier also imply GGP^2_2.

To prove the equivalence, note that, on the one hand, an argument like the one in Theorem 4.8 proves ℓ-EM in $\mathsf{RCA}^*_0 + \mathsf{GGP}^2_2 + \mathsf{EM}$. Given $c\colon [\mathbb{N}]^2 \to 2$, we can use GGP^2_2 to obtain $(G_i)_{i\in I}$ such that $c \restriction (G_i \times G_j)$ is constant for each $i < j \in I$, thin out each G_i at most exponentially to obtain G'_i on which c is constant, and then apply EM to $c \restriction \{\min(G'_i \mid i \in I\}$ in order to find $S = \{\min(G'_{i_j} \mid j \in J\}$ on which c is transitive. Then $\bigcup_{j\in J} G'_{i_j}$ is a set of cardinality $\mathbb{N}$ on which c is transitive. On the other hand, $\mathsf{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1 + \ell$-$\mathsf{EM}$ implies GGP^2_2. Given a colouring $c\colon [\mathbb{N}]^2 \to 2$, we can apply ℓ-EM to obtain a set S of cardinality $\mathbb{N}$ such that c is transitive on S. Then Lemma 4.5 applied to $c \restriction [S]^2$ provides a solution to GGP^2_2.

5. The curious case of COH

In the final section of the paper, we consider the behaviour over RCA^*_0 of the cohesion principle COH. Recall that a set $C \subseteq \mathbb{N}$ is *cohesive* for a sequence $(R_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$ if, for each $n \in N$, either all but finitely many elements of C belong to R_n or all but finitely many elements of C belong to $\mathbb{N} \setminus R_n$. We write $C \subseteq^* R_n$ in the former case and $C \subseteq^* \overline{R_n}$ in the latter.

COH **:** *For each sequence $(R_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$, there exists an unbounded set C which is cohesive for $(R_n)_{n\in\mathbb{N}}$.*

ℓ-COH**:** *For each sequence $(R_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$, there exists a set C of cardinality $\mathbb{N}$ which is cohesive for $(R_n)_{n\in\mathbb{N}}$.*

Belanger [2] asked whether COH is Π^1_1-conservative over RCA^*_0. A negative answer to this question follows from the results of Section 3. This is because COH implies CRT^2_2, and the implication remains provable in RCA^*_0: for a colouring $c\colon [\mathbb{N}]^2 \to 2$, any set C that is cohesive for the sequence $(\{y \mid c(n,y) = 1\})_{n\in\mathbb{N}}$ is also stable for c. Thus, Corollary 3.14 immediately implies the following result.

Corollary 5.1. $\mathsf{RCA}^*_0 + \mathsf{COH}$ *is not Π_5-conservative over* RCA^*_0.

Of course, ℓ-COH implies ℓ-CRT^2_2 over RCA^*_0 in an analogous way. Below we focus on COH, as we have no results to report on ℓ-COH beyond immediate consequences of the easy implications from ℓ-COH to COH and to ℓ-CRT^2_2.

In terms of our classification of Ramsey-theoretic statements into normal and long principles, COH has some aspects of both. On the one hand, the solution C is only required to be unbounded but not to have cardinality $\mathbb{N}$. On the other hand, C is required to behave in a certain way with respect to *each* element of the sequence $(R_n)_{n\in\mathbb{N}}$, which obviously has length $\mathbb{N}$. We will show that the latter feature of COH has an interesting consequence: the well-known implication from RT^2_2 to COH [3][1] is not provable over RCA^*_0. *A fortiori*, this means that neither the implications from CAC and ADS to COH

[1]The proof of $\mathsf{RT}^2_2 \to \mathsf{COH}$ given in [3] actually requires $\mathrm{I}\Sigma^0_2$ but Mileti [16] gave another proof which goes through in RCA_0.

known to hold over RCA_0 nor the equivalence between COH and CRT^2_2 known to hold over $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$ [9] are provable in RCA^*_0.

