A New 3-color Criterion for Planar Graphs (extended abstract)

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Abstract. We present a new general 3-color criterion for planar graphs. Applying this criterion we characterize a broad class of 3-colorable planar graphs and provide a corresponding linear time 3-coloring algorithm. We also characterize fully infinite 3-colorable planar triangulations.

1 Introduction

The problem of vertex coloring of a graph using few colors has given rise to one of the most intensively studied areas of the graph theory. A frequently encountered special case is that in which the graph to be colored is planar. Computing a coloring that uses the smallest possible number of colors is known to be an NP-complete problem, even when restricted to the class of planar graphs and 3 colors. More precisely it is an NP-complete problem to decide whether a given planar graph is 3-colorable [GJS]. On the other hand the famous „4-color theorem" says that every planar graph is 4-colorable. Hence it is natural to characterize those planar graphs which are 3-colorable. The first 3-color criterion was formulated by Heawood in 1898 and it is known as the Three Color Theorem [Hea,Ste]: *A maximal planar graph is vertex colorable in three colors if and only if all its vertices have even degrees.* Obviously this theorem implies a very simple algorithm for checking 3-colorability of maximal planar graphs. As the problem of 3-colorability of planar graphs is NP-complete one cannot rather expect any „effective" 3-color criterion for general planar graphs. On the other hand there is a „simple", general 3-color criterion which does not lead to an efficient 3-coloring algorithms: *A planar graph is vertex colorable in three colors if and only if it is a subgraph of a maximal planar graph in which all vertices have even degrees.* This theorem was already known to Heawood as well as it was discovered independently by several authors – see the comprehensive survey written by Steinberg [Ste].

In this paper we introduce a new general 3-color criterion which can be efficiently checked for a broad class of planar graphs. Our criterion generalizes the Heawood's Three Color Theorem. In order to get this result we define a new class of planar graph colorings, so called *edge-side colorings*, and prove that the new type of coloring is equivalent to the ordinary vertex 3-coloring. The criterion allows to characterize 3-colorable triangulations with holes, i.e. plane graphs in which each vertex touches at most one non-triangular face. We provide a linear time algorithm for 3-coloring such graphs. Our criterion allows also to formulate sufficient and necessary conditions for 3-coloring infinite planar triangulations.

2 Basic Definitions and Notation

It is known that a graph is 3-colorable iff all its biconnected components are 3-colorable. In the sequel, if it is not stated explicitly, saying a graph we mean a biconnected, finite multigraph of at least three vertices but without selfloops.

A *plane graph* is a graph whose vertex set is a point set in the plane and the edges are Jordan curves such that two different edges have at most end points in common. A graph is called *planar* if it can be embedded in the plane, i.e. if it is isomorphic to a plane graph.

Let C be a simple cycle in a plane graph G. The cycle C divides the plane into two disjoint open domains – the *interior C-domain* (homeomorphic to an open disc) and the *exterior C-domain.* The set consisting of all vertices of G belonging to the interior C-domain and of all edges crossing this domain is denoted by $\operatorname{Int} C$. If v is a vertex on C then the number of the graph neighbors of v lying in the interior C-domain is called the *internal degree* of v with respect to C and it is denoted by $dIn(C,v) = |\{(v,w) \in E(G) \ : \ w \in \operatorname{Int} C\}|$. We define the internal degree of the cycle C as the sum of the internal degrees of all its vertices. This sum is denoted by $dIn(C) = \sum_{v \in C} dIn(C,v)$.

A *face* in a plane graph G is a C-domain (interior or exterior), for some cycle C, without any vertices and edges inside. Only one face is unbounded and it is called the *outer* face. Similarly, its boundary cycle is called the *outer* one.

A *triangulation* is a plane graph in which the boundary cycle of every face is a triangle (3-cycle). A biconnected plane graph in which all the boundary cycles, except at most one, are triangles is called a *near-triangulation.* W.l.o.g. we will consider this boundary cycle to be the outer one.

