A $9k$ kernel for nonseparating independent set in planar graphs

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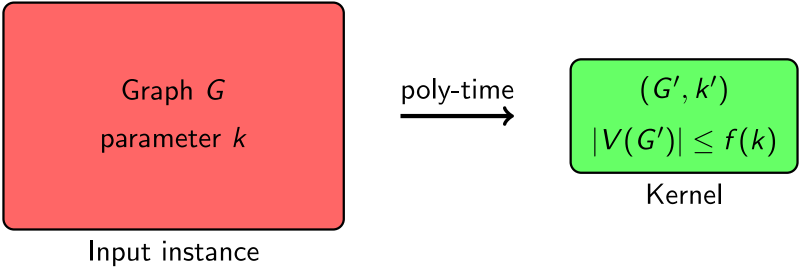
Kernelization (of graph problems)

Let (G, k) be an instance of a decision problem (k is a parameter).



- (G, k) is a YES-instance iff (G', k') is a YES-instance.
- $k' \leq k$,
- $|V(G')| \leq f(k)$.

Some examples of kernels

General graphs:

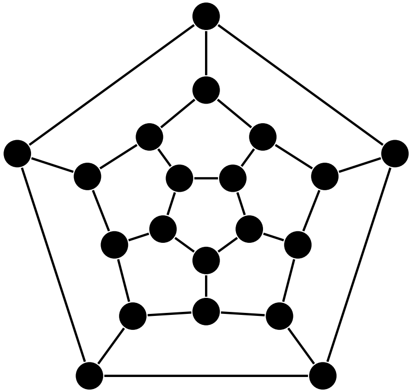
- VERTEX COVER $2k$,
- FEEDBACK VERTEX SET $O(k^2)$,
- ODD CYCLE TRANSVERSAL $k^{O(1)}$,
- ...

Planar graphs:

- DOMINATING SET $67k$,
- FEEDBACK VERTEX SET $112k$,
- INDUCED MATCHING $28k$,
- CONNECTED VERTEX COVER $\frac{11}{3}k$,
- ...

Vertex Cover and Independent Set

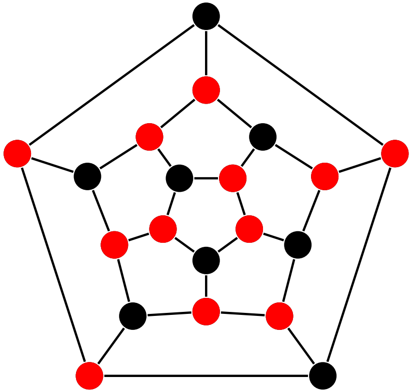
Let $G = (V, E)$ be a graph.



Vertex Cover and Independent Set

Let $G = (V, E)$ be a graph.

- C is a vertex cover (●) when every edge has **at least one endpoint** in C.



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Let $G = (V, E)$ be a graph.

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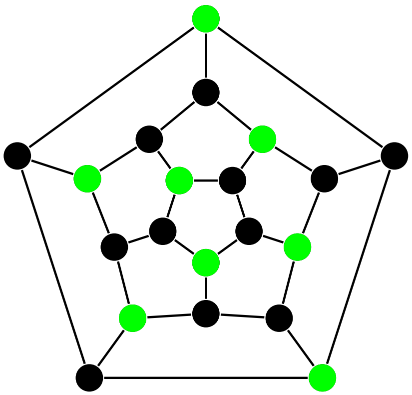
- S is an independent set (●) when every edge has **at most one endpoint** in S.



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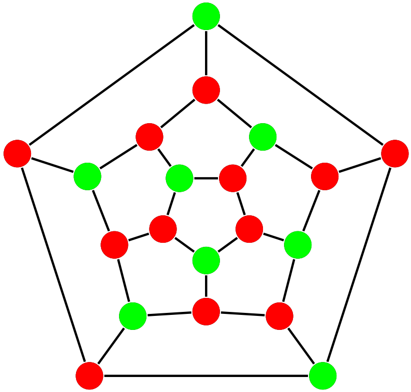


Observation

- C is a vertex cover iff $V \setminus C$ is an independent set.
- G has a vertex cover of size k iff G has independent set of size $|V| - k$.

