A $14k$-kernel for Planar Feedback Vertex Set via Region Decomposition

Marthe Bonamy[1] and Łukasz Kowalik[2] (speaker)

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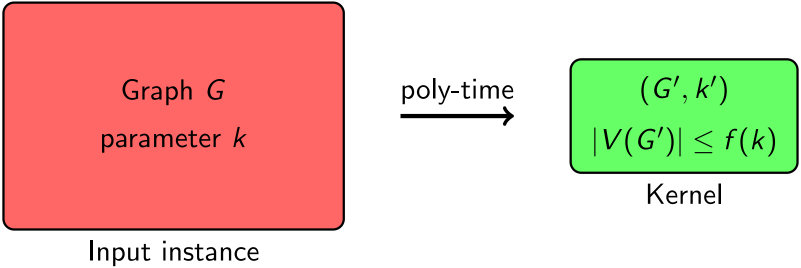
10th September 2014

Kernelization (of graph problems)

Let (G, k) be an instance of a decision problem (k is a parameter).

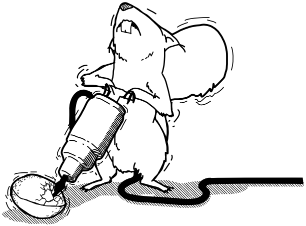


- (G, k) is a YES-instance iff (G', k') is a YES-instance.
- $k' \leq k$,
- $|V(G')| \leq f(k)$.

Some examples of kernels

General graphs:

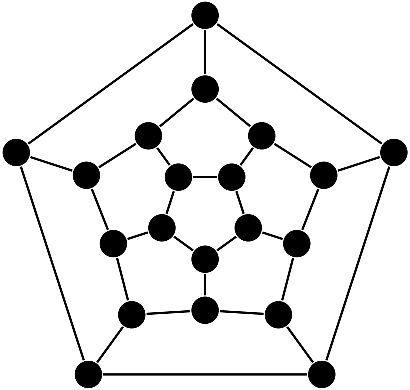
- Vertex Cover $2k$,
- Feedback Vertex Set $O(k^2)$,
- Odd Cycle Transversal $k^{O(1)}$,
- ...



Planar graphs:

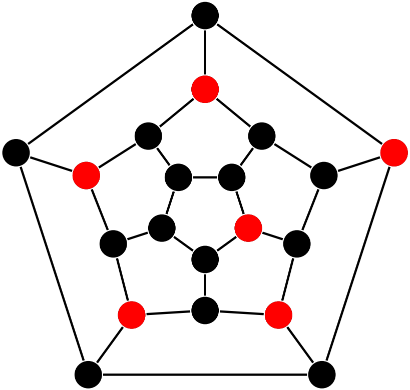
- Dominating Set $335k \rightarrow 67k$,
- Induced Matching $40k \rightarrow 28k$,
- Connected Vertex Cover $14k \rightarrow \frac{11}{3}k$,
- Connected Dominating Set $3968187k \rightarrow 130k$,
- ...

Feedback Vertex Set: hit every cycle



Feedback Vertex Set: hit every cycle

$S \subseteq V$ is a feedback vertex set (●) in graph $G = (V, E)$ when every cycle in G contains at least one vertex from S.



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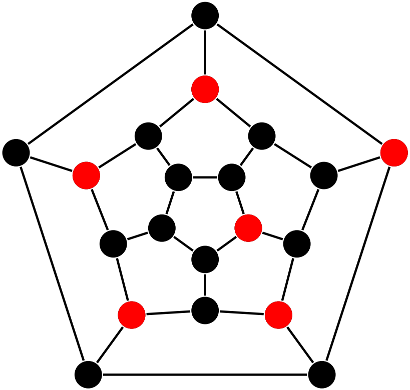
PLANAR FEEDBACK VERTEX SET

INSTANCE: planar graph G, $k \in \mathbb{N}$
PARAMETER: $k \in \mathbb{N}$
QUESTION: Does G contain a feedback vertex set of size k?



