Mathematical Logic midterm
December 15, 2010

1. Prove the following sequent in the sequent calculus:

$$\to \neg(p \Rightarrow q) \Rightarrow (p \wedge \neg q).$$

2. Write down a first-order sentence in the signature $\{\leq\}$ which is true in

$$\{(\{0\} \times \mathbb{Z}) \cup (\{1\} \times \mathbb{Q}), \leq_{\text{lex}}\},$$

but false in

$$\{(\{0\} \times \mathbb{Q}) \cup (\{1\} \times \mathbb{Z}), \leq_{\text{lex}}\}.$$

3. Let $\mathbb{A} = (A, E)$, where $card(A) = \mathfrak{c}$ and E is an equivalence relation with continuum many equivalence class of each finite cardinality, countably many equivalence classes of size $\mathfrak{c}$, and no other equivalence classes. Prove that no (exactly) countable subset of A is definable in $\mathbb{A}$ (with parameters).

4. Let T be a theory in a countable language and let $\varphi(x), \psi(x)$ be two formulas with one free variable x. Assume that there exists $\mathbb{A} \models T$ such that $card(\varphi^{\mathbb{A}}) = \aleph_0$, $card(\psi^{\mathbb{A}}) = \mathfrak{c}$. Show that T has at least two non-isomorphic models of cardinality $\mathfrak{c}$.