Parametric Duality

Corollary

G has independent set of size k iff G has a vertex cover of size $|V| - k$.

VERTEX COVER

INSTANCE: Graph $G = (V, E)$, $k \in \mathbb{N}$
QUESTION: Does G contain a vertex cover of size k?

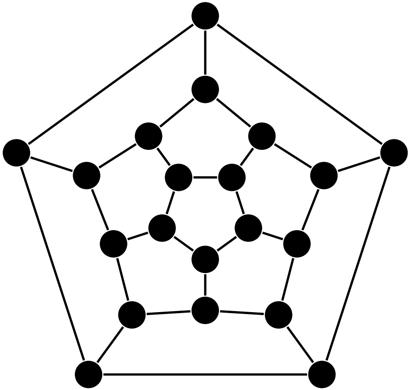
INDEPENDENT SET

INSTANCE: Graph $G = (V, E)$, $k \in \mathbb{N}$
QUESTION: Does G contain an independent set of size k?

- We can treat INDEPENDENT SET as VERTEX COVER with $|V| - k$ as a parameter.
- Then, VERTEX COVER is a **parametric dual** of INDEPENDENT SET.
- **But** a small kernel for one problem does not give a small kernel for another.

Connected Vertex Cover & Nonseparating Independent Set

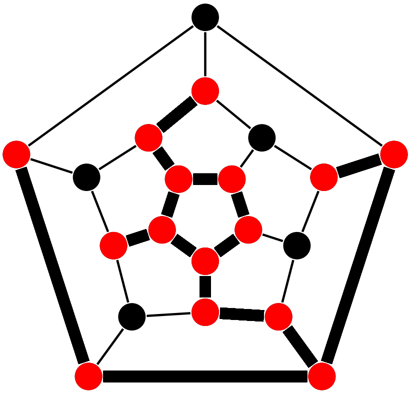
Let $G = (V, E)$ be a connected graph.



Connected Vertex Cover & Nonseparating Independent Set

Let $G = (V, E)$ be a connected graph.

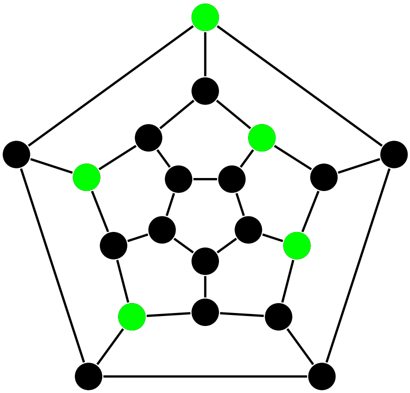
- C is a connected vertex cover (●) when C is a vertex cover and C induces a connected subgraph of G.



Connected Vertex Cover & Nonseparating Independent Set

Let $G = (V, E)$ be a connected graph.

- C is a connected vertex cover (●) when C is a vertex cover and C induces a connected subgraph of G.
- S is a nonseparating independent set (●) when S is an independent set and $G - S$ is connected.



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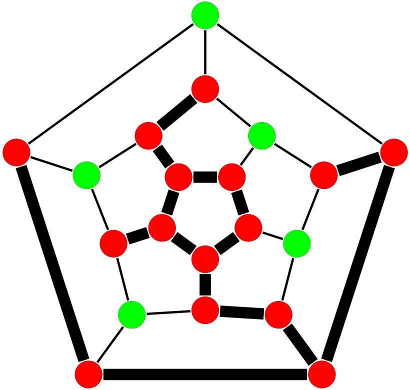


Observation

- C is a connected vertex cover iff $V \setminus C$ is a nonseparating independent set.
- G has a connected vertex cover of size k iff G has a nonseparating independent set of size $|V| - k$.