Previous results:

- Bodlaender and Penninkx IWPEC 2008: $112k$-kernel
- Abu-Khzam and Khuzam IPEC 2012: $97k$-kernel

Kernels for Planar Feedback Vertex Set

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Parallel result(s):

- Xiao AAIM 2014: $29k$-kernel

Kernels for Planar Feedback Vertex Set

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Parallel result(s):

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Our result:

- A $14k$-kernel
- Region decomposition technique applied (tight analysis)

Linear kernel? Let's do it!

Let S be a feedback vertex set of size k in planar graph G.

Can we bound $|V(G)|$?

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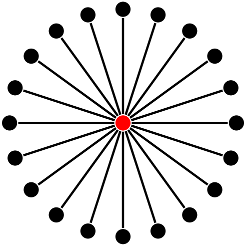

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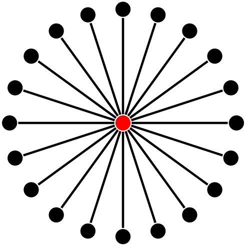

Reduction rule:

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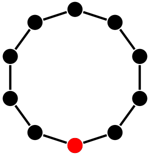

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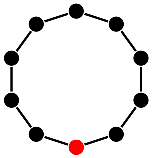
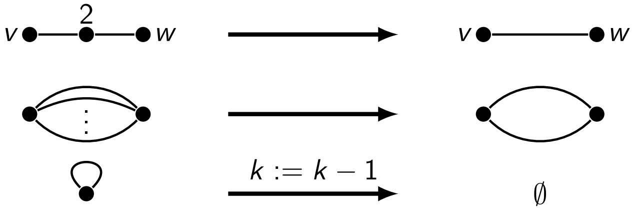


New reduction rules:



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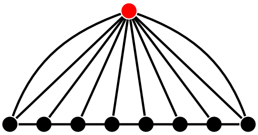

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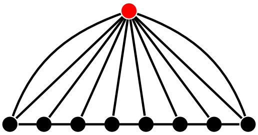

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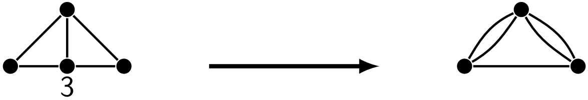

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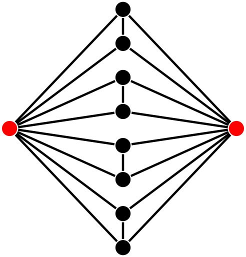

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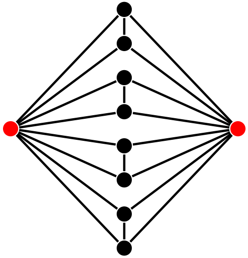

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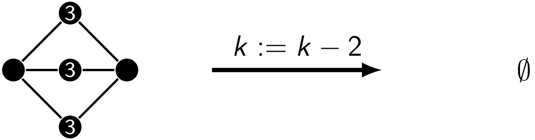

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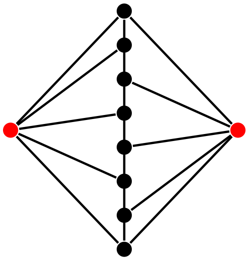

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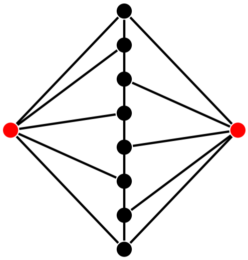

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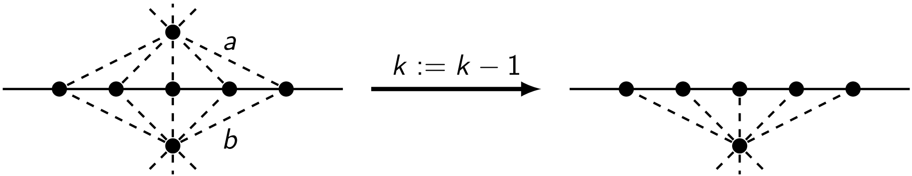
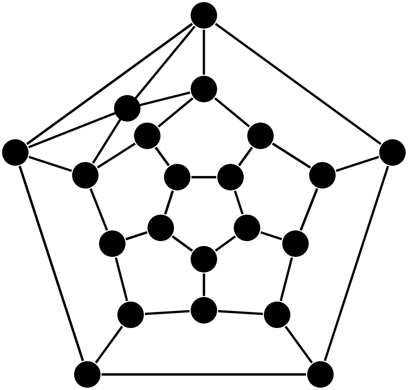


