Exam in Mathematical Logic
January 27, 2011

Reminder: an $\exists_1$ (or purely existential) formula has the form $\exists y_1 \ldots \exists y_n\, \psi$, where ψ is quantifier-free. A $\forall_2$ formula has the form $\forall x_1 \ldots \forall x_k \exists y_1 \ldots \exists y_n\, \psi$, where ψ is quantifier-free.

1. Find a formula in DNF and a formula in CNF logically equivalent to:

$$\neg\left((p \Rightarrow q) \Leftrightarrow \neg(q \Leftrightarrow (p \wedge \neg r))\right).$$

2. Let φ be the sentence:

$$\neg\forall x\,(\neg\forall z\, R(x,z) \wedge \forall z \exists z\, \neg S(x,z)).$$

(a) Find a sentence in prenex normal form logically equivalent to φ,

(b) Prove that φ is not logically equivalent to any $\forall_2$ sentence.

3. Let σ be a finite signature containing only relation symbols. Let $\mathbb{A}$ be a countable structure over σ and let T be a theory over σ. Assume that every finite substructure of $\mathbb{A}$ can be embedded in a model of T. Show that $\mathbb{A}$ can be embedded in a countable model of T.

4. A model $\mathbb{A}$ of a theory T is called existentially closed if for every $\mathbb{B} \supseteq \mathbb{A}$, $\mathbb{B} \models T$, every tuple $\bar{a} \in A$ and every $\exists_1$ formula $\varphi(\bar{x})$,

$$\text{if } \mathbb{B} \models \varphi(\bar{a}), \text{ then } \mathbb{A} \models \varphi(\bar{a}).$$

(a) Give an example of an infinite boolean algebra which is not existentially closed (as a model of the theory of boolean algebras in the signature $\{\wedge, \vee, -, 0, 1\}$).

(b) Let T be a theory in a finite signature. Show that if T has an infinite model that is not existentially closed, then T has a model of cardinality $\mathfrak{c}$ that is not existentially closed.

5. Recall that $Spec(\varphi) = \{n \in \mathbb{N} :\ \text{there is } \mathbb{A} \models \varphi,\ card(A) = n\}$.

(a) Show that there exists a sentence φ such that
$Spec(\varphi) = \{n \in \mathbb{N} \setminus \{0\} : n \equiv 0 \pmod 4\}$.

(b) Show that there exists a sentence ψ such that
$Spec(\psi) = \{n \in \mathbb{N} \setminus \{0\} : n \equiv 2 \pmod 8\}$.

(c) Show that there is no sentence ξ in the empty signature which is true in all models of cardinality divisible by 4 and false in all models of cardinality congruent to 2 (mod 8).

(d) Explain how (a)-(c) imply that "Craig's interpolation theorem fails over finite structures".