Parametric Duality

Connected Vertex Cover (CVC)

Instance: Graph $G = (V, E)$, $k \in \mathbb{N}$
Question: Does G contain a vertex cover of size k?

Nonseparating Independent Set (NSIS)

Instance: Graph $G = (V, E)$, $k \in \mathbb{N}$
Question: Does G contain an independent set of size k?

Connected Vertex Cover is a parametric dual of Nonseparating Independent Set.

Known complexity results for CONNECTED VERTEX COVER (CVC) and NONSEPARATING INDEPENDENT SET (NSIS)

Both problems

- NP-complete even for planar graphs,
- in P for graphs of maximum degree 3 (Ueno 1988).

CVC: kernelization

- CVC has no kernel of polynomial size (Dom et al 2009),
- PLANAR CVC has a $\frac{11}{3}k$-kernel (Kowalik et al 2011).

NSIS: kernelization

- NSIS is $W[1]$-hard, so no kernel at all (folklore),
- PLANAR NSIS: $O(k)$-sized kernel (Fomin et al. 2010)

Our results: kernel upper bounds

Main Result

There is a $9k$-kernel for PLANAR NONSEPARATING INDEPENDENT SET.

A Bonus Result (skipped in this presentation)

There is a $5k$-kernel for PLANAR MAX LEAF.

Our results: kernel lower bounds

Theorem (Chen et al. 2007)

If a problem admits a kernel of size at most αk, then the dual problem has no kernel of size at most $(\frac{\alpha}{\alpha-1} - \epsilon)k$, for any $\epsilon > 0$, unless P=NP.

Two corollaries

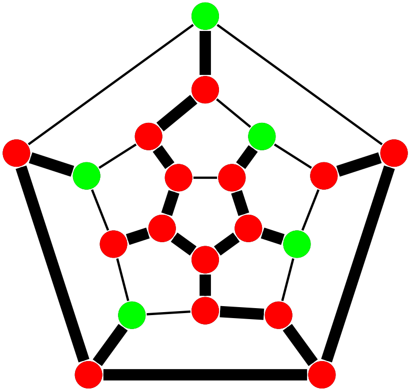
- Planar CVC has no kernel of size at most $(\frac{9}{8} - \epsilon)k$, unless P=NP,
- Planar Connected Dominating Set has no kernel of size at most $(\frac{5}{4} - \epsilon)k$, unless P=NP,

A simple $12k$ kernel for Planar NSIS

For a tree T let $L(T)$ denote the set of leaves of T.

Maximum Independent Leaf

Instance: Graph $G = (V, E)$, $k \in \mathbb{N}$
Question: Is there a spanning tree T such that $L(T)$ contains a subset of size k that is independent in G?



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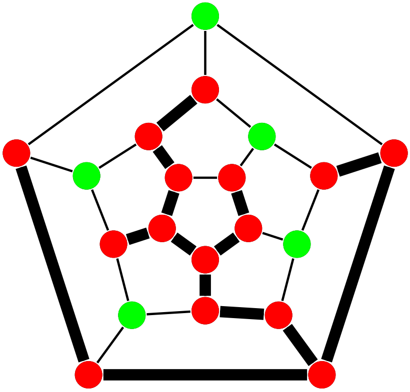

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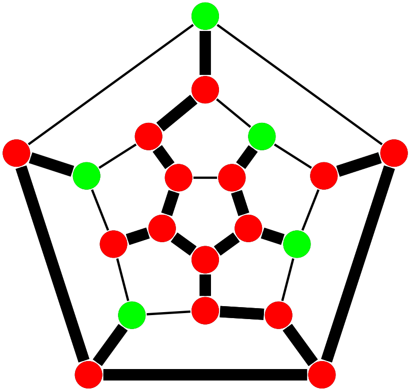