To prove that the implication $\mathsf{RT}^2_2 \to \mathsf{COH}$ breaks down over RCA^*_0, we will show that, in contrast to all the "normal" Ramsey-theoretic principles considered in Section 3, COH is never computably true, i.e. it never holds in a model of the form $(M, \Delta_1\text{-Def}(M))$. We will prove this by means of a detour through what is called the Σ^0_2-separation principle in [2].

Σ^0_2**-separation:** *For every two disjoint Σ^0_2-sets A_0, A_1 there exists a Δ^0_2-set B such that $A_0 \subseteq B$ and $A_1 \subseteq \overline{B}$.*

It was shown in [2] that COH is equivalent to Σ^0_2-separation over $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$ and that the implication from COH to Σ^0_2-separation works over RCA_0. Below, we verify that this implication remains valid over RCA^*_0. On the other hand, we show that $\mathrm{B}\Sigma_1 + \exp$ is enough to prove the existence of two disjoint lightface Σ_2-sets that cannot be separated by a Δ_2-set. That is the same thing as saying that in any structure $M \vDash \mathrm{B}\Sigma_1 + \exp$, the second-orded universe consisting exclusively of the Δ_1-definable sets satisfies the negation of the Σ^0_2-separation principle and hence also $\neg\mathsf{COH}$.

Lemma 5.2. RCA^*_0 *proves that* COH *implies* Σ^0_2-separation.

Proof. We will follow the structure of the proof in RCA_0 described in [2] (which is based on [10]), pointing out where we have to depart from it. We work in $\mathsf{RCA}^*_0 + \mathsf{COH}$ and prove the dual formulation of Σ^0_2-separation: if A_0 and A_1 are Π^0_2 sets such that $A_0 \cup A_1 = \mathbb{N}$, then there exists a Δ^0_2-set B such that $B \subseteq A_0$ and $\overline{B} \subseteq A_1$.

Assume that:

$$A_0 = \{x \mid \forall y\, \exists z\, \theta_0(x, y, z)\},$$
$$A_1 = \{x \mid \forall y\, \exists z\, \theta_1(x, y, z)\},$$

where θ_0, θ_1 are Δ^0_0, and for each $n \in \mathbb{N}$ it holds that $n \in A_0$ or $n \in A_1$.

The argument in RCA_0 would now make use of a Δ^0_1-definable function $f \colon \mathbb{N} \times \mathbb{N} \to 2$ such that for every n,

$$\{s \mid f(n, s) = i\} \text{ is infinite iff } n \in A_i.$$

It seems unclear whether we can have access to such a function in RCA^*_0. However, we can use a witness comparison argument to find a Δ^0_1-definable $f \colon \mathbb{N} \times \mathbb{N} \to 2$ such that for every n,

$$\text{if } \{s \mid f(n, s) = i\} \text{ is infinite, then } n \in A_i.$$

Namely, for every n at least one of $\forall y\, \exists z\, \theta_0(n, y, z)$ and $\forall y\, \exists z\, \theta_1(n, y, z)$ holds. So, by $\mathrm{B}\Sigma^0_1$, for every n and s there must exist some w_0 such that $\forall y \le s\, \exists z \le w_0\, \theta_0(n, y, z)$ or some w_1 such that $\forall y \le s\, \exists z \le w_1\, \theta_1(n, y, z)$. Define $f(n, s) = 0$ if the smallest such w_0 is at most equal to the smallest such w_1, and $f(n, s) = 1$ otherwise.

Now consider the Δ^0_1-definable sequence of sets $(R_n)_{n\in\mathbb{N}}$ where $R_n = \{s \mid f(n, s) = 0\}$. Let C be a cohesive set for this sequence. Notice that if $C \subseteq^* R_n$, then R_n is infinite and hence $n \in A_0$, and analogously if $C \subseteq^* \overline{R}_n$ then $n \in A_1$.