A graph is *even* if all its vertices have even degrees. A near-triangulation is *internally even* if all its vertices different from those on the outer cycle have even degrees.

3 A New 3-color Criterion

In 1898 Heawood [Hea] proved a theorem characterizing (finite) 3-colorable triangulations:

Theorem 1 (Three Color Theorem). *A (finite) triangulation is 3-colorable if and only if it is even.*

This criterion applies only to the *maximal* planar graphs which are isomorphic to triangulations. It allows to check in a very simple manner whether a given maximal planar graph is 3-colorable. One can ask a natural question: can the criterion be generalized to all planar graphs? Unfortunately, since checking

3-colorability is an NP-complete problem even in the planar case, we cannot rather expect any polynomially checkable criterion for general planar graphs. However there are general criteria which allow checking 3-colorability of a given planar graph in some special cases. As stated in [Ste] such a general criterion was already known to Heawood [Hea]. Nevertheless it was not widely known and has been independently discovered and proved several times, e.g. in [Kr1], [Kr2], [Mar]. The criterion follows:

Theorem 2 (Heawood's 3-color Criterion). *Let G be a plane graph. The following two conditions are equivalent:*

(i) G is 3-colorable.
(ii) There exists an even triangulation H such that G is a subgraph of H, i.e. $H \supseteq G$.

Moreover, every 3-coloring of a plane graph G can be extended to a 3-coloring of some even triangulation $H \supseteq G$.

As we can see the 3-color criterion stated above tells us nothing about the structure of the graph under consideration. In this section we provide a new type of graph coloring, called *edge-side coloring*, which is equivalent to the vertex 3-coloring but additionally reflects some structural properties of a given graph. This new feature will allow us to characterize a new, broad class of 3-colorable planar graphs which are recognizable and 3-colorable in a linear time.

We start from a few indispensable definitions.

Let G be a plane graph, f a face in G and e an edge on the boundary cycle of f. The pair $s = (e, f)$ is called a *side of edge e in face f* (or shortly a *side*). We say also that side s touches face f. If vertex v is an end point of e then side s is said to be incident with v. Observe that in a biconnected graph every edge has exactly two sides.

Let G be a plane graph and S be the set of all sides in G. *Edge-side coloring* of G is an arbitrary function

$$m : S \longrightarrow \{black, white\}.$$

Edges with one side black and the other side white are called *b-w edges*. The other edges are called *one-color edges* and can be of type *b-b* or *w-w* depending on the colors of their sides, black or white respectively.

We say that an edge-side coloring of a plane graph G is *proper* if and only if the following two conditions are satisfied:

(i) for each face f in G the numbers of white and black sides touching f are congruent (equal) mod 3;
(ii) each vertex v in G is incident with an even number of one-color edges.

We say that a plane graph G is *edge-side colorable* if its edge-sides can be properly colored.

Now we can state the main theorem of the paper.

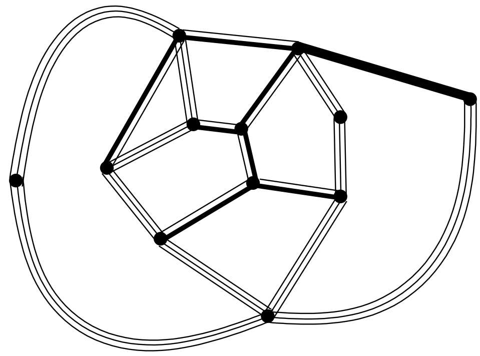

Fig. 1. A proper edge-side coloring. Light (dark) lines indicate white (black) sides

Theorem 3 (3-color Criteria). *Let G be a (biconnected) plane graph. The following three conditions are equivalent:*

(i) G is 3-colorable.
(ii) There exists an even triangulation $H \supseteq G$.
(iii) G is edge-side colorable.

The equivalence of conditions (i) and (ii) was proved by Heawood (see Theorem 2). The proof of the equivalence of (ii) and (iii) is our main contribution to this paper. We start from a few observations on internally even near-triangulations.