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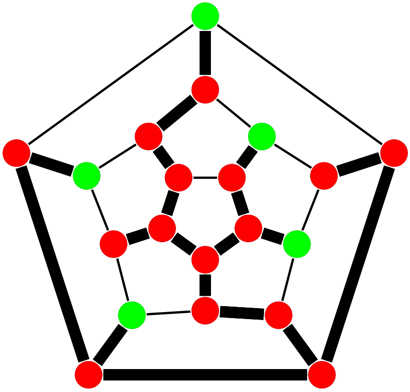

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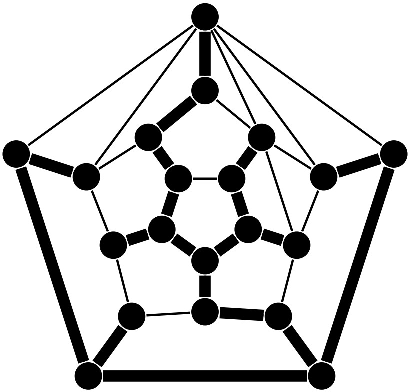
In what follows...

we focus on Maximum Independent Leaf!

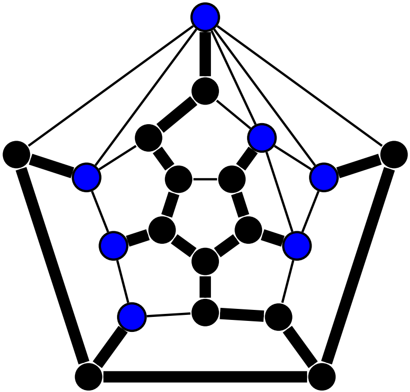
A 12k-kernel for Planar NSIS for minimum degree 3



A 12k-kernel for Planar NSIS for minimum degree 3

Theorem (Kleitman, West 1991)

Every n-vertex graph of minimum degree 3 has a spanning tree of $\geq n/4$ leaves.



 $=$ leaves.

A $12k$-kernel for PLANAR NSIS for minimum degree 3

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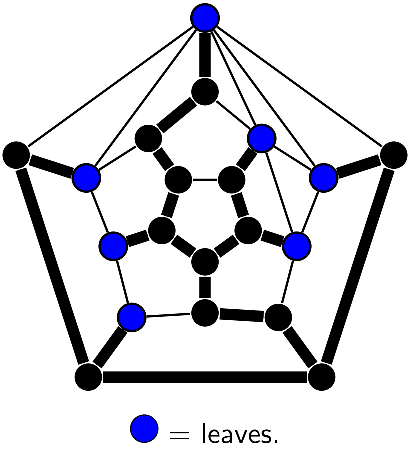
Observation

If G is planar and T is a spanning tree of G then $G[L(T)]$ is outerplanar.



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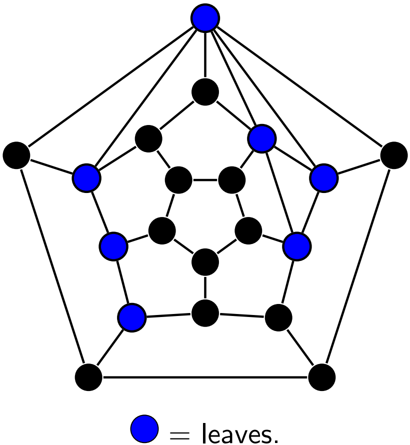

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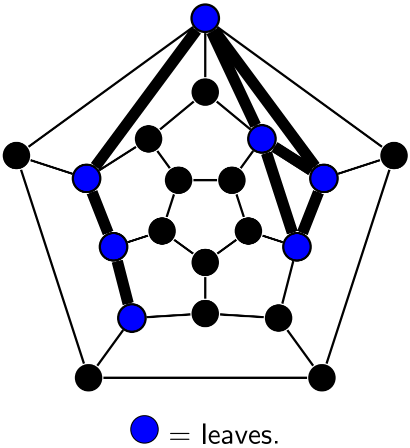

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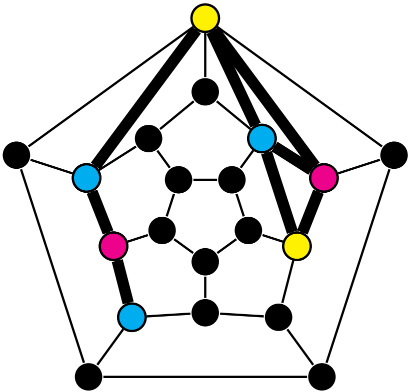
Theorem (folklore)

Every outerplanar graph is 3-colorable.



