# A note on the $\Sigma_1$ collection scheme and fragments of bounded arithmetic


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**Abstract**

We show that for each $n \geq 1$, if $T^n_2$ does not prove the weak pigeonhole principle for $\Sigma^b_n$ functions, then the collection scheme $B\Sigma_1$ is not finitely axiomatizable over $T^n_2$. The same result holds with $S^n_2$ in place of $T^n_2$.


The collection scheme $B\Sigma_1$ is

$$\forall v\,(\forall x < v\,\exists y\,\varphi(x,y) \Rightarrow \exists w\,\forall x < v\,\exists y < w\,\varphi(x,y))$$

for all bounded formulae $\varphi$ (or equivalently, for all $\varphi \in \Sigma_1$, as the initial existential quantifiers may be absorbed by $\exists y$).

An intriguing open problem, mentioned already in [WP89], concerns the provability of $B\Sigma_1$ in $I\Delta_0 + \neg\exp$. It is well known that $I\Delta_0$ does not


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prove $B\Sigma_1$, but all known proofs (cf. e.g. [PK78], [CFL07]) make use of the universal formula for $\Sigma_1$, and hence need the totality of exponentiation. It is widely believed that $B\Sigma_1$ remains unprovable even if we assume $\neg\exp$, but so far, no proof or even promising proof strategy has emerged. $B\Sigma_1$ is, however, known to be unprovable in $T^n_2 + \neg\exp$, where $T^n_2$ is the finite fragment of Buss' $S_2$ (essentially a notational variant of $I\Delta_0 + \Omega_1$) axiomatized by induction for $\Sigma^b_n$ formulae. Here, the universal formula for a restricted fragment of $\Sigma_1$ is enough.

The proofs of independence of $B\Sigma_1$ from $I\Delta_0 + \exp$ and from $T^n_2 + \neg\exp$ have very much in common. In this note, we point out that the behaviour of $B\Sigma_1$ in these theories is nevertheless probably quite different. In $I\Delta_0 + \exp$, collection is finitely axiomatizable, by the existence of the universal $\Sigma_1$ formula. On the other hand, we show that under a plausible assumption, $B\Sigma_1$ is not only unprovable, but even not finitely axiomatizable over $T^n_2$.

The "plausible assumption" we need is that $T^n_2$ does not prove the weak pigeonhole principle WPHP($\Sigma^b_n$), i.e. that the existence of a $\Sigma^b_n$ definable injection from $a^2$ to $a$ for some $a > 1$ is consistent with $T^n_2$. Our proof goes through if $T^n_2$ is replaced in both the assumption and the conclusion by the presumably weaker theory $S^n_2$. It is worth noting that until a breakthrough occurs, we cannot hope to prove non-finite axiomatizability of $B\Sigma_1$ unconditionally: it follows easily from [Bus95] and [Zam96] that if $T^n_2 \vdash S^{n+1}_2$, then $B\Sigma_1$ is finitely axiomatized over $T^n_2$. The assumption about unprovability of WPHP seems reasonable, as it is true for all $n$ in the relativized world ([BK94]) and, for $n = 1$ and $S^1_2$, follows from the hardness of integer factoring ([Jeř07]).

Our result does have some bearing on the problem whether $I\Delta_0 + \neg\exp \vdash B\Sigma_1$, in that it casts doubt on one possible line of attack. If $B\Sigma_1$ were finitely axiomatized over $T^n_2$ for some $n$, the answer to the problem would be negative. This is because the unprovability of $B\Sigma_1$ in $T^m_2 + \neg\exp$ for each $m \geq n$ would imply unprovability of a fixed finite fragment, which would then be independent from $S_2 + \neg\exp$ by compactness. However, if we are to believe the assumption about WPHP, then finite axiomatizability should not be hoped for.

We assume familiarity with basic notions and results concerning bounded arithmetic, which may be found e.g. in [HP93], [Kra95] or [Bus98]. For a brief review of relevant facts about WPHP, see e.g. [KT08] or [Tha02]. One important fact we need is that in $S^n_2$ the failure of WPHP($\Sigma^b_n$) can be amplified, that is, a $\Sigma^b_n$ injection from $a^2$ into $a$ can be used to obtain an injection from $b$ into $a$ for larger $b$.