Let

$$B = \{n \mid \exists k\, \forall \ell \ge k\, (\ell \in C \to \ell \in R_n)\}.$$

Since C is cohesive for $(R_n)_{n\in\mathbb{N}}$, both B and its complement are Σ^0_2-definable. Moreover, it follows from the construction that if $n \in B$ then $n \in A_0$ and if $n \notin B$ then $n \in A_1$. □

Lemma 5.3. $\mathrm{B}\Sigma_1 + \exp$ *proves that there exist two disjoint Σ_2-sets that cannot be separated by a Δ_2-set.*

Proof. We verify that an essentially standard proof of the existence of Δ_2-inseparable disjoint Σ_2-sets goes through in $\mathrm{B}\Sigma_1 + \exp$. The recursion-theoretic facts and notions needed for the proof to work were formalized within $\mathrm{B}\Sigma_1 + \exp$ in [6].

A *Turing functional* Φ is a Σ_1-set of tuples $\langle x, y, P, N\rangle$, where $x, y \in \mathbb{N}$ and P, N are disjoint finite sets. Turing functionals are constrained to be well-defined in the sense that for fixed x, P, N there is at most one y such that $\langle x, y, P, N\rangle \in \Phi$, and to be monotone in the sense that increasing P or N preserves membership in Φ. Given a Turing functional Φ, we say that $\Phi^{0'}(x) = y$ if there exist $P \subseteq 0'$ and $N \subseteq \overline{0'}$ such that $\langle x, y, P, N\rangle \in \Phi$.

Work in $\mathrm{B}\Sigma_1 + \exp$, and let $(\Phi_e)_{e\in\mathbb{N}}$ be an effective listing of all Turing functionals. Let A_0 be the Σ_2-set $\{e \in \mathbb{N} : \Phi_e^{0'}(e) = 0\}$, and let A_1 be the Σ_2-set $\{e \in \mathbb{N} : \Phi_e^{0'}(e) = 1\}$. Clearly, A_0 and A_1 are disjoint. We claim that they cannot be separated by a Δ_2-set.

Suppose that B is a Δ_2-set such that $A_0 \subseteq B$ and $A_1 \subseteq \overline{B}$. By [6, Corollary 3.1], provably in $\mathrm{B}\Sigma_1 + \exp$ the Δ_2-set B is *weakly recursive* in $0'$ in the following sense: there is some Turing functional Φ_{e_0} such that for every x, if $x \in B$ then $\Phi_{e_0}^{0'}(x) = 1$, and if $x \notin B$ then $\Phi_{e_0}^{0'}(x) = 0$. By the definition of A_0 and A_1, this implies that $\Phi_{e_0}^{0'}(e_0) = 0$ iff $\Phi_{e_0}^{0'}(e_0) = 1$, which is a contradiction because $\Phi_{e_0}^{0'}$ is defined on every input and takes 0/1 values. $\square$

Theorem 5.4. *Any model $(M, \Delta_1^0\text{-}\mathrm{Def}(M, A)) \vDash \mathsf{RCA}_0^*$, where $A \subseteq M$, satisfies $\neg\mathsf{COH}$.*

Proof. This is an immediate consequence of Lemma 5.2 and Lemma 5.3 relativized to A. Lemma 5.2 says that if the structure $(M, \Delta_1^0\text{-}\mathrm{Def}(M, A))$ satisfied COH, then it would also satisfy the Σ_2^0-separation principle. The latter would contradict Lemma 5.3, because in $(M, \Delta_1^0\text{-}\mathrm{Def}(M, A))$ the Σ_2^0-sets are exactly the $\Sigma_2(A)$-definable sets and the Δ_2^0-sets are exactly the $\Delta_2(A)$-definable sets. $\square$

Corollary 5.5. $\mathsf{RCA}_0^* + \mathsf{RT}_2^2$ *does not imply* COH.

