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Toric Mori Theory and Fano Manifolds

Grenoble lectures – by J. A. Wiśniewski

These are the notes to five lectures which I gave during the school on toric geometry in Grenoble in the Summer of 2000. The first week of the three week long school was meant to introduce the basics of toric geometry to the students while the other two weeks were devoted to advanced topics. Therefore the idea of the present notes is to give a brief and self-contained introduction to an advanced and broad topic to students who have just learned the fundamentals of toric language.

I claim no originality on the contents of these notes. Actually, they are primarily based on [Reid83]. An exposition of Mori theory in general can be found in [CKM88]. Moreover Lecture 3 uses ideas of [Reid92] while Lecture 5 is related to [Batyrev99].

All varieties are algebraic and defined over **C**.

Plan of lectures:

0. Short Introduction: Minimal Model Program.
1. Cone Theorem.
2. Contraction Theorem.
3. Flip and Flop.
4. Canonical Divisor.
5. Fano Manifolds.

Short introduction: Minimal Model Program.

In the course of the first week's lectures you have learned basics of toric geometry. You must have noticed that the theory is nice, clear and elegant, even too good to be true. And, indeed, that's right: toric varieties are very, very rare among algebraic varieties, so don't be confused: toric geometry is less than tip of the iceberg of algebraic geometry. Nevertheless it is very, very useful. Firstly because you can test your theories and conjectures (wisely posed!) in the toric environment. Secondly, because, as special as it is, toric geometry gives a very close insight in the local structure of varieties, where "local" is in analytic or formal neighborhood sense (not Zariski!). I'll try to illustrate these two principles in the course of my lectures.

We set for the classification of complex projective varieties of given dimension. Our primary example are complex curves (or Riemann surfaces). The contents of the following table is referred to frequently when it comes to explaining principles of classification theory which includes the apparent trichotomy.

Sphere with g handles:	$g = 0$	$g = 1$	$g \geq 2$
Fundamental group:	trivial	$\mathbf{Z}^2$	$2g$ generators
Curvature:	positive	zero	negative
Holomorphic forms:	none	non-vanishing	g independent

Holomorphic vector fields:	2 independent	non-vanishing	none
Canonical divisor K_X:	negative	zero	positive

I shall focus on the canonical divisor. Let me recall the following:

Definition. Let X be a normal variety of dimension n, with $X_0 \subset X$ denoting its smooth part. The canonical divisor K_X is a Weil divisor obtained by extending the divisor K_{X_0} associated to the sheaf of holomorphic n-forms $\Omega^n_{X_0} = \Lambda^n(\Omega_{X_0})$.

We will need moreover the following.

Definition. Let L be a $\mathbf{Q}$-Cartier divisor on a normal variety X, that is, a multiple mL, with $m \in \mathbf{Z}$ is a Cartier divisor. We say that L is nef (numerically effective) if the intersection $L \cdot C = (1/m) deg_C(mL_{|C})$ is non-negative for any compact curve $C \subset X$.

Apart from the curve case we have the following observation in dimension 2 which is an easy corollary to Enriques-Kodaira classification of smooth complex surfaces: each projective surface can be modified birationally – using blow-ups and blow-downs – either to a $\mathbf{P}^1$ bundle over a curve, or to a surface with nef canonical divisor.

Based on this (very roughly presented) evidence one can state

Minimal Model Conjecture. Any projective normal variety X is birationally equivalent to a normal projective variety X' which satisfies one of the following:

(i) X' admits Fano-Mori fibration $\varphi : X' \longrightarrow Y$, that is: φ is a projective morphism with connected fibers, $\varphi_* \mathcal{O}_{X'} = \mathcal{O}_Y$, onto a normal variety Y, with $dimY < dimX'$, and $-K_{X'}$ ample on fibers of φ, or

(ii) X' is minimal which means that $K_{X'}$ is nef (such X' is then called a minimal model of X).

At this point I am rather vague about possible singularities of the involved varieties, however we have to assume that $K_{X'}$ is $\mathbf{Q}$-Cartier at least.

Here is an idea how to approach the Conjecture:

(1) Locate curves which have negative intersection with canonical divisor, understand their position in homology of X: use Cone Theorem [Mori, Kawamata].

(2) Eliminate some of these curves by contracting them to points: use Contraction Theorem [Kawamata, Shokurov]; chances are that we shall get Fano-Mori fibration, or we get a birational morphism to a simpler variety; unfortunately the birational map may also lead to a variety with very bad singularities (case of small contractions), so that the canonical divisor is not $\mathbf{Q}$-Cartier.

(3) If the contraction leads to bad singularities use birational surgery (flips) to replace curves which have negative intersection with K_X by curves which have positive intersection with K_X: this should be possible by Flip Conjecture (proved by Mori in dimension 3).

Although the Minimal Model Conjecture is void for toric varieties (they are rational, hence birational to a Fano-Mori fibration), they can be used effectively to test steps of the Program and possibly to describe local (in the analytic, or formal sense) geometry of non-minimal varieties. In the course of the present lectures I will review the main ideas of Minimal Model Program in the situation of toric varieties.

Let me recall toric notation.

$M \simeq \mathbf{Z}^n$	lattice of characters of a torus $T \simeq (\mathbf{C}^*)^n$
$N = Hom(M, \mathbf{Z})$	lattice of 1-dimensional subgroups of T
$M_{\mathbf{R}}$ and $N_{\mathbf{R}}$	vector spaces in which they live
$\langle v_1, \ldots, v_k \rangle$	convex cone spanned on vectors $v_1, \ldots, v_k$
$conv(v_1, \ldots, v_k)$	(affine) convex hull of points $v_1, \ldots, v_k$
$X = X(\Delta)$	toric variety associated to a fan Δ in $N_{\mathbf{R}}$
$\Delta(k)$	the set of k dimensional cones in Δ
$V(\sigma) \subset X(\Delta)$	stratum (= closure of the orbit) associated to a cone $\sigma \in \Delta$

Moreover, I will frequently confuse rays in $\Delta(1)$ with primitive elements from N generating them: for a ray $\rho \in \Delta(1)$ I will always consider the (unique) primitive element $e \in N \cap \rho$.