We recall that the class $\hat{\Sigma}^b_n$, the prenex version of $\Sigma^b_n$, consists of formulae of the form

$$\exists y_1 < t_1 \, \forall y_2 < t_2 \ldots Q y_n < t_n \, \psi,$$

where $\psi$ is sharply bounded. The classes $\Sigma^b_n$ and $\hat{\Sigma}^b_n$ coincide w.r.t. equivalence in $S^n_2$, but in weaker theories $\Sigma^b_n$ might be strictly larger. $\hat{\Pi}^b_n$ is defined dually to $\hat{\Sigma}^b_n$, and $\exists\hat{\Pi}^b_n$ is the class of $\hat{\Pi}^b_n$ formulae preceded by existential quantifiers. It is easily checked that collection for $\exists\hat{\Pi}^b_n$ formulae, $B\exists\hat{\Pi}^b_n$, is equivalent to $B\hat{\Pi}^b_n$.

We also introduce one piece of notation: for a number $a$, $\#^m a$ is $a\#a \ldots \#a$, where $a$ appears $m$ times. Given a model $\mathcal{A}$, $\#^{\mathbb{N}} a$ is the cut in $\mathcal{A}$ determined by the numbers $\#^m a$ for standard $m$.

We now state and prove our theorem. Our proof is essentially a combination of slightly stronger variants of arguments from [AK07] and [KT08], and we assume the reader has access to those two papers.

**Theorem 1.** *Let $T_n$ be $T^n_2$ or $S^n_2$. If $T_n \not\vdash \mathrm{WPHP}(\Sigma^b_n)$, then the collection scheme $B\Sigma_1$ is not finitely axiomatizable over $T_n$.*

We prove the theorem through a series of lemmas. Our starting point is a countable model $\mathcal{A} \models T_n$ containing a number $a$ such that $\mathcal{A} = \#^{\mathbb{N}} a$ and the WPHP for $\Sigma^b_n$ functions fails at $a$, in the sense that there is a $\Sigma^b_n$ definable injection from $a^2$ into $a$. Such a model exists if $T_n$ does not prove $\mathrm{WPHP}(\Sigma^b_n)$.

To prove that $B\Sigma_1$ is not finitely axiomatizable over $T_n$, we need to show that there is no $k$ such that over $T^n_2$, collection for $\Sigma_1$ formulae follows from collection for $\exists\hat{\Pi}^b_k$ formulae. W.l.o.g., we may consider only $k$ for which $k + 2 \geq n + 1$.

There is a standard way of building a cofinal and $\hat{\Sigma}^b_{k+2}$-elementary extension of $\mathcal{A}$ to a $\hat{\Sigma}^b_{k+3}$-maximal model $\mathcal{A}_+$ of $T_n$, i.e. one all of whose $\hat{\Sigma}^b_{k+2}$-elementary extensions to models of $T_n$ are actually $\hat{\Sigma}^b_{k+3}$-elementary. By tweaking the construction a little, we are able to make $\mathcal{A}_+$ satisfy $\exists\hat{\Pi}^b_k$ collection (Lemma 2). By $\hat{\Sigma}^b_{n+1}$-elementarity, $\mathrm{WPHP}(\Sigma^b_n)$ still fails at $a$ in $\mathcal{A}_+$.

To complete the proof of the theorem, we show that $\mathcal{A}_+$ does not satisfy $B\exists\hat{\Pi}^b_{k+2}$. The argument is as follows. We observe that in $\mathcal{A}_+$, each $\hat{\Sigma}^b_{k+3}$ formula is equivalent to a $\forall\hat{\Sigma}^b_{k+2}$ formula, with $a$ as parameter (Lemma 3). If $\mathcal{A}_+$ satisfied $B\exists\hat{\Pi}^b_{k+2}$, this collapse would translate into a "local" collapse of $\hat{\Sigma}^b_{k+2}$ to $\hat{\Pi}^b_{k+2}$ (Corollary 4), which falls just short of implying $\Sigma^p_{k+2} \subseteq \Pi^p_{k+2}/\text{poly}$. But in any model of $S^n_2$, such a collapse is incompatible with $\neg\text{WPHP}(\Sigma^b_n)$ (Lemma 5).