$a > b$

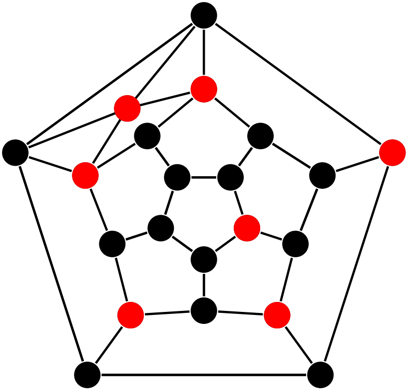
We have some more rules, but they only help to improve the constant further, let's skip them here.

Analysis: partition of the vertices



Analysis: partition of the vertices

- (●) S : a solution (fvs).



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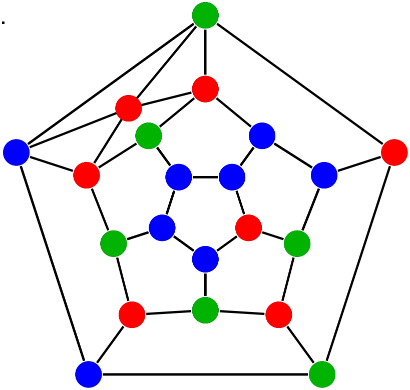

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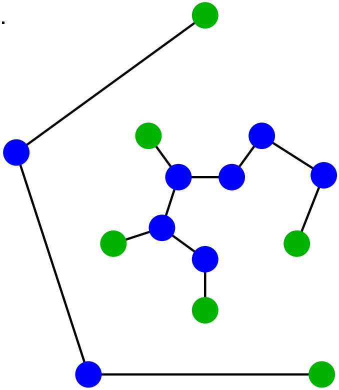
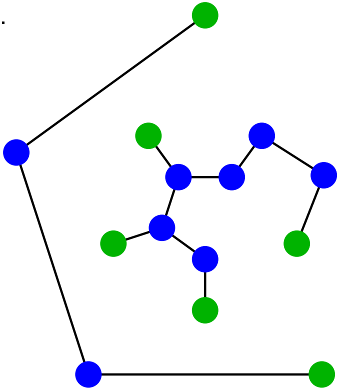
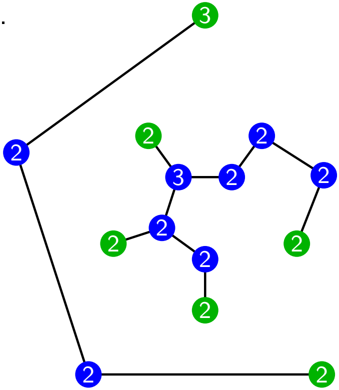


Analysis: partition of the vertices

- (●) S : a solution (fvs).
- (● ●) $F = V \setminus S$: an induced forest.
- (●) L: leaves in F.
- (●) I: inner vertices in F.



- (red) S : a solution (fvs).
- (green blue) $F = V \setminus S$: an induced forest.
- (green) L: leaves in F.
- (blue) I: inner vertices in F.
- (blue 2) $I_2 \subseteq I$: 2 neighbors in F.
- (blue 3) $I_{3+} \subseteq I$: 3+ neighbors in F.
- (green 2) $L_2 \subseteq L$: 2 neighbors in S.
- (green 3) $L_{3+} \subseteq L$: 3+ neighbors in S.



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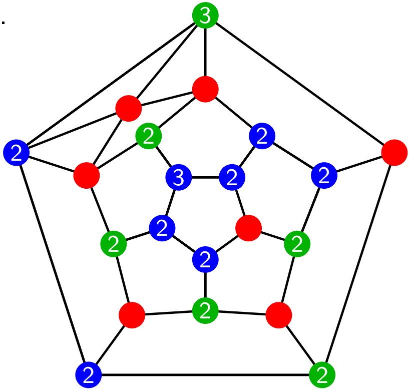


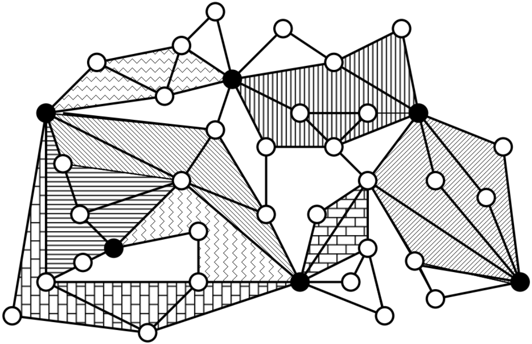
Goal:
show $|L_2| + |L_{3+}| + |I_2| + |I_{3+}| = O(k)$.