We usually assume that fans are *non-degenerate*, that is any cone $\sigma \in \Delta$ is strictly convex: $\sigma \cap -\sigma = 0$. Now I shall consider a slightly more general situation. Let $V \subset N_R$ be a rational vector subspace, then I call Δ^* *a fan with vertex* V if it satisfies the usual conditions of a fan with strict convexity of cones replaced by

$$\forall \sigma \in \Delta^* : \sigma \cap -\sigma = V$$

The star * in Δ^* will indicate that the fan Δ^* has possibly non-trivial vertex. (The fans in the usual sense have vertices equal to $\{0\}$.)

If Δ^* is a fan in $N_{\mathbf{R}}$ with a vertex V then we can define a lattice $N' = N/(N \cap V)$, so that $N'_{\mathbf{R}} = N_{\mathbf{R}}/V$. Then the fan Δ^* descends to a nondegenerate fan Δ^*/V in $N'_{\mathbf{R}}$ and $X(\Delta^*/V)$ is a toric variety of dimension $n - dimV$.

Let me recall that Γ is a sub-division of Δ if $|\Delta| = |\Gamma|$ and any cone in Δ is a union of cones from Γ. If both fans are non-degenerate then this defines a birational morphism $X(\Gamma) \longrightarrow X(\Delta)$. If a fan Δ^* with a vertex V has a sub-division to a non-degenerate fan Γ then we have a morphism $X(\Gamma) \to X(\Delta^*/V)$, general fiber of which is of dimension $dimV$.

Lecture 1: Cone Theorem.

First, let me recall basic facts about the intersection on toric varieties. We start with a complete algebraic variety X. Let $N^1(X) \subset H^2(X, \mathbf{R})$ and $N_1(X) \subset H_2(X, \mathbf{R})$ be the $\mathbf{R}$-linear subspaces spanned by, respectively, cohomology and homology classes of, respectively, Cartier divisors and holomorphic curves on X. The class of a curve C in $N_1(X)$ will be denoted by $[C]$.

The intersection of cycles and cocycles restricts to $N_1(X) \times N^1(X)$ and provides a non-degenerate pairing. Thus we can identify any space in question with the dual of its pairing partner.

The following definition describes a convenient class of varieties.

Definition. A normal variety X is called $\mathbf{Q}$-factorial if some multiple of any Weil divisor is a Cartier divisor.

For toric varieties we have a clear description of $\mathbf{Q}$ factoriality.

Proposition. A toric variety $X = X(\Delta)$ is $\mathbf{Q}$-factorial if and only if the fan Δ is simplicial, that is all the cones in Δ are simplicial.

Note that if $X = X(\Delta)$ is $\mathbf{Q}$-factorial then for any $\rho_i \in \Delta(1)$ the Weil divisor $V(\rho_i)$ is $\mathbf{Q}$-Cartier. Let $\mathbf{R}^{\Delta(1)}$ be an (abstract) real vector space in which vectors called $\tilde{e}_i$, with e_i primitive in $\rho_i \in \Delta(1)$, form an orthonormal basis. We have the following exact sequences of vector spaces, dual each to the other,

$$0 \longrightarrow M_{\mathbf{R}} \longrightarrow \mathbf{R}^{\Delta(1)} \longrightarrow N^1(X) \longrightarrow 0$$
$$0 \longrightarrow N_1(X) \longrightarrow \mathbf{R}^{\Delta(1)} \longrightarrow N_{\mathbf{R}} \longrightarrow 0$$

with arrows in the first sequence defined as $M_{\mathbf{R}} \ni m \longrightarrow \sum e_i(m) \cdot \tilde{e}_i$ and $\tilde{e}_i \longrightarrow V(e_i)$ while the maps in the second sequence are as follows $N_1(X) \ni Z \longrightarrow \sum (Z \cdot V(\rho_i)) \cdot \tilde{e}_i$ and $\tilde{e}_i \longrightarrow e_i$.

Corollary. If $X = X(\Delta)$ is a $\mathbf{Q}$-factorial toric variety defined by a fan Δ then $N_1(X)$ can be interpreted as the space of linear relations between primitive vectors e_i in rays $\rho_i \in \Delta(1)$.

Now, for an arbitrary variety X, we consider the following cones in the linear spaces defined above: the cone of curves (called also the cone of effective 1-cycles, or Mori cone) $NE(X) \subset N_1(X)$ and the cone of nef divisors $\mathcal{P} = \mathcal{P}(X) \subset N^1(X)$; they are $\mathbf{R}_{\geq 0}$-spanned by, respectively, the classes of curves and numerically effective divisors. Note that $\mathcal{P}$ and $\overline{NE}$ (the closure of NE) are — by their very definition — dual each to the other in the sense of the intersection pairing of $N^1(X)$ and $N_1(X)$. If X is projective then, by Kleiman criterion of ampleness, the cone $NE(X)$ is strictly convex.

Let me explain one of the starting points of the Program: Mori's move–bend–and–break argument. In toric case this is particularly explicit: if X is a complete toric variety then every effective cycle on X is numerically equivalent to a positive linear combination of some 1-dimensional strata of the big torus action.

Let $C \subset X(\Delta)$ be an irreducible curve. Suppose that C is contained in a stratum $V(\sigma)$ which is of the smallest dimension among the strata containing C. If $dim V(\sigma) = 1$ then there is nothing to be done, otherwise we want to deform C to a union of curves belonging to lower-dimensional strata. We may assume — possibly by passing to a smaller dimensional toric variety — that $V(\sigma) = X(\Delta)$ which means that the general point of C is contained in the open orbit of $X(\Delta)$. If $dim X(\Delta) = 2$ then we note that fixed points of the action of T on the linear system $|C|$ are associated to combination of 1-dimensional strata of $X(\Delta)$, hence we are done in this case.