The remainder of the note contains proofs of the lemmas and a concluding remark.

**Lemma 2.** *Let $m \geq n$. Let $\mathcal{A} \models T_n$ be countable and of the form $\#^{\mathbb{N}}a$ for some $a$. There exists a cofinal countable extension $\mathcal{A}_+ \succeq_{\Sigma^b_m} \mathcal{A}$ which is a $\hat{\Sigma}^b_{m+1}$-maximal model $\mathcal{A}_+$ of $T_n$ and (if $m \geq 2$) satisfies $\exists\hat{\Pi}^b_{m-2}$ collection.*

*Proof.* The construction of a cofinal $\hat{\Sigma}^b_m$-elementary $\hat{\Sigma}^b_{m+1}$-maximal extension of $\mathcal{A}$ is a routine variant of the general model-theoretic construction of an existentially closed model. Starting with $\mathcal{A}_0 = \mathcal{A}$, we build a chain $\mathcal{A}_0 \preceq_{\hat{\Sigma}^b_m} \mathcal{A}_1 \preceq_{\hat{\Sigma}^b_m} \ldots$ of countable cofinal models of $T_n$. $\mathcal{A}_{l+1}$ arises from $\mathcal{A}_l$ by adding a witness for the initial existential quantifier in a given $\hat{\Sigma}^b_{m+1}$ formula with a given choice of parameters from $\mathcal{A}_l$, whenever that is possible without losing $\hat{\Sigma}^b_m$-elementarity. $\mathcal{A}_+$ is the union of the chain, and $\hat{\Sigma}^b_m$-elementarity guarantees that $\mathcal{A}_+$ satisfies $T_n$. (See the proof of Lemma 2.2 in [AK07] for details.)

To ensure that $\mathcal{A}_+$ satisfies the right amount of collection, we slightly modify our method of constructing $\mathcal{A}_{l+1}$ from $\mathcal{A}_l$. As before, we add a witness for a given $\hat{\Sigma}^b_{m+1}$ formula with given parameters in a $\hat{\Sigma}^b_m$-elementary way. However, we also make sure that the model thus obtained, say $\tilde{\mathcal{A}}_{l+1}$, is *not* a cofinal extension of $\mathcal{A}_l$, but satisfies overspill for all bounded formulae. This can be achieved by a standard compactness argument. We then take $\mathcal{A}_{l+1}$ to be the cut $\#^{\mathbb{N}}a$ in $\tilde{\mathcal{A}}_{l+1}$.

In this way, $\mathcal{A}_{l+1}$ is a proper initial segment of the form $\#^{\mathbb{N}}a$ in a model of $\Sigma^b_\infty$ overspill. It is now easy to show $\mathcal{A}_{l+1} \models B\Sigma_1$. This is done by mirroring the well-known proof that a proper initial segment of a model of $I\Delta_0$ satisfies $B\Sigma_1$ (cf. [WP89]).

It remains to check that the fact that $\mathcal{A}_l \models B\Sigma_1$ for all $l$ implies $\mathcal{A}_+ \models B\exists\hat{\Pi}^b_{m-2}$. Let $b, \bar{p} \in \mathcal{A}_+$, let $\psi(x, y, \bar{p})$ be a $\hat{\Pi}^b_{m-2}$ formula, and assume that

$$\mathcal{A}_+ \models \forall w\, \exists x < b\, \forall y < w\, \neg\psi(x, y, \bar{p}).$$

In particular, for each $i \in \mathbb{N}$ we have

$$\mathcal{A}_+ \models \exists x < b\, \forall y < \#^i a\, \neg\psi(x, y, \bar{p}).$$

