Sequent calculus

Sequents have the form $\Gamma \longrightarrow \Delta$, where Γ, Δ are finite sequences of formulas. The intuitive meaning of a sequent $\Gamma \longrightarrow \Delta$ is: "the conjunction of all formulas in Γ implies the disjunction of all formulas in Δ".

Axiom

$$\frac{}{\varphi \longrightarrow \varphi}\ \text{(axiom)}$$

Structural rules

$$\frac{\Gamma_1, \varphi, \psi, \Gamma_2 \longrightarrow \Delta}{\Gamma_1, \psi, \varphi, \Gamma_2 \longrightarrow \Delta}\ \text{(exchange:L)} \qquad \frac{\Gamma \longrightarrow \Delta_1, \varphi, \psi, \Delta_2}{\Gamma \longrightarrow \Delta_1, \psi, \varphi, \Delta_2}\ \text{(exchange:R)}$$

$$\frac{\Gamma, \varphi, \varphi \longrightarrow \Delta}{\Gamma, \varphi \longrightarrow \Delta}\ \text{(contraction:L)} \qquad \frac{\Gamma \longrightarrow \Delta, \varphi, \varphi}{\Gamma \longrightarrow \Delta, \varphi}\ \text{(contraction:R)}$$

$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \varphi \longrightarrow \Delta}\ \text{(weakening:L)} \qquad \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \varphi}\ \text{(weakening:R)}$$

Rules for connectives

$$\frac{\Gamma, \varphi, \psi \longrightarrow \Delta}{\Gamma, \varphi \wedge \psi \longrightarrow \Delta}\ (\wedge\text{:L}) \qquad \frac{\Gamma \longrightarrow \Delta, \varphi \qquad \Gamma \longrightarrow \Delta, \psi}{\Gamma \longrightarrow \Delta, \varphi \wedge \psi}\ (\wedge\text{:R})$$

$$\frac{\Gamma, \varphi \longrightarrow \Delta \qquad \Gamma, \psi \longrightarrow \Delta}{\Gamma, \varphi \vee \psi \longrightarrow \Delta}\ (\vee\text{:L}) \qquad \frac{\Gamma \longrightarrow \Delta, \varphi, \psi}{\Gamma \longrightarrow \Delta, \varphi \vee \psi}\ (\vee\text{:R})$$

$$\frac{\Gamma \longrightarrow \Delta, \varphi}{\Gamma, \neg\varphi \longrightarrow \Delta}\ (\neg\text{:L}) \qquad \frac{\Gamma, \varphi \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \neg\varphi}\ (\neg\text{:R})$$

$$\frac{\Gamma \longrightarrow \Delta, \varphi \qquad \Gamma, \psi \longrightarrow \Delta}{\Gamma, \varphi \Rightarrow \psi \longrightarrow \Delta}\ (\Rightarrow\text{:L}) \qquad \frac{\Gamma, \varphi \longrightarrow \Delta, \psi}{\Gamma \longrightarrow \Delta, \varphi \Rightarrow \psi}\ (\Rightarrow\text{:R})$$

The cut rule

$$\frac{\Gamma, \varphi \longrightarrow \Delta \qquad \Gamma \longrightarrow \Delta, \varphi}{\Gamma \longrightarrow \Delta}\ \text{(cut)}$$

Rules for quantifiers

$$\frac{\Gamma, \varphi(t/x) \longrightarrow \Delta}{\Gamma, \forall x\, \varphi \longrightarrow \Delta}\ (\forall{:}\mathrm{L}) \qquad \frac{\Gamma \longrightarrow \Delta, \varphi(y/x)}{\Gamma \longrightarrow \Delta, \forall x\, \varphi}\ (\forall{:}\mathrm{R})$$

$$\frac{\Gamma, \varphi(y/x) \longrightarrow \Delta}{\Gamma, \exists x\, \varphi \longrightarrow \Delta}\ (\exists{:}\mathrm{L}) \qquad \frac{\Gamma \longrightarrow \Delta, \varphi(t/x)}{\Gamma \longrightarrow \Delta, \exists x\, \varphi}\ (\forall{:}\mathrm{L})$$

(restriction: in the (∀:R) and (∃:L) rules, the variable y does not appear in $\Gamma \cup \Delta \cup \{\varphi\}$)

Equality axioms (in first-order logic with equality)

$$\frac{}{\longrightarrow t = t}\ (\mathrm{AE1})$$

$$\frac{}{t = s \longrightarrow s = t}\ (\mathrm{AE2})$$

$$\frac{}{t = s, s = u \longrightarrow t = u}\ (\mathrm{AE3})$$

$$\frac{}{t_1 = s_1, \ldots t_n = s_n, R(\bar{t}) \longrightarrow R(\bar{s})}\ (\mathrm{AE4})$$

$$\frac{}{t_1 = s_1, \ldots t_n = s_n \longrightarrow f(\bar{t}) = f(\bar{s})}\ (\mathrm{AE5})$$