Now, let $dim X(\Delta) > 2$ and $C \subset X$ be an irreducible curve. Let $\lambda \in N$ be general and consider the action $\mathbf{C}^* \times X(\Delta) \to X(\Delta)$ of the 1-parameter group coming from λ, we denote it $(t, x) \mapsto t^\lambda \cdot x$. We may assume that the action has only a finite number of fixed points. The action gives a morphism $\mathbf{C}^* \times C \to X(\Delta)$ and hence a rational map $\mathbf{C} \times C - \to X(\Delta)$. Blowing up the points of indeterminacy we resolve this map, that is we find a surface S, a regular morphism $\psi : S \to X(\Delta)$ and a projection $\pi : S \to \mathbf{C}$, such that $\psi(\pi^{-1}(1)) = C$. Over $\mathbf{C}^* \times C$ we have a natural $\mathbf{C}^*$-action which lifts up to S so that both ψ and π are $\mathbf{C}^*$ equivariant. The (reducible) curve $\psi(\pi^{-1}(0))$ is invariant with respect to the action of λ, thus it is a union of closures of 1-dimensional orbits of λ. Note that to make it numerically equivalent to the original C the components of the curve

$\psi(\pi^{-1}(0))$ may have to be assigned multiplicities depending on the degree of the map ψ on components of $\pi^{-1}(0)$; moreover, via the action of the group the generic point of C is moved toward a fixed point of the action (to so-called sink, or source, of the action on $X(\Delta)$) and thus the strict transform of $\{0\} \times C$ in S gets contracted to this point.

Now to conclude the argument we consider the case when C is the closure of an orbit of λ contained in the open orbit of $X(\Delta)$. After choosing appropriate identification of the open orbit with the torus we can assume that C is the image of $\mathbf{P}^1$ under the morphism of toric varieties $\mathbf{P}^1 \to X(\Delta)$ given by the inclusion of the line $\mathbf{R} \cdot \lambda \to N_{\mathbf{R}}$. Now let us choose a rational plane $\Pi \subset N_{\mathbf{R}}$ which contains $\mathbf{R} \cdot \lambda$ and meets any cone from Δ of dimension $\leq n-2$ only at the origin. On the plane Π we consider the lattice $N_\Pi = \Pi \cap N$ and the fan Δ_Π obtained by restricting Δ to Π. This defines a toric surfaces $X(\Delta_\Pi)$ together with a toric morphism $X(\Delta_\Pi) \to X(\Delta)$. Moreover, by our assumptions on Π, 1-dimensional strata of the surface $X(\Delta_\Pi)$ are mapped to 1-dimensional strata of $X(\Delta)$ (that is, images of Π invariant curves are Δ invariant) and the morphism $\mathbf{P}^1 \to C \subset X(\Delta)$ factors equivariantly through $X(\Delta_\Pi)$. We already have noticed that on a toric surface each curve is linearly equivalent to a sum of 1 dimensional strata so we are done. Thus we have the following result:

Toric Cone Theorem. Let $X = X(\Delta)$ be a complete toric variety. Then

$$NE(X) = \sum_{\omega \in \Delta(n-1)} \mathbf{R}_{\geq 0} \cdot [V(\omega)]$$

In particular $NE(X)$ is a closed rational polyhedral cone and it is strictly convex if and only if X is projective.

Now if we compare it with the Kleiman criterion for ampleness we get: a Cartier divisor (line bundle) on a toric variety is ample if and only if its intersection with any 1-dimensional stratum is positive.

There is a special name for the edges of the cone $NE(X)$.

Definition. 1-dimensional faces (half-lines, rays) of the cone $NE(X)$ are called extremal rays. More precisely, a ray $R \subset NE(X)$ is an extremal ray if given $Z_1,\ Z_2 \in NE(X)$ such that $Z_1 + Z_2 \in R$ then $Z_1,\ Z_2 \in R$. If an extremal ray R satisfies $R \cdot K_X < 0$ then it is called Mori extremal ray.

Note that in the toric case the existence of extremal rays of $NE(X)$ implies that the cone in question is strictly convex (does not contain any non-zero linear subspace), hence X is projective. Thus, whenever we talk about extremal rays then X is assumed projective.

The Mori extremal rays are the only ones which appear in the general version of the cone theorem.

General Cone Theorem. [Mori, Kawamata] Let X be a complex projective variety with canonical singularities (singularities will be explained later in my 4th lecture). Then

$$NE(X) = NE(X) \cap \{Z \in N_1(X) : Z \cdot K_X \geq 0\} + \sum_\nu R_\nu$$

where R_ν are Mori extremal rays.

Note that the above theorem does not give any information on $NE(X) \cap \{Z \in N_1(X) : Z \cdot K_X \geq 0\}$. In fact, this part of the cone may be neither polyhedral nor closed, see examples in [CKM88]. Mori's proof of the cone theorem for smooth varieties is based on a version of move–bend–and–break argument which was explained above in the toric case. The hard part is to make a curve to move – the property which was given in the toric case for free. For this purpose Mori used deformation of morphisms and, in addition, positive characteristic arguments, see [CKM88] for an exposition of the method.

Example. The case of toric surfaces is elementary but important (it will be used in the subsequent lecture). Let $X = X(\Delta)$ be a complete toric surface. Suppose that b is a positive number and $\langle e_1, e_2\rangle$ and $\langle e_1, ae_1 - be_2\rangle$ are in Δ. Assume that $V(\langle e_1\rangle)$ is in an extremal ray. Then:

(i) if $a < 0$ then $\Delta(1) = \{\langle e_1\rangle, \langle e_2\rangle, \langle ae_1 - be_2\rangle\}$ and therefore $N^1(X)$ is of dimension 1; such X can be shown to be a quotient of the projective plane,

(ii) if $a = 0$ then X is a "generalized Hirzebruch surface", that is: its fan is generated by the following rays: $\langle e_1\rangle, \langle e_2\rangle, \langle -e_2\rangle, \langle -ce_1 + de_2\rangle$ and the projection along the line $\mathbf{R} \cdot e_2$ produces a regular map (ruling) onto $\mathbf{P}^1$,

(iii) if $a > 0$ then the fan Δ is a sub-division of a fan Δ' which is obtained from Δ by forgetting the ray $\langle e_1\rangle$ and replacing two cones $\langle e_1, e_2\rangle$ and $\langle e_1, ae_1 - be_2\rangle$ by one cone $\langle e_2, ae_1 - be_2\rangle$; the associated birational morphism $X(\Delta) \to X(\Delta')$ is a weighted blow-down.