Lemma 1. *Every internally even near-triangulation is 3-colorable.*

Proof. Let C be the outer cycle of a near-triangulation G. Let us take a separate embedding G' of G in which the cycle C' corresponding to C is not longer the outer one. Now we build a new plane graph H from G' placing the entire graph G in the interior C'-domain and identifying the corresponding vertices and edges of the cycles C and C' as shown in Fig. 2. One can easily check that the graph H is an even plane triangulation and hence it is 3-colorable by the Heawood's Three Color Theorem. Since $G \subseteq H$ it is also 3-colorable. □

Lemma 2. *Let G be an even near-triangulation with the outer cycle C. Then* $|C| \equiv 0 \pmod 3$.

Proof. By Lemma 1, G is 3-colorable. Let $C = v_0 v_1 \ldots v_{|C|-1} v_0$ and let $\mathcal{K}$ be an arbitrary 3-coloring of G. We will show that one can rename the colors in $\mathcal{K}$ in such a way that $\mathcal{K}(v_i) = (i \bmod 3) + 1$, for every $i = 0 \ldots |C| - 1$.

Let v be an arbitrary vertex in C and let x and y be its neighbors such that x, y and v are incident with the same internal (not unbounded) triangular face in G. Vertices x, y, and v have different colors. Now one can observe that

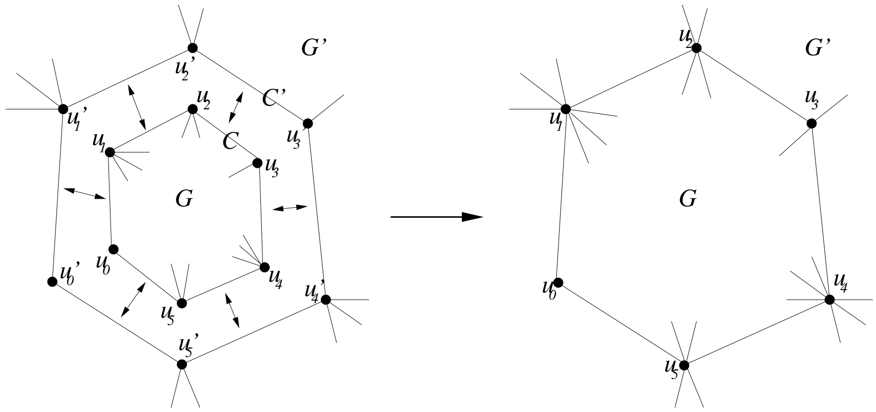

Fig. 2. Proof of lemma 1

every two successive neighbors of v, in a sequence of all neighbors listed in the clockwise order, have different colors. Since the degree of v is even its neighbors on C have different colors. As a result we get $\mathcal{K}(v_i) = (i \bmod 3) + 1$ (possibly after renaming the colors) what implies $|C| \equiv 0 \pmod 3$. □

Lemma 3. *For every $i \geq 3$ such that $i \equiv 0 \pmod 3$ there exists a finite even near-triangulation with the outer cycle of length i.*

Proof. The proof is by induction on i. For $i = 3$ it suffices to take K_3.

Inductive step: by the induction hypothesis there exists a finite, even near-triangulation G_0 with the outer cycle of length $i - 3$. Let v_1, v_2 be arbitrary adjacent vertices in the outer cycle of G_0. Then $G = G_0 \cup \{v_3, v_4, v_5\} \cup \{v_2 - v_3, v_3 - v_4, v_4 - v_5, v_5 - v_1, v_2 - v_4, v_1 - v_4\}$ is an even near-triangulation and its outer cycle has length i. □

Let C be a simple cycle and let m_C be an arbitrary edge 2-coloring of C, $m_C : E(C) \longrightarrow \{black, white\}$. We say that coloring m_C is *balanced* if and only if $|m_C^{-1}(black)| \equiv |m_C^{-1}(white)| \pmod 3$.

Let G be a graph and let C be a simple cycle in G. We say that a balanced coloring m_C of C *corresponds* with G if the following holds: for every vertex v in C the edges of $E(C)$ incident with v have different colors if and only if the degree $d_G(v)$ of v in G is odd.