Take $l$ such that $\mathcal{A}_l$ contains $b$ and $\bar{p}$. By $\hat{\Sigma}^b_m$-elementarity, we get

$$\mathcal{A}_l \models \exists x < b \, \forall y < \#^i a \, \neg\psi(x, y, \bar{p})$$

for each $i$, and thus

$$\mathcal{A}_l \models \forall w \, \exists x < b \, \forall y < w \, \neg\psi(x, y, \bar{p})$$

since $\mathcal{A}_l$ is of the form $\#^{\mathbb{N}} a$. By $B\exists\hat{\Pi}^b_{m-2}$ in $\mathcal{A}_l$, there exists $c < b$ such that $\mathcal{A}_l \models \forall y \, \neg\psi(c, y, \bar{p})$. Applying $\hat{\Sigma}^b_m$-elementarity once again, we obtain $\mathcal{A}_+ \models \forall y < \#^i a \, \neg\psi(c, y, \bar{p})$ for each $i$. But $\mathcal{A}_+$ is also of the form $\#^{\mathbb{N}} a$, which means that $\mathcal{A}_+ \models \forall y \, \neg\psi(c, y, \bar{p})$. □

**Lemma 3.** *Let $m \in \mathbb{N}$ and let $\mathcal{A}$ be a $\hat{\Sigma}^b_{m+1}$-maximal model of $T_n$ of the form $\#^{\mathbb{N}} a$. Then each $\hat{\Sigma}^b_{m+1}$ formula is equivalent in $\mathcal{A}$ to a $\forall\hat{\Sigma}^b_m$ formula with $a$ as an additional parameter.*

*Proof.* We sketch the proof omitting some details which are essentially the same as in Section 3 of [AK07].

It is easy to see that if $\mathcal{A}$ is $\hat{\Sigma}^b_{m+1}$-maximal for $T_n$, $\psi(x)$ is a $\hat{\Sigma}^b_{m+1}$ formula, and $d \in \mathcal{A}$, then $\psi(d)$ holds iff it is consistent with $T_n$ plus the $\hat{\Pi}^b_m$ theory of $\mathcal{A}$ in the language $L(\mathcal{A})$ (that is, $L_{BA}$ expanded by constants for all elements of $\mathcal{A}$). Thus, it remains to check that "$\psi(x)$ is consistent with $T_n$ plus the $\hat{\Pi}^b_m$ theory of $\mathcal{A}_{L(\mathcal{A})}$" can be expressed in $\mathcal{A}$ using a $\forall\hat{\Sigma}^b_m$ formula with $a$ as a parameter.

Formalize $L(\mathcal{A})$ is some reasonable way, e.g. by letting the first few odd numbers represent the symbols of $L_{BA}$, letting $2d$ represent a constant symbol $\underline{d}$ standing for $d \in \mathcal{A}$, and then coding syntax as usual. Our formula will say the following:

$$\begin{gathered}\forall y \, \forall l \, \forall s \, [l \in \mathbb{N} \,\&\, y = 2^{|a|^l} \\ \&\ \text{“}s \text{ is a sequence of formulae”} \,\&\, \textstyle\sum_{i<\mathrm{lh}(s)} \mathrm{lh}((s)_i)) \leq |l| \\ \&\ \text{“no } (s)_i \text{ contains a constant for a number greater than } y\text{”} \\ \&\, \forall i < \mathrm{lh}(s) \ ((s)_i \in T_n \vee \text{“}(s)_i \text{ is a true } \hat{\Pi}^b_m \text{ formula”} \\ \vee\ \text{“}(s)_i \text{ is derived from previous elements of } s \text{ by an inference rule”}) \\ \Rightarrow (s)_{\mathrm{lh}(s)-1} \neq \ulcorner\neg\psi(\underline{x})\urcorner]\end{gathered}$$

We need to see that this is equivalent in $\mathcal{A}$ to a $\forall\hat{\Sigma}^b_m$ formula, which amounts to checking that each conjunct in the antecedent of the implication may be stated in $\exists\hat{\Pi}^b_m$ form. The conjunct $l \in \mathbb{N}$ is not really needed, as it is implied by $y = 2^{|a|^l}$ because $\mathcal{A}$ is of the form $\#^{\mathbb{N}}a$. The only other problematic conjunct is:

$$\forall i < \mathrm{lh}(s)\ (\ldots \vee \text{“}(s)_i \text{ is a true } \hat{\Pi}^b_m \text{ formula”} \vee \ldots),$$

but a universal formula for $\hat{\Pi}^b_m$ formulae of length $\leq |l|$ and arguments below $2^{|a|^l}$ is $\hat{\Pi}^b_m$ with a bounding parameter, which can be any number above $2^{|a|^{l^2}}$. □