The knowledge of the cone $NE(X)$ is valuable because it gives information on morphisms of X. That is, a morphism of projective normal varieties $\varphi : X \to Y$ determines a face $NE(X/Y)$ of the cone $NE(X)$ which consists of 1-cycles contracted by φ or, equivalently $NE(X/Y) = NE(X) \cap \{Z : \varphi^* L \cdot Z = 0\}$ for any ample L over Y. If in addition φ has connected fibers then it is called a *contraction of the face* $NE(X/Y)$ and its target Y is determined uniquely by $\varphi^* L$. Namely, in this case the variety Y can be recovered by the formula $Y = Proj(\bigoplus_{m\geq 0} H^0(X, m\varphi^* L))$. This property, called sometimes "fundamental triviality of Mori's program", occasionally can be inverted, that is: given a face we can produce its contraction. We will discuss it in the next lecture.

Let me finish this lecture with a series of exercises related to ampleness of line bundles over toric varieties. Let Δ be simplicial complete fan in $N_\mathbf{R}$ with the set of rays $\Delta(1) = \{\rho_1, \dots, \rho_k\}$, each ray ρ_i generated by a primitive $e_i \in \rho_i \cap N$. Let $L = \mathcal{O}_X(\sum b_i V(\rho_i))$ be a line bundle on $X = X(\Delta)$. Recall that the total space of the bundle L is defined as the relative spectrum $\mathbf{V}(L) = Spec_X(\bigoplus_{m\geq 0} L^{\otimes -m})$.

1. Prove that the total space of the line bundle L is a toric variety whose fan Δ' in $N'_\mathbf{R}$, with $N' = N \oplus \mathbf{Z} \cdot e_0$, is constructed as follows:
   (i) its rays are generated by $e'_0 = e_0$ and $e'_i = e_i - b_i e_0$, for $i = 1, \dots, k$,
   (ii) if $\langle e_{i_1}, \dots, e_{i_r}\rangle \in \Delta$ then $\langle e'_{i_1}, \dots, e'_{i_r}\rangle \in \Delta'$ and $\langle e'_0, e'_{i_1}, \dots, e'_{i_r}\rangle \in \Delta'$

Let us set $\gamma = \gamma(L) = |\Delta'|$, where Δ' is the fan defined above.

2. Let $\psi_L : N_\mathbf{R} \to \mathbf{R}$ be a continuous function which is linear on any cone $\sigma \in \Delta$ and such that $\psi_L(e_i) = -b_i$. Prove that
   (i) The graph of ψ_L in $N_\mathbf{R} \times \mathbf{R}$ coincides with the boundary of $\gamma(L)$.

(ii) The line bundle L is spanned by global sections if and only if the function ψ_L is convex, that is $\psi_L(u+v) \geq \psi_L(u) + \psi_L(v)$.

3. Prove that if $\gamma(L)$ is a strictly convex cone then the affine toric variety $U_{\gamma(L)}$ is isomorphic to $Spec(\bigoplus_{m\geq 0} H^0(X, L^{-m}))$.
4. Prove that L^{-1} is ample if and only if $\gamma(L)$ is a strictly convex cone whose 1-dimensional faces are spanned by e_i', where $i = 1, \ldots, k$, (equivalently, $\langle e'_{i_1}, \ldots e'_{i_r}\rangle$ is a proper face of $\gamma(L)$ if and only if $\langle e_{i_1}, \ldots, e_{i_r}\rangle \in \Delta$).
5. Prove Grauert criterion for toric varieties: A line bundle L^{-1} is ample if and only if the morphism of the total space of the dual bundle $\mathbf{V}(L) \to Spec(\bigoplus_m H^0(X, L^{-m}))$, defined by the evaluation of sections, is an isomorphism outside of the zero section in $\mathbf{V}(L)$ and contracts the zero section to the vertex of the cone.

Lecture 2: Contraction Theorem.

From now on I will always assume that $X = X(\Delta)$ is a $\mathbf{Q}$-factorial complete toric variety, that is: the fan Δ is simplicial and $|\Delta| = N_{\mathbf{R}}$. I will also frequently refer to the following

Set-up:
We consider 1-stratum $V(\omega) \subset X(\Delta)$ associated to a cone $\omega = \langle e_1, e_2, \ldots e_{n-1}\rangle$ where e_i are primitive lattice elements on rays spanning ω. The cone ω separates two n-dimensional cones from Δ

$$\delta_{n+1} = \langle e_1, e_2, \ldots, e_n\rangle, \quad \delta_n = \langle e_1, e_2, \ldots, e_{n-1}, e_{n+1}\rangle$$

where e_n and e_{n+1} are primitive on rays on opposite sides of ω. We write ρ_i for the ray generated by e_i. We have a relation

$$\sum_{i=1}^{n+1} a_i e_i = 0$$

with $a_{n+1} = 1$, $a_i \in \mathbf{Q}$ and indexes ordered so that

$$\begin{array}{lll} a_i < 0 & \text{for} & 1 \leq i \leq \alpha \\ a_i = 0 & \text{for} & \alpha+1 \leq i \leq \beta \\ a_i > 0 & \text{for} & \beta+1 \leq i \leq n+1 \end{array}$$

If $X(\Delta)$ is smooth along $V(\omega)$, or – equivalently – if $\{e_1, \ldots, e_n\}$ and $\{e_1, \ldots, e_{n-1}, e_{n+1}\}$ are bases for N then $V(\omega)$ is the complete intersection $\bigcap_{i=1}^{n-1} V(\rho_i)$ and the numbers a_i form the splitting type of the normal bundle of $V(\omega)$ in X, that is $N_{V(\omega)/X} \simeq \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_{n-1})$. If $X(\Delta)$ is only $\mathbf{Q}$-factorial then for a cone $\sigma \in \Delta$ we define the number $mult(\sigma)$ as the index of the sub-lattice generated by the primitive generators of rays of σ inside the lattice $N \cap (\sigma - \sigma)$. Then we have the following fact about intersection numbers.