Analysis: Region Decomposition

Fix a solution S of size k.

Goal: Show that $n = O(k)$.

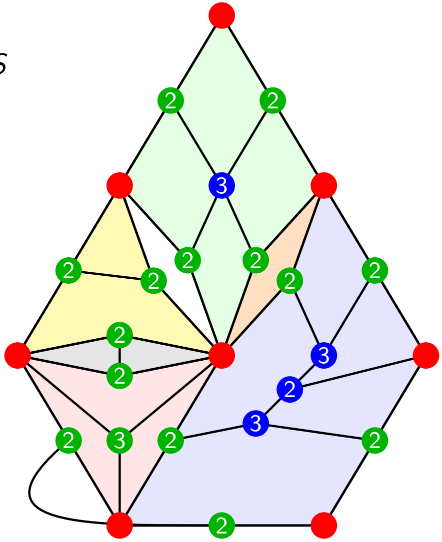
Region Decomposition Technique (Alber, Fellows, Niedermeier 2002)



Decompose plane into $O(k)$ regions, each contains $O(1)$ vertices.

Our regions: faces of multigraph $H_S = (S, E_S)$

For every path uxv such that $u, v \in S$ and $x \in L_2$ put an edge uv in E_S.



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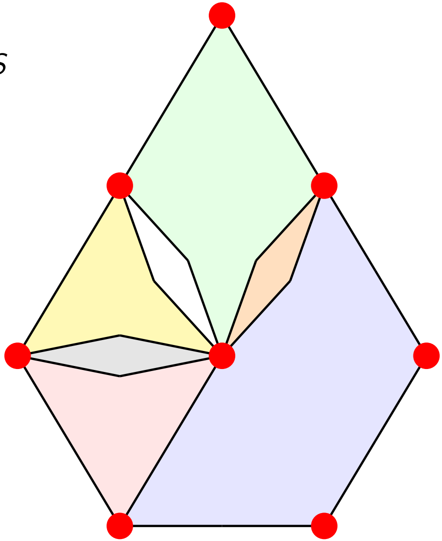

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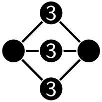
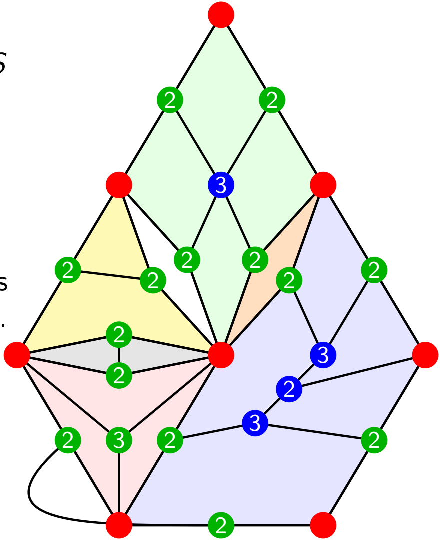
Because 

is excluded, H_S has no triple (or larger multiplicity) edges.

Then,

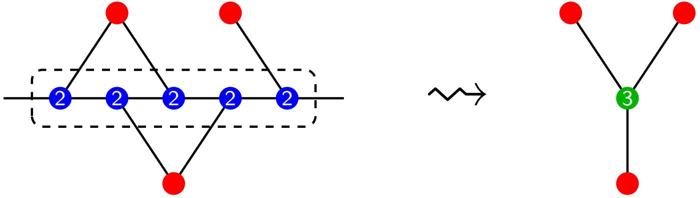
$$|L_2| \leq 2E_S \leq 2(3|S| - 6) < 6k.$$



Bounding $|I_2|$

Observation [Abu-Khzam, Khuzam]

Induced paths of I_2 vertices with at least three neighbors in S behave similarly as vertices in L_3.