**Corollary 4.** *Let $\mathcal{A}$ be a $\hat{\Sigma}^b_{m+1}$-maximal model of $T_n$ of the form $\#^{\mathbb{N}}a$ and $\mathcal{A} \models B\exists\hat{\Pi}^b_m$. Then for each $d \in \mathcal{A}$, each $\hat{\Sigma}^b_{m+1}$ formula is equivalent on $[0, d]$ to a $\hat{\Pi}^b_{m+1}$ formula with $a$ as an additional parameter.*

*Proof.* Let $\psi(x)$ be a $\hat{\Sigma}^b_{m+1}$ formula and $d \in \mathcal{A}$. By Lemma 3, $\psi(x)$ is equivalent in $\mathcal{A}$ to $\forall y\, \eta(x, y, a)$, where $\eta$ is $\hat{\Sigma}^b_m$. Thus, we have

$$\mathcal{A} \models \forall x\, (\psi(x) \vee \exists y\, \neg\eta(x, y, a)).$$

If $\mathcal{A} \models B\exists\hat{\Pi}^b_m$, then there exists $w$ such that for $x \in [0, d]$, a witness for either $\psi(x)$ or $\exists y\, \neg\eta(x, y)$ may be bounded by $w$. We may take $w$ to be of the form $\#^l a$ for some $l \in \mathbb{N}$, so on $[0, d]$, $\psi(x)$ is equivalent to $\forall y < \#^l a\, \eta(x, y, a)$. □

**Lemma 5.** *Let $\mathcal{A} \models S^n_2 + \neg\mathrm{WPHP}(\Sigma^b_n)$ be of the form $\#^{\mathbb{N}}a$. Then for each $m \geq 1$ there exists $d \in \mathcal{A}$ and a $\hat{\Sigma}^b_m$ formula $\psi(x)$ which is not equivalent on $[0, d]$ to a $\hat{\Pi}^b_m$ formula, even with parameters.*

*Proof.* The proof is based on an argument from Sections 4 and 5 of [KT08], used there to show that in a model of $S^n_2 + \neg\mathrm{WPHP}(\Sigma^b_n)$ the bounded formula hierarchy does not collapse, even with parameters (Theorem 5.1 of [KT08]). We will check that the argument is actually strong enough to show that in the case of models of the form $\#^{\mathbb{N}}a$, failure of $\mathrm{WPHP}(\Sigma^b_n)$ excludes even a collapse on a large enough initial segment with a top.

Assume that $m$ is such that on each interval $[0, d]$, every $\hat{\Sigma}^b_m$ formula is equivalent to a $\hat{\Pi}^b_m$ formula with a parameter (which may depend on $d$). It can then be easily checked that on each $[0, d]$, every bounded formula is equivalent to a $\hat{\Pi}^b_m$ formula with a parameter.

By increasing $a$ and amplifying the function violating WPHP if necessary, we may assume that $a$ is a power of 2, $f$ is a $\Sigma^b_n$ injection from $a\#a$ into $a$ and that the definition of $f$ involves only a parameter $q < a$ and quantifiers bounded by $a\#a$. We now use compactness to extend $\mathcal{A}$ elementarily to a model $\mathcal{A}'$ additionally containing an element $t > \mathcal{A}$ and an element $b > \#^{\mathbb{N}}t$ of the form $\#^c a$ for some small nonstandard $c$. Let $\mathcal{B}$ be the cut $\#^{\mathbb{N}}a$ in $\mathcal{A}'$. The reason for the amplification and for introducing the new models is that the relation between $\mathcal{A}'$ and $\mathcal{B}$ is now exactly the same as between the models $\mathcal{A}$ and $\mathcal{B}$ in Section 4 of [KT08], and we will be able to apply the results of that section.