Proposition. Let $V(\rho) \subset X(\Delta)$ be a divisor corresponding to a ray $\rho = \mathbf{R}_{\geq 0} \cdot e \in \Delta(1)$, with $e \in N$ primitive in ρ. Then
(i) $V(\rho) \cdot V(\omega) = 0$ if $e \notin \{e_1, e_2, \ldots, e_n, e_{n+1}\}$,
(ii) $V(\rho_{n+1}) \cdot V(\omega) = mult(\omega)/mult(\delta_n) > 0$,

(iii) $V(\rho_i) \cdot V(\omega) = a_i V(\rho_{n+1}) \cdot V(\omega)$ for $i = 1, \ldots, n$.

As an immediate consequence we get

Corollary. Suppose that $R = \mathbf{R}_{\geq 0} \cdot [V(\omega)]$ is an extremal ray. Then the numbers α and β, as well as the primitive vectors $e_1, \ldots, e_\alpha, e_{\beta+1}, \ldots e_{n+1}$ (and the rays ρ_i spanned on them) defined above depend on the ray R only. Moreover, if we choose a cone $\omega' \in \Delta(n-1)$ such that $[V(\omega')] \in R$ then $\rho_1, \ldots \rho_\alpha$ are among the edges of ω' and for $k \geq \beta + 1$ the ray ρ_k is either an edge of ω' or of one of the two adjacent n-dimensional cones.

Proof. If we choose ω' such that the curve $V(\omega')$ is numerically proportional to $V(\omega)$ then the divisors which have negative or, respectively, positive intersection with them are the same — hence the result.

We have an explicit description of the Mori cone in terms of the linear spaces which we have introduced in the previous lecture.

Proposition. If $N_1(X) \hookrightarrow \mathbf{R}^{\Delta(1)}$ is identified with the subspace of linear relations between primitive generators of rays in $\Delta(1)$ then

$$N_1(X) \supset NE(X) = \sum_{\omega \in \Delta(n-1)} \mathbf{R}_{\geq 0} \cdot [a_1 \tilde{e}_1 + \ldots + a_\alpha \tilde{e}_\alpha + a_{\beta+1} \tilde{e}_{\beta+1} + \ldots + a_{n+1} \tilde{e}_{n+1}]$$

with e_i and a_i defined for ω as in our set-up.

Proof. We can evaluate the product

$$(a_1 \tilde{e}_1 + \ldots + a_\alpha \tilde{e}_\alpha + a_{\beta+1} \tilde{e}_{\beta+1} + \ldots + a_{n+1} \tilde{e}_{n+1},\ \tilde{e}_j) = c \cdot V(\omega) \cdot V(\rho_j)$$

where c is a positive constant which depends on ω and the choice of e_{n+1}. Thus the class of $V(\omega)$ lies in the ray spanned by the vector $a_1 \tilde{e}_1 + \ldots + a_\alpha \tilde{e}_\alpha + a_{\beta+1} \tilde{e}_{\beta+1} + \ldots + a_{n+1} \tilde{e}_{n+1}$.

We need to introduce more notation:

$$\begin{aligned} \delta_i &= \langle e_1, e_2, \ldots \overset{\frown i}{\ } \ldots e_n, e_{n+1} \rangle, \\ \omega_i' &= \delta_i \cap \delta_{n+1} = \langle e_1, e_2, \ldots \overset{\frown i}{\ } \ldots, e_{n-1}, e_n \rangle, \\ \omega_i'' &= \delta_i \cap \delta_n = \langle e_1, e_2, \ldots \overset{\frown i}{\ } \ldots, e_{n-1}, e_{n+1} \rangle, \\ \sigma_i &= \omega_i' \cap \omega_i'' = \langle e_1, e_2, \ldots \overset{\frown i}{\ } \ldots, e_{n-1} \rangle \end{aligned}$$

Note that δ_i does not have to belong to the fan Δ.

Now we assume that $[V(\omega)] \in R$, where $R \subset NE(X)$ is an extremal ray, and therefore X is projective. The following crucial result is proved by reducing to surfaces strata $V(\sigma_i)$, which is then essentially the example from the previous lecture.

Proposition. In the above situation if $a_i > 0$ then $\delta_i \in \Delta$ and the 1-dimensional strata associated to w_i' and w_i'' are in R. If $a_i = 0$ then there exists $e_i' \in \Delta(1)$ such that the star of σ_i in Δ consists of four n-dimensional cones: the two which we already have δ_n, δ_{n+1} and in addition

$$\begin{aligned} \delta_n' &= \langle e_1, \ldots e_{i-1}, e_i', e_{i+1}, \ldots, e_{n-1}, e_{n+1} \rangle, \\ \delta_{n+1}' &= \langle e_1, \ldots, e_{i-1}, e_i', e_{i+1}, \ldots, e_{n-1}, e_n \rangle, \end{aligned}$$

and their faces. Moreover the 1-stratum $V(\omega') := V(\delta'_n \cap \delta'_{n+1})$ is in R.

We have just noted that if in our set-up the number a_i is positive then, from the point of view of the extremal ray R, the roles of indexes i, n and $n+1$ can be exchanged. On the other hand the numbers a_i have appeared in measuring the intersection of 1-dimensional strata with divisors. Thus we can choose a curve in the ray which is minimal in this respect. That is, we can renumber e_i for $i = \beta+1, \ldots, n+1$ in such a way that $V(\rho_{n+1}) \cdot V(\omega) \geq V(\rho_i) \cdot V(\omega)$ for all i. Then the curve $C = V(\omega)$ will have its class in the same ray as before and in addition $a_i \leq 1$, for $i = 1, \ldots, n+1$. (We call such a curve extremal.)