Lemma 4 (Key lemma).

(i) For every internally even near-triangulation G with the outer cycle C there exists a balanced 2-coloring m_C of C corresponding with G.

(ii) For every balanced 2-coloring m_C of a cycle C there exists an internally even near-triangulation G with the outer cycle C and such that m_C corresponds with G.

Proof (i). Since G is internally even, the number of vertices of C with odd degrees is even. Let $v_1, v_2, \ldots, v_{2k-1}, v_{2k}$ be a list of all such vertices given in the clockwise order. For each $i = 1, 2, \ldots, k$, we color black edges on C between v_{2i-1} and v_{2i}. The remaining edges are colored white (see Fig. 3). Observe that vertices in C are incident with edges of different colors if and only if they have odd degrees.

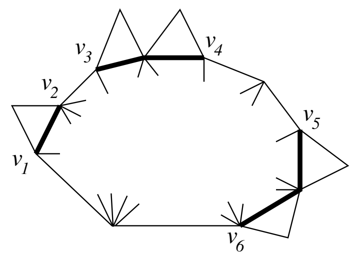

Fig. 3. Constructing a balanced 2-coloring

Let b be the number of black edges on C and let w be the number of white edges on C. After extending G by triangles formed in the outer (unbounded) face and with the black edges as the triangle bases (as shown in Fig. 3) we get an even near-triangulation. By Lemma 2 the outer cycle of this triangulation has length $\equiv 0 \pmod 3$. Hence $2b + w \equiv 0 \pmod 3$ and finally $b \equiv w \pmod 3$. □

Proof (ii). Denote the number of black and white edges of C by b and w respectively. We form a triangle on each black edge e in the interior C-domain as shown in Fig. 4. As the result we get a graph H. Observe that vertex v of H has odd degree if and only if it is incident on C with edges of different colors.

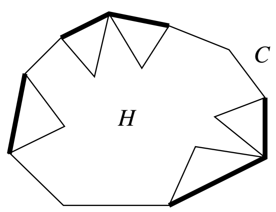

Fig. 4. The graph H with the outer cycle C – constructing a near-triangulation

Let f be the only face of H different from the added triangles and placed in the interior C-domain. The length of the f's boundary cycle is $2b + w \equiv 0 \pmod 3$. By Lemma 3, one can triangulate this face and obtain a required internally even near-triangulation G. □

Lemma 5. *Let G be a finite, biconnected plane graph. Let H be an even triangulation (possibly infinite but locally finite) such that $G \subseteq H$. If for every face f of G with facial cycle C_f there exists a balanced edge 2-coloring corresponding with $C_f \cup \mathrm{Int}_H C_f$, the graph G is edge-side colorable.*

Proof. For every face f of G with the facial cycle C_f let m_{C_f} be a balanced edge 2-coloring corresponding with $C_f \cup \mathrm{Int}_H C_f$. Let us take an edge-side coloring assigning each side (f, e) the color $m_{C_f}(e)$. Obviously this coloring satisfies the first condition of the definition of the proper edge-side coloring. In order to prove the other one let us consider an arbitrary vertex v in G. Let f' be an arbitrary face in G with v on its facial cycle. Denote this cycle by C'. Let e_1, e_2 be the edges of C' incident with v. Since $m_{C_{f'}}$ corresponds with $C_{f'} \cup \mathrm{Int}_H C_{f'}$ the sides of e_1, e_2 in f' have the same color if and only if the degree $dIn_H(C', v)$ is even. Let $B(v)$ denote the number of black sides incident with v. If $d_G(v)$ is odd then there is an odd number of faces f' incident with v and such that $dIn_H(C', v)$ is odd. On the other hand if $d_G(v)$ is even the number of faces f' incident with v and such that $dIn_H(C', v)$ is odd is even. It follows that $d_G(v) + B(v)$ is always even which is equivalent to the statement that the number of one-color edges incident with v is even.

