Exam in Mathematical Logic
January 29, 2019

Problem 1. Give a proof of the sequent $\longrightarrow \forall x\, R(x), \exists x\, \neg R(x)$ in sequent calculus.

Problem 2. Let ξ be the sentence:

$$\Big[\forall y\, \exists x\, R(x,y) \wedge \forall x\, \forall y\, (R(x,y) \Rightarrow y = f(x))\Big] \Rightarrow \forall x\, \forall y\, \Big[f(x) = f(y) \Rightarrow x = y\Big].$$

(a) Find a sentence ψ in prenex normal form logically equivalent to ξ.

(b) Prove that ψ is true in all finite structures.

(c) Prove that $\neg\psi$ is satisfiable.

Problem 3. Prove that the set of pairs $\{\langle q, 2q\rangle : q \in \mathbb{Q}\}$ is:

(a) definable without parameters in the structure $(\mathbb{Q}, +)$,

(b) not definable without parameters in the structure $(\mathbb{Q}^2, +)$, where $+$ is vector addition.

Problem 4. Let T be a theory over the finite signature σ. Let $\mathbb{A}$ be a countable structure over σ such that for each finite tuple of elements $a_1, \ldots, a_k \in A$, there is a substructure $\mathbb{A}_0$ of $\mathbb{A}$ containing $a_1, \ldots, a_k$ such that $\mathbb{A}_0$ is also a substructure of some model of T.

Show that there is a countable $\mathbb{B} \models T$ such that $\mathbb{A} \subseteq \mathbb{B}$.

Problem 5. Let $\mathbb{A} = (A, \leqslant^{\mathbb{A}})$ be a linear order and let $\mathbb{A}^*$ be the ultrapower $\mathbb{A}^{\mathbb{N}}/\mathcal{U}$, where $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. We identify $\mathbb{A}$ with its canonical copy inside $\mathbb{A}^*$ given by constant sequences. Prove that for all $a_1, a_2 \in A$ with $a_1 < a_2$ the following conditions are equivalent:

(a) there is $b \in A^* \setminus A$ such that $a_1 < b < a_2$,

(b) there are infinitely many $d \in A$ such that $a_1 < d < a_2$.

Problem 6. Let $\mathbb{A} = (A, E^{\mathbb{A}})$ be a structure in which A is a countable set and $E^{\mathbb{A}}$ is an equivalence relation on A with exactly one equivalence class of each finite cardinality $n \in \mathbb{N} \setminus \{0\}$ and no infinite equivalence classes.

(a) How many countable models does $\mathrm{Th}(\mathbb{A})$ have (up to isomorphism)?

(b) Decide whether $\mathrm{Th}(\mathbb{A})$ has quantifier elimination.