For an extremal ray $R \subset NE(X)$ we define

$$Locus(R) = \bigcup \{C \subset X : [C] \in R\}$$

Proposition. In the above set-up suppose that $V(\omega)$ is in an extremal ray R, then

$$Locus(R) = V(\langle e_1, \ldots, e_\alpha \rangle)$$

Proof. $Locus(R)$ is a closed subset of X, and it is invariant with respect to the action of T_N, hence it is of the form $V(\sigma)$ for some $\sigma \in \Delta$. Since $V(\rho_i) \cdot R < 0$, for $i = 1, \ldots \alpha$, it follows that $V(\rho_i) \supset Locus(R)$ hence $V(\langle e_1, \ldots e_\alpha \rangle) \supset Locus(R)$ and $\sigma \supset \langle e_1, \ldots e_\alpha \rangle$. On the other hand $Locus(R) = V(\sigma) \supset V(\omega)$, hence σ is a face of $\langle e_1, \ldots, e_{n-1} \rangle$. However, from the previous proposition it follows that, switching from ω to another ω', whose class is in R, we can replace each of vertices $e_{\beta+1}, \ldots, e_{n-1}$ by e_n, while each of vertices e_i, with $i = \alpha+1, \ldots \beta$ can be replaced by e'_i which is not in the original collection. Thus σ is contained in $\langle e_1, \ldots, e_\alpha \rangle$ and we are done.

Corollary. In the previous set-up, $V(\omega)$ has non-negative intersection with any effective divisor if and only if $Locus(R) = X$ (in this case we call R a nef extremal ray). If $\alpha = 1$ then $Locus(R) = V(\rho_1)$ is the only effective divisor whose intersection with R is negative.

In the non-toric case the locus of a ray may be not as nice as the toric case would suggest: for example it can be disconnected.

Again, we need more notation:

$$\delta(\omega) = \delta_n + \delta_{n+1} = \langle e_1, e_2, \ldots, e_n, e_{n+1} \rangle$$
$$\mu(\omega) = \langle e_1, \ldots, e_\alpha, e_{\beta+1}, \ldots, e_{n+1} \rangle$$

Since we already have noticed that the set of rays spanning $\mu(\omega)$ is common for all $\omega \in \Delta(n-1)$, such that $[V(\omega)] \in R$, we can actually write $\mu(R)$ instead of $\mu(\omega)$. We note moreover that if $\alpha = 0$ then $\mu(R)$ is a vector space.

Lemma. There are the following two simplicial subdivisions

$$\delta(w) = \bigcup_{i=\beta+1}^{n+1} \delta_i = \bigcup_{i=1}^{\alpha} \delta_i$$

where the second equality makes sense if $\alpha > 0$.

Proof. Take $x = \sum_i t_i e_i \in \Delta(\omega)$. Choose $j \geq \beta+1$ (or $j \leq \alpha$) for which t_j/a_j is minimum (maximum, respectively). Then we can write $x = \sum_i [t_i - (t_j/a_j)a_i]e_i$ and all coefficients of e_i are non-negative, and zero if $i = j$. So $x \in \delta_j$ and this proves the simplicial sub-division.

Now let me note that $N_{\mathbf{R}} \simeq \mathbf{R}^n$ can be written as the Cartesian product of two linear spaces $\mathbf{R}^n = \mathbf{R}^{\beta-\alpha} \times \mathbf{R}^{n+\alpha-\beta}$, where the first factor is spanned on the (linearly independent) vectors $e_{\alpha+1}, \ldots, e_\beta$ and the second factor is spanned on $e_1, \ldots, e_\alpha, e_{\beta+1}, \ldots e_{n+1}$. Subsequently, $\delta(\omega)$ can be written as the product of the respective two cones, $\delta(\omega) = \langle e_{\alpha+1}, \ldots, e_\beta \rangle \times \mu(R)$. In particular $\mu(R)$ can be seen as the face of $\delta(\omega)$ cut by the linear subspace $\{0\} \times \mathbf{R}^{n+\alpha-\beta}$. If $\alpha = 0$ then $-e_{n+1} \in \langle e_{\beta+1}, \ldots, e_n \rangle$ hence $\langle e_{\beta+1}, \ldots e_{n+1} \rangle = \{0\} \times \mathbf{R}^{n-\beta}$.

In the proposition at the beginning of the lecture we have noticed that for $i = \alpha + 1, \ldots \beta$ each e_i has a twin e_i' on the other side of the hyperplane spanned by the remaining rays. Then $\omega' = \langle e_1, \ldots, e_i', \ldots, e_{n-1} \rangle$ defines another curve in the ray R and $\delta(\omega') = \langle e_1, \ldots, e_i', \ldots, e_{n+1} \rangle$, is a cone which has $\mu(R)$ as a face and lies on the other side of the hyperplane spanned by e_j, with $j \neq i$. Now let me set $\mathcal{V}(R) = \bigcup \delta(\omega)$, where the sum is taken over $\omega \in \Delta(n-1)$ such that $[V(\omega)] \in R$. Then $\mathcal{V}(R)$ is a convex neighborhood of $\mu(R)$ in $N_{\mathbf{R}}$.

The above discussion is needed for the proof of the following

Toric Contraction Theorem I. Let Δ be a complete simplicial fan, $X = X(\Delta)$ its toric variety, and suppose that R is an extremal ray of X. Let us remove from $\Delta(n-1)$ all $(n-1)$-dimensional cones ω associated to curves from R and for each such ω replace the two adjacent cones δ_n and δ_{n+1} from $\Delta(n)$ by the cone $\delta(w)$. Then, taking respectively their faces in $\Delta(i)$, where $i \leq n-2$, we get a complete fan Δ_R^*, degenerate with vertex $\mu(R)$ if $\alpha = 0$, nondegenerate if $\alpha \neq 0$. Moreover, if $\alpha = 0$ then $\Delta_R := \Delta_R^*/\mu(R)$ is a complete simplicial fan. If $\alpha = 1$ then $\Delta_R := \Delta_R^*$ is simplicial as well.