We have just showed that the second condition in the definition of proper edge-side coloring is satisfied, which completes the proof. □

Now we are ready to prove the part (ii)$\leftrightarrow$(iii) of our main theorem.

Proof.
(ii)$\longrightarrow$(iii)
Assume that there exists an even triangulation $H \supseteq G$. Observe that since G is biconnected, every face is bounded by a simple cycle. For each face f with the facial cycle C_f we can apply lemma 4 to get a balanced edge 2-coloring m_C of $E(C)$ corresponding with the near-triangulation $C_f \cup \mathrm{Int}_H C_f$. Now we can apply lemma 5 to obtain a proper edge-side coloring of G, what completes the proof of (ii)$\longrightarrow$(iii).

(ii)$\longleftarrow$(iii)
Assume that G is properly edge-side colored. By Lemma 4 one can triangulate (i.e. divide into triangles) each face into an internally even near-triangulation getting a triangulation $H \supseteq G$. Let v be an arbitrary vertex of G. Denote by $F(v)$ the number of faces incident with v for which the odd number of edges ending in v was added during the process of triangulation. Similarly as in the proof of lemma 5 one can show that $d_G(v) + F(v)$ is even. It implies finally that for every vertex v, $d_H(v)$ is even, what means that H is an even triangulation. □

As the result we get a new 3-color criterion for general planar graphs. In fact, using this criterion for checking whether an arbitrary plane graph is 3-colorable is equally hard as trying to find a proper 3-coloring of a given graph. However we can apply our theorem to show a few classes of planar graphs for which the new criterion can be effectively checked.

4 Applications

One can expect that the criterion formulated in section 3 can be effectively checked for graphs that are "highly triangulated", i. e. when *a lot* of faces are triangles. Moreover, if such a graph has a special structure it can be colored using a greedy algorithm. We define formally a class of graphs for which the greedy algorithm works well. A plane graph G is called *triangle connected* if each vertex of G is incident with a triangular face and the subgraph of the graph dual to G induced by the triangular faces is connected.

In the following subsections we present the greedy algorithm and three classes of graphs for which effective 3-color criteria can be formulated. Our general criterion can be also used to show that plane graphs with face lengths of multiple of three are 3-colorable.

4.1 The Greedy Algorithm

Given a planar, triangle connected graph (without its planar embedding) the algorithm below computes its 3-coloring or reports that such a coloring doesn't exist. The algorithm runs in a linear time. For each vertex v set PossibleColors(v) contains colors which are still admissible for v; S represents the set of vertices for which set PossibleColor contains at most one color. Algorithm uses operation `RESTRICT(v)` which restricts the set of admissible colors for neighbors of v.

```
OPERATION RESTRICT(v)::
for each u in Neighbors(v) do
  if Col(u) = -1 then
  begin
    PossibleColors(u).Remove(Col(v))
    if |PossibleColors(u)| ≤ 1 then S.Add(u)
  end
```

```
ALGORITHM GREEDY::
for each v in V(G) do
begin
  PossibleColors(v) := {1, 2, 3}
  Col(v) := -1 {undefined}
end
S := ∅
(p, q) := an arbitrary edge of an arbitrary triangle in G
Col(p) := 1
Col(q) := 2
RESTRICT(p)
RESTRICT(q)

while not S.Empty do
```

```
begin
  v := S.Remove
  if |PossibleColors(v)| ≠ 1 then
    Exit{G is not 3-colorable}
  else
  begin
    Col(v) := PossibleColors(v).Get
    RESTRICT(v)
  end
end while
```

Finding a triangle in a planar graph without its embedding in the plane can be easily done in a linear time, see [Chr]. During each iteration algorithm chooses a triangle with two vertices already colored and colors the third vertex. It can be easily shown that in every embedding one of edges of the initial triangle bounds a triangular face. As graph is triangle connected algorithms stops when all vertices are properly colored.

4.2 Triangulations with Holes

A biconnected plane graph is called a *triangulation with holes* if every of its vertices is incident with at most one non-triangular face, i.e. a face of length at least 4.

