LECTURE 6

SOME FINITE DIFFERENCE SCHEMES FOR TRANSPORT AND HEAT EQUATIONS IN 1D

TRANSPORT EQUATION $u_t + \alpha u_x = 0$.
Grid functions: $u_k^n \approx u(t_n, x_k)$, $t_n = \tau n$, $x_k = kh$, $\lambda = \frac{\tau}{h}$.

1. **BOX LEFT=>RIGHT (implicit-explicit)**

   $$au_{k+1}^{n+1} + bu_k^{n+1} = bu_{k+1}^n + au_k^n \tag{6.1}$$

   where $a = (1+\alpha\lambda)$, $b = (1-\alpha\lambda)$. For $\alpha > 0$ equation (6.1) is solved in u_{k+1}^n. Scheme of order $O(\tau) = O(h)$, stability test $u_k^n = \gamma^n e^{-isk}$ unconditionally succesful.

2. **UPWIND SCHEMES (explicit)**

   $$u_k^{n+1} = u_k^n(1+\alpha\lambda) - \alpha\lambda u_{k+1}^n, \ \alpha < 0 \tag{6.2}$$

   $$u_k^{n+1} = u_k^n(1-\alpha\lambda) + \alpha\lambda u_{k-1}^n, \ \alpha \geq 0 \tag{6.3}$$

   Schemes of order $O(\tau) = O(h)$. Stability test $u_k^n = \gamma^n e^{-isk}$ succesful if $\lambda \leq \frac{1}{|\alpha|}$.

3. **LAX-FRIEDRICHS (explicit)**

   $$\frac{1}{\tau}(u_k^{n+1} - \frac{u_{k-1}^n + u_{k+1}^n}{2}) + \alpha\frac{u_{k+1}^n - u_{k-1}^n}{2h} = 0 \tag{6.4}$$

   or

   $$u_k^{n+1} = \frac{(1+\alpha\lambda)}{2}u_{k-1}^n + \frac{(1-\alpha\lambda)}{2}u_{k+1}^n = 0 \tag{6.5}$$

   Scheme is of order $O(\tau) = O(h)$, stability test $u_k^n\gamma^n e^{-isk}$ is succesful if $\lambda \leq \frac{1}{|\alpha|}$, **independently of the sign of** α**.**

4. **Be careful!** In the following scheme

$$\frac{u_k^{n+1} - u_k^n}{\tau} + \alpha \frac{u_{k+1}^n - u_{k-1}^n}{2h} = 0 \tag{6.6}$$

the derivative u_x is approximated by the so called **central finite difference of the order $O(h^2)$**, hence it seems to be "beter" then, for example, the schemes of the type **"upwind" (6.2) (6.3)**. Unfortunatelly, the stability test $u_k^n = \gamma^n e^{-isk}$ for the scheme (6.6) **never passes with succes.**

HEAT EQUATION $u_t = \nu u_{xx}$, $\nu > 0$ (will follow)