Proof. By Theorem 5.4, it is enough to note that there exists a model of $\mathsf{RCA}_0^* + \mathsf{RT}_2^2$ of the form $(M, \Delta_1^0\text{-}\mathrm{Def}(M, A))$ for some $A \subseteq M$. The existence of such a model follows from the existence of a model of $\mathsf{RCA}_0^* + \mathsf{RT}_2^2 + \neg\mathrm{I}\Sigma_1^0$ [11] and Corollary 3.6. $\square$

Corollary 5.6. RT_2^2, CAC, *and* ADS *are incomparable with* COH *with respect to implications over* RCA_0^*.

Another consequence of Theorem 5.4 is that an analogue of Theorem 3.5 does not hold for COH. In particular, it is not true that if $(M, \mathcal{X}) \vDash \mathsf{RCA}_0^*$ and $(I, \mathrm{Cod}(M/I)) \vDash \mathsf{COH}$ for some Σ_1^0-cut I of M, then $(M, \mathcal{X}) \vDash \mathsf{COH}$, since in a model of $\neg\mathrm{I}\Sigma_1$ this would work in particular for $\mathcal{X} = \Delta_1\text{-}\mathrm{Def}(M)$. On the other hand, using methods in the style of Section 3 it is easy to show that the converse implication still holds.

Proposition 5.7. *For every $(M, \mathcal{X}) \vDash \mathsf{RCA}_0^*$ and every proper Σ_1^0-cut I in $(M, \mathcal{X})$, if $(M, \mathcal{X}) \vDash \mathsf{COH}$, then $(I, \mathrm{Cod}(M/I)) \vDash \mathsf{COH}$.*

Proof. Suppose $(M, \mathcal{X}) \vDash \mathsf{RCA}_0^* + \mathsf{COH}$ and I is a proper Σ_1^0-cut in $(M, \mathcal{X})$. Let $A \in \mathcal{X}$ be a cofinal subset of M enumerated as $A = \{a_i \mid i \in I\}$, as in Proposition 2.1.

Let $(R_i)_{i\in I}$ be a sequence of subsets of I that belongs to $\mathrm{Cod}(M/I)$. Define a sequence $(R'_n)_{n\in M}$ in the following way. If $n \in (a_{i-1}, a_i]$ for some $i \in I$, let

$$R'_n = \{x \in M \mid \exists j \in I\, (x \in (a_j, a_{j+1}]) \wedge j \in R_i)\}.$$

The sequence $(R'_n)_{n\in M}$ is Δ_1-definable in A and the code for $(R_i)_{i\in I}$, so it belongs to $\mathcal{X}$. By COH in $(M, \mathcal{X})$, there exists $C' \in \mathcal{X}$ such that $C' \subseteq_{\mathrm{cf}} M$ and C' is cohesive for $(R'_n)_{n\in M}$. Define $C = \{i \in I \mid C' \cap (a_i, a_{i+1}] \neq \emptyset\}$. Both C and $I \setminus C$ are Σ_1-definable in

C' and A, so $C \in \mathrm{Cod}(M/I)$ by Lemma 2.2. Moreover, $C \subseteq_{\mathrm{cf}} I$ and it is easy to check that C is cohesive for $(R_i)_{i \in I}$. □

Results such as Theorem 5.4 and Corollary 5.5 provide some new information about COH, but the strength of this principle in RCA_0^* is still to a large extent mysterious. Some rather basic problems remain open.

Question 5.8. Does COH, or at least ℓ-COH, imply $\mathrm{I}\Sigma^0_1$ over RCA_0^*? Is ℓ-COH, or at least COH, $\forall\Pi^0_3$-conservative over RCA_0^*?

Question 5.9. Does RCA_0^*, or at least WKL_0^*, prove $\mathsf{COH} \leftrightarrow \ell$-$\mathsf{COH}$?

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