Proposition 1. *Every triangulation with holes is triangle connected.*

Proof. Let G be a triangulation with holes. Consider two triangular faces f, h sharing a common vertex v. Vertex v is incident with at most one non-triangular face and subsequently f and h can be connected by a sequence formed of triangular faces, where each two successive faces share a common edge.

Now let f and h be two arbitrary triangular faces of G. Since G is connected, f and h can be connected by a path $e_1, e_2, \ldots, e_k$, where e_i are edges of G. Each edge belongs to at least one triangular face. Denote such face for e_i by t_i. Every two successive faces t_i, t_{i+1} share a vertex. Additionally t_1 shares a vertex with f and t_k shares a vertex with h. Hence we conclude that f and h are connected by a path formed of triangular faces, where each two successive faces share a common edge. □

Triangulations with holes have the following interesting property. Let G a triangulation with holes and let f be a face in G. Let C_f be the facial cycle of f. Then for every vertex v in C_f and an arbitrary even triangulation $H \supseteq G$ the parity of $dIn_H(C_f, v)$ is the same. It follows that there is exactly one edge-side coloring for every triangulation with holes (if not to consider isomorphic ones). This implies a very simple 3-color criterion for triangulations with holes.

We say that a triangulation with holes is *internally even* when the degree of every vertex incident with triangular faces only is even.

Theorem 4. *A triangulation with holes G is 3-colorable if and only if*

(i) it is internally even,
(ii) for every non-triangular face f with the facial cycle C_f there exists a balanced edge 2-coloring m_{C_f} of C_f corresponding with G.

Proof. The proof follows easily from Theorem 3. Let G be a triangulation with holes satisfying conditions *(i)* and *(ii)*. We shall show that G is 3-colorable. For each triangular face f, we color all its sides (e, f) black. For each non-triangular face f with facial cycle C_f we color every side (e, f) with color $m_{C_f}(e)$. The constructed edge-side coloring is balanced for each face and it is easy to check that every vertex is incident with an even number of one-color edges. Now it suffices to use theorem 3.

Now we will show the other implication. Let G be a 3-colorable triangulation with holes. Using theorem 3 we can obtain its proper edge-side coloring m. We recolor black all sides of all triangular faces obtaining a new edge-side coloring m'. One can see that m' is also proper. Now all vertices touching only triangular faces are ends of only one-color (black) edges. Hence they have even degrees. Moreover one can verify that for every non-triangular face f with the facial cycle C_f the coloring $m_{C_f}(e) = m'(f, e)$ corresponds with G.

□

4.3 Near-triangulations and Outerplanar Graphs

Obviously near-triangulations are triangulations with holes. Informally speaking a near-triangulation is a *triangulation with only one hole*. It gives a very simple 3-color criterion for near-triangulations:

Theorem 5. *A near-triangulation is 3-colorable if and only if it is internally even.*

Proof. Lemma 4 implies that the second condition of the Theorem 4 is always satisfied for near-triangulations. □

As every outerplanar graph is a subgraph of a certain internally even near-triangulation we immediately get the following known result:

Corollary 1. *Outerplanar graphs are 3-colorable.*

4.4 Plane Graphs with Faces which Lengths Are of Multiple of 3

The following theorem ([Ore], [Ste]) easily follows from our criterion:

Theorem 6. *Let G be a graph embedded in the plane in such a way that the number of edges in the boundary of each face is a multiple of 3. If G is even then G is 3-colorable.*

Proof. It suffices to color all edge-sides in G black and to apply Theorem 3. □

4.5 Infinite Triangulations

It is surprising that we can apply our criterion to infinite plane graphs. An *infinite triangulation* is an infinite plane graph with all faces being triangles. We will consider only locally finite triangulation where degrees of all vertices are finite. An *edge accumulation point* (shortly EAP) of an infinite plane graph G is a point P such that for every positive real number ϵ there are infinitely many edges of G with Euclidean distance from P less than ϵ. We will show that the Three Color Theorem holds also for EAP-free infinite triangulations.