Note that because $\mathcal{A}' \succeq \mathcal{A}$ and $\mathcal{A}$ is cofinal in $\mathcal{B}$, on each interval $[0, d]$ in $\mathcal{B}$ every bounded formula is equivalent to a $\hat{\Sigma}^b_m$ formula with a parameter from $\mathcal{B}$ (even from $\mathcal{A}$).

Since $t > \mathcal{B}$, there is a universal $\hat{\Sigma}^b_m$ formula $U_m$ such that for all $x, y \in \mathcal{B}$ and $\hat{\Sigma}^b_m$ formulae $\psi$, $\psi(x, y)$ is equivalent in $\mathcal{A}'$ to $U_m(x, \langle \ulcorner\psi\urcorner, y\rangle, t)$. Now, $b > \#^{\mathbb{N}}t$ and $U_m$ is bounded, so for $x, y \in \mathcal{B}$ and for standard $\psi$ the quantifiers in $U_m(x, \langle \ulcorner\psi\urcorner, y\rangle, t)$ range only over numbers below $b$. By Lemma 4.3 of [KT08], this means that we can use the failure of WPHP to translate $U_m(x, \langle \ulcorner\psi\urcorner, y\rangle, t)$ into a bounded formula with parameters from $\mathcal{B}$. More precisely, there is a bounded (even linearly bounded) formula $U^{\mathrm{lin}}_m$ such that for all $x, y \in \mathcal{B}$ and all $\psi$, $U_m(x, \langle \ulcorner\psi\urcorner, y\rangle, t)$ is equivalent to $U^{\mathrm{lin}}_m(x, \langle \ulcorner\psi\urcorner, y\rangle, p)$, where $p$ is a parameter bounded by some standard power of $a\#a$ ($p$ is actually a tuple $(\hat{t}, a\#a, c, q)$, where $c, q$ are as above and $\hat{t}$ is a number below $a$ which codes $t$ in a certain way).

On each interval $[0, d]$ in $\mathcal{B}$, the bounded formula $\neg U^{\mathrm{lin}}_m(x, x, p)$ is equivalent to a $\hat{\Sigma}^b_m$ formula $\varphi(x, r)$. A priori, the size of $r$ depends on $d$, but for sufficiently large $d$, we can assume $\langle \ulcorner\varphi\urcorner, r, p\rangle < d$. This is because $r$ is certainly bounded by $\#^i a$ for some $i$, and $\#^i a$ can be mapped injectively into $a$ by an amplified version $f^{(i-1)}$ of $f$, which is also $\Sigma^b_n$ definable; thus, we can replace the original $\varphi(x, r)$ by $\tilde{\varphi}(x, \tilde{r}) := \exists y < \#^m a\, (f^{(i-1)}(y) = \tilde{r} \,\&\, \varphi(x, y))$, where $\tilde{r} = f^{(i-1)}(r)$ is a number below $a$.

Consider $\varphi(\langle \ulcorner\varphi\urcorner, r\rangle, r)$. By the properties of $U_m$, this is equivalent to $U_m(\langle \ulcorner\varphi\urcorner, r\rangle, \langle \ulcorner\varphi\urcorner, r\rangle, t)$ and hence to $U^{\mathrm{lin}}_m(\langle \ulcorner\varphi\urcorner, r\rangle, \langle \ulcorner\varphi\urcorner, r\rangle, p)$. On the other hand, for a large enough $d$, $\neg U^{\mathrm{lin}}_m(\langle \ulcorner\varphi\urcorner, r\rangle, \langle \ulcorner\varphi\urcorner, r\rangle, p)$ is also equivalent to $\varphi(\langle \ulcorner\varphi\urcorner, r\rangle, r)$, which gives a contradiction. $\square$

*Remark.* The model $\mathcal{A}_+$ obtained in our construction has the property that each $\exists\hat{\Pi}^b_{k+3}$ formula is equivalent to an $\exists\hat{\Pi}^b_{k+2}$ formula with a parameter

(Lemma 3), but is not in general equivalent to an $\exists\hat{\Pi}^b_k$ formula, even with parameters (because collection holds for the latter class).

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