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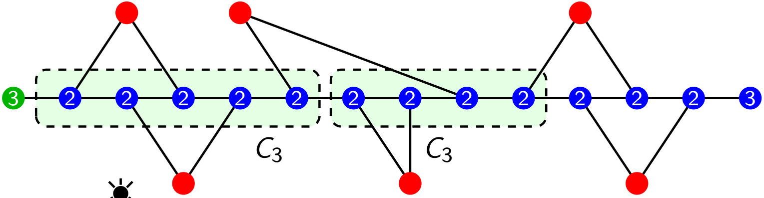
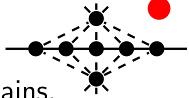
Dealing with I_2:

- Identify set C_3 of chains of I_2 vertices with 3+ neighbors in S:



- Since  is excluded, there are at most $5|C_3|$ vertices in C_3 chains,
- There are few remaining I_2 vertices: $4(|L_2| + |L_3| + |I_3|)$.

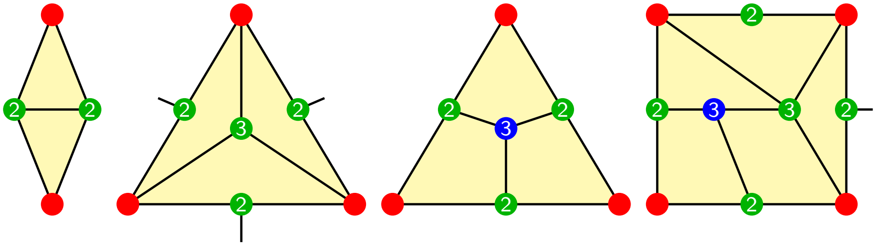
Bounding $|L_3|$, $|I_3|$ and $|C_3|$.

- Let f be a face of the region graph H_S.
- Let L_3^f, I_3^f and C_3^f denote elements of relevant sets inside f.
- $d(f)$ denotes the length of f.

Lemma (perhaps *key* lemma)

$|L_3^f| + |I_3^f| + |C_3^f| \leq d(f) - 2$

Some handwaving:



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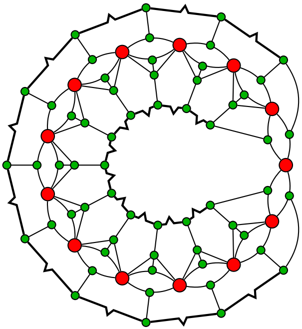
Then:

$$\begin{aligned}
|L_3| + |I_3| + |I_2| &\leq |L_3| + |I_3| + |V(C_3)| + 4(|L_2| + |L_3| + |I_3|) \\
&\leq 5(|L_3| + |I_3| + |C_3|) + 4|L_2| \leq 5(|L_3| + |I_3| + |C_3|) + 24k \\
&= 5 \sum_{f \in F_S} (|L_3^f| + |I_3^f| + |C_3^f|) + 24k \\
&= 5 \sum_{f \in F_S} (d(f) - 2) + 24k \\
&= 5(2|E_S| - 2|F_S|) + 24k = 5 \cdot 2(|S| - 2) + 24k < 34k.
\end{aligned}$$

The final bound

- From the bounds we have seen one can get, $|S| + |L_2| + |L_3| + |I_3| + |I_2| \leq 41k$.
- For $14k$ we used a few (minor) analysis tricks plus a few more kernelization rules.
- We have a tight example:



where  is a chain of four I_2 vertices.

Conclusions

- $14k$ kernel
- (journal version: $13k$ kernel)
- Region decomposition technique applied
- Unlike in the Alber et al. paper, our regions are not of $O(1)$ size.
- Analysis is tight.

A prize to win!

WANTED

A **single-digit** kernel for Planar FVS

A prize to win!

WANTED

A **single-digit** kernel for PLANAR FVS

Reward:

and

a bottle of French wine

a bottle of Polish vodka

Thank you!

The presentation contains some figures of Felix Reidl from a book

Cygan, Fomin, Marx, Kowalik, Lokshtanov, Pilipczuk, Pilipczuk, Saurabh
Parameterized Algorithms

(to appear in early 2015)