Theorem 7. *An EAP-free infinite triangulation is 3-colorable if and only it is even.*

Proof. Assume that EAP-free infinite triangulation G is 3-colorable. Let v be a vertex of odd degree. Since arbitrary two successive neighbors (in clockwise order) of v are adjacent they have different colors. As there is an odd number of neighbors of v we need 3 colors to color them and there is no color left for v. We have just proved implication $(\longrightarrow)$.

Assume that G is even. Let v_0 be an arbitrary vertex of G. We define a sequence of graphs

$$G_0 \subset G_1 \subset G_2 \subset G_3 \ldots$$

Let $V(G_0) = \{v_0\}$ and $E(G_0) = \emptyset$. Let W_i be the set of vertices with the graph distance at most i from v. Since graph $G(W_i)$ is finite, it has the outer face f_i with the facial cycle C_i. Obviously there are no cut vertices in $G(W_i)$. Therefore C_i is a simple cycle. We define G_i as $C_i \cup \mathrm{Int}_G\, C_i$. Since G is EAP-free for every natural i, G_i is a finite graph. Moreover G_i is an internally even near-triangulation. It follows from Theorem 5 that graphs G_i are 3-colorable. Since $G_{i-1} \subseteq G_i$, for $i > 1$, and 3-colorings of G_i and G_{i-1} are unique (i. e. they define the unique partition of the vertices into 3 independent subsets) we can construct 3-colorings $\mathcal{K}_i$ for graphs G_i, $i = 0, 1, 2, \ldots$, such that $\mathcal{K}_{i|G_{i-1}} = \mathcal{K}_{i-1}$. Now we can define a 3-coloring of G as $\mathcal{K}(u) = \mathcal{K}_{d(v_0,u)}(u)$, where $d(v_0, u)$ denotes the graph distance from v_0 to u. □

It is easy to show examples of infinite even triangulations with EAP that are even, but *not* 3-colorable. The construction of such triangulation is shown in Fig. 5. One can see that even first graph in this sequence is not 3-colorable.

In the sequel we use the following well-known fact.

Fact. Let G be an infinite graph. If every finite subgraph of G is k-colorable then G is k-colorable.

Theorem 8. *Let G be an infinite but locally finite triangulation (not necessarily EAP-free). G is 3-colorable if and only if*

(i) G is even,
(ii) for every simple cycle C in G there exists a balanced edge 2-coloring of C corresponding with $C \cup \mathrm{Int}_G\, C$.

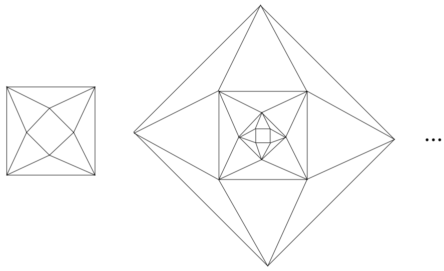

Fig. 5. A construction of not 3-colorable even infinite triangulation

Proof. Assume that G is 3-colorable. Then obviously G must be even. Now let us consider an arbitrary cycle C in G. Let $V_H \subset V$ be a set of vertices defined as follows: $v \in V_H$ if and only if v has a neighbor in $V(C)$ and $v \in C \cup \mathrm{Int}_G C$. Let $H = G(V_H)$. As $H \subset G$, H is 3-colorable. It is easy to see that H is a biconnected graph. Therefore we can apply Theorem 3 and get a required edge 2-coloring m_C corresponding with $C \cup \mathrm{Int}_G C$.

Now we prove that if (i) and (ii) hold then G is 3-colorable. It suffices to prove that if G satisfies (i) and (ii) then every finite subgraph of G is 3-colorable. Let F be a finite subgraph of G. W.l.o.g. one can assume that F is biconnected. If not, F is a subgraph of a certain biconnected graph $G(W_i)$ defined in the proof of the previous theorem. Now we can use Lemma 5 to get a proper edge-side coloring of F and finish the proof using Theorem 3. □

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