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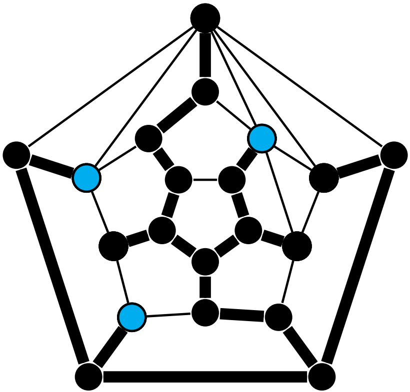

Corollary

Every n-vertex planar graph G of minimum degree 3 has a spanning tree with a subset of leaves of size at least $n/12$ which is independent in G.

A $12k$-kernel for Planar NSIS for minimum degree 3

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Kernelization algorithm

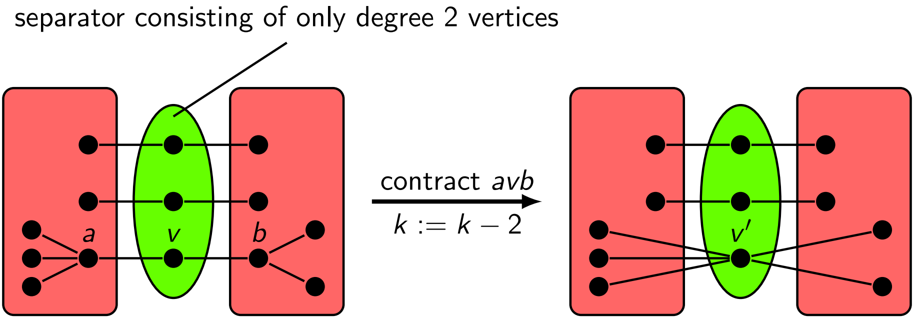
1. If $k \leq n/12$ answer YES,
2. Otherwise (i.e. $n < 12k$) return (G, k).

What about vertices of degree 1 and 2?

Generalization of the theorem of Kleitman and West

Let G be a connected n-vertex graph that does not contain a separator consisting of only degree 2 vertices. Then G has a spanning tree with at least $n/4$ leaves.

Degree 2 vertices: the separator rule



$12k$-kernelization algorithm

Kernelization algorithm

1. Apply the separator rule as long as possible,
2. If $k \leq n/12$ answer YES,
3. Otherwise (i.e. $n < 12k$) return (G, k).

A glimpse at the $9k$-kernel

Theorem 1

Let G be a connected n-vertex graph that does not contain a separator consisting of only degree 2 vertices. Then G has a spanning tree T such that if $\mathcal{C}$ is a collection of vertex-disjoint cycles in $G[L(T)]$, then

$$|L(T)| \geq \frac{n + 3|\mathcal{C}|}{4}.$$

Theorem 2

Every ℓ-vertex outerplanar graph contains

- an independent set I, and
- a collection of vertex-disjoint cycles $\mathcal{C}$

such that $9|I| \geq 4\ell - 3|\mathcal{C}|$.

Some open questions

- Get a smaller kernel for PLANAR NSIS. $8k$?
- Are there linear kernels for the parametric duals of the following problems:
  - (PLANAR) CONNECTED FEEDBACK VERTEX SET,
  - (PLANAR) CONNECTED ODD CYCLE TRANSVERSAL,
  - (PLANAR) STEINER TREE?