The result follows from the above discussion and lemma we already proved: our construction leaves the fan Δ unchanged outside $\mathcal{V}(R)$, while $\mathcal{V}(R)$ itself is divided into cones of the type $\delta(\omega)$, each of them containing the cone $\mu(R)$. The induced morphism $\varphi_R : X(\Delta) \to X(\Delta_R)$ is what we have called the contraction of the ray R and it has all the features listed in the following theorem (the projectivity will be apparent later).

Contraction Theorem. [Kawamata, Shokurov] Let X be a projective variety with terminal singularities. Then for any Mori ray R of X there exists a morphism (the contraction of R) $\varphi_R : X \to X_R$ such that:

(i) X_R is a normal projective variety;
(ii) φ_R is a morphism with connected fibers: $(\varphi_R)_* \mathcal{O}_X = \mathcal{O}_{X_R}$;
(iii) a curve $C \subset X$ is contracted to a point by φ_R if and only if $[C] \in R$.

Remark. (notation as before) Let $\varphi_R : X = X(\Delta) \to X_R = X(\Delta_R)$ be the toric contraction coming from the above theorem. Then the exceptional set of φ_R is $V(\langle e_1, \ldots, e_\alpha \rangle)$ and φ_R contracts it to $V(\mu(R)) \subset X(\Delta_R)$ (if $\alpha = 0$ then both are equal to $V(0)$). Moreover,

(i) if $\alpha = 0$ then $dim X_R = \beta$ and φ_R is a bundle whose fiber is a quotient of a projective space.

(ii) if $\alpha = 1$ then φ_R contracts the exceptional divisor $V(\rho_1)$ to a set in X_R of dimension $\beta - 1$.
(iii) if $\alpha > 1$ then φ_R is birational and it is an isomorphism in codimension 1. The ray R is then called small.

The construction of the toric contraction was pretty explicit and involved understanding the local structure of the variety. In general however one gets the contraction as the result of the following theorem.

Base Point Free Theorem. [Kawamata, Shokurov] Let X be a projective variety with terminal singularities and L a nef Cartier divisor such that $tL - K_X$ is ample for $t \gg 0$. Then the linear system $|mL|$ has no base points for sufficiently large m.

The contraction theorem follows from the base point freeness quite easily: given a Mori ray R we have to choose a *good supporting* line bundle L such that the hyperplane $\{Z \in N_1(X) : Z{\cdot}L = 0\}$ meets $NE(X)$ along R and $NE(X)\backslash R$ is in the half-space which has positive intersection with L. Now we can take the morphism defined by the linear system $|mL|$, call it $\psi : X \to \mathbf{P}^r$. It is clear that ψ contracts (only!) curves from R, so if we take its connected part coming from the Stein factorization $X \to X_R \to \psi(X) \subset \mathbf{P}^r$ then we get the contraction of the ray R. Equivalently, we can define X_R as the projective spectrum $Proj(\bigoplus_{M\geq 0} H^0(X, mL))$ and the morphism $X \to X_R$ is then defined by the evaluation of sections $H^0(X, mL) \otimes \mathcal{O}'_X \to mL$. Let me note that the target of the contraction is projective since by the construction it comes with an ample line bundle whose pull-back to X is the original bundle L.

The toric case has very nice base-point-free properties which is one of the main reasons why our construction works that well. Namely, combining the results of exercises which were at the end of the previous lecture with the numerical properties of the intersection which has been discussed today we get.

Lemma. Let $X = X(\Delta)$ be a $\mathbf{Q}$-factorial complete toric variety and let consider a line bundle $L = \mathcal{O}_X(\sum_i b_i V(e_i))$ over X. Then L is nef if and only if it is spanned by global sections.

Proof. In the situation of our set-up we have $L \cdot V(\omega) = b_1 a_1 + \ldots + b_\alpha a_\alpha + b_{\beta+1} a_{\beta+1} + \ldots + b_{n+1} a_{n+1}$. On the other hand the function $\psi_L : N_{\mathbf{R}} \to \mathbf{R}$ defined at the end of the previous lecture is convex on $\delta_n \cup \delta_{n+1}$ if and only if

$$\begin{aligned}\psi_L(a_n e_n + a_{n+1} e_{n+1}) = -\psi_L(a_1 e_1 + \ldots + a_{n-1} e_{n-1}) = a_1 b_1 + \ldots + a_{n-1} b_{n-1} \geq \\ \geq \psi_L(a_n e_n) + \psi_L(a_{n+1} e_{n+1}) = -a_n b_n - a_{n+1} b_{n+1}\end{aligned}$$

and this is equivalent to $L \cdot V(\omega) \geq 0$. Thus our lemma follows by the characterization of the spanned bundles which was discussed in the previous lecture.

Now, let me sketch an argument which shows the equivalence of these two constructions (the set-up is as usual).

First we choose a line bundle $L = \mathcal{O}(\sum_i b_i V(\rho_i))$ which is good supporting for R, that is the sum $\sum_i a_i b_i$ is non-negative for a_i arising — as in the set-up — for any $\omega \in \Delta(n-1)$ and it is zero only if $[V(w)] \in R$. Now, as in the exercises finishing the previous lecture we consider the support $\gamma(L^{-1})$ of the fan Δ' defining the total space of the dual L^{-1}. When L was ample then the n-dimensional faces of $\gamma(L^{-1})$ were in $1-1$ correspondence with n-dimensional cones of Δ. Now this is not the case because the vectors $e'_1, \ldots, e'_{n+1}$ (notation as in the construction of the fan Δ') are on a hyperplane if the relation $\sum_i a_i \tilde{e}_i$ is in R. Indeed,

$$\sum_{i=1}^{n+1} a_i e'_i = \sum_{i=1}^{n+1} (a_i e_i - a_i b_i e_0)$$

and therefore the left hand side vanishes if and only if both $\sum a_i e_i$ and $\sum a_i b_i$ are zero.

Exercise. Prove that the fan in $N_{\mathbf{R}}$ obtained by projecting along $\mathbf{R} \cdot e_0$ the faces of the cone $\gamma(L^{-1})$ is equal to Δ_R^* and therefore X_R constructed in the first part of the lecture is equal to $Proj(\bigoplus_{M \geq 0} H^0(X, mL))$.

An advantage of the arguments using general base-point-freeness is that — although being less constructive than in the case of a ray — it works for an arbitrary face of the cone $NE(X)$. Thus we have the result which we predicted at the end of the previous lecture.

Toric Contraction Theorem II. Let $X = X(\Delta)$ be a projective $\mathbf{Q}$-factorial toric variety. Then any face $F \subset NE(X)$ can be contracted. That is, there exits a morphism $\varphi_F : X \to X_F$ such that:

(i) X_F is a toric projective variety;
(ii) φ_F has connected fibers;
(iii) a curve $C \subset X$ is contracted to a point by φ_F if and only if $[C] \in F$.

The construction of X_F and φ_F is obtained as it was explained above: we choose a good supporting line bundle L_F for the face F. We know that L_F is spanned and therefore we can set $X_F := Proj(\bigoplus_{m \geq 0} H^0(X, mL_F))$, and we define φ_F via evaluation. Finally, the resulting variety X_F is toric because the action of T on X gives an action on $H^0(X, mL_F)$, for each m, and thus an action on the projective spectrum $Proj(\bigoplus_{m \geq 0} H^0(X, mL_F))$ so that $\varphi_F : X \to X_F$ is T-equivariant. The action of T on X_F has an open orbit, thus if we divide T by the isotropy group of a general point of X_F then the resulting torus T_F acts on X_F with an open orbit and therefore X_F is toric, see e.g. [Oda88, 1.5]. In the conclusion let us note that if F' is a sub-face of F then the contraction of F factors through the contraction of F'.

Lecture 3: Flip and Flop.

We deal with a situation as before, that is: $X = X(\Delta)$ is a complete $\mathbf{Q}$-factorial toric variety, R is an extremal ray of X. Assume that R is a small ray, that is, as in our set-up: $\alpha \geq 2$. Let $\varphi_R : X \to X_R = X(\Delta_R)$ be the extremal ray contraction. Every $\delta \in \Delta_R(n) \setminus \Delta(n)$ is of the form $\delta = \langle e_1, \ldots, e_{n+1} \rangle$, where $e_1, \ldots, e_\alpha$ and $e_{\beta+1}, \ldots, e_{n+1}$ are fixed and satisfy the relation:

$$\sum_{i=1}^{\alpha} (-a_i) e_i = \sum_{i=\beta+1}^{n+1} a_i e_i$$

and we have proved during the previous lecture that

$$\delta = \bigcup_{i=\beta+1}^{n+1} \delta_i = \bigcup_{i=1}^{\alpha} \delta_i$$

are simplicial sub-divisions and the first one of them leads to the birational morphism φ_R.

Before presenting the main result of the lecture let us formulate a general observation concerning birational maps which are isomorphisms in codimension 1.

Lemma. If $\psi : X - \to Y$ is a birational map of $\mathbf{Q}$-factorial varieties which is an isomorphism in codimension 1 then the strict transform of Weil divisors provides a natural identification $N^1(X) = N^1(Y)$ and hence a dual isomorphism $N_1(X) \simeq N_1(Y)$.

Toric Flip Theorem. Let $X = X(\Delta)$ be as in our set-up with $R = \mathbf{R}_{\geq 0}V(\omega)$ an extremal ray and $\varphi_R : X \to X_R = X(\Delta_R)$ its contraction. Assume moreover that φ_R is small. Let Δ_1 be a simplicial subdivision of Δ_R such that

$$\Delta_1(n) = \Delta_R(n) \setminus \{\delta(\omega) : \ [V(\omega)] \in R\} \cup \{\delta_i(\omega) : \ [V(\omega)] \in R, i = 1 \dots \alpha\}$$

then $X_1 = X(\Delta_1)$ is a $\mathbf{Q}$ factorial projective variety and the resulting birational morphism $\varphi_1 : X_1 \to X_R = X(\Delta_R)$ is isomorphism in codimension 1. If, using the resulting birational map $\psi : X - \to X_1$, we identify $N_1(X) \simeq N_1(X_1)$ then $R_1 = -R$ is an extremal ray of X_1 and φ_1 is its contraction.

Proof. All features of X_1 and φ_1 follow by the symmetry of the above construction. In particular, all $n-1$ dimensional cones introduced in subdividing Δ into Δ_1 lead to 1-dimensional strata which are numerically proportional and they have opposite sign intersection numbers with respect to the 1-dimensional strata which we contracted by φ_R. Thus, if we take an ample divisor H on X and a good supporting divisor L_R for R then, by toric Kleiman criterion of ampleness, $mL_R - H$ is ample on X_1 for m sufficiently large; hence X_1 is projective. Other features are immediate.

Thus, in case of toric geometry we have an affirmative answer to the following

Flip Conjecture. Let $\varphi_R : X \to X_R$ be a small contraction of a Mori ray R of a variety X with terminal singularities. Then there exists a variety X_+ with terminal singularities, together with a birational morphism $\varphi_+ : X_+ \to X_R$ which is isomorphism in codimension 1. Moreover the induced birational map $(\varphi_+)^{-1} \circ \varphi_R : X - \to X_+$ is an isomorphism outside $Locus(R)$ and K_{X_+} is φ_+-ample.

As explained in the introduction, a proof of Flip Conjecture, together with Contraction and Cone theorems would complete the Minimal Model Program. This was successfully done in dimension 3 by Mori.

Complements: flips as Mumford's quotients, Morelli-Włodarczyk cobordisms.

Consider an action of $\mathbf{C}^*$ on $\mathcal{B} := \mathbf{C}^{n+1}$:

$$\mathbf{C}^* \times \mathbf{C}^{n+1} \ni (t, z) = (t, (z_1, \dots, z_{n+1})) \longrightarrow t \cdot z := (t^{a_1} \cdot z_1, \dots, t^{a_{n+1}} \cdot z_{n+1}) \in \mathbf{C}^{n+1}$$

where $(a_1, \dots a_{n+1})$ is a sequence of non-zero (for simplicity) coprime integers and

$$\begin{array}{lll} a_i < 0 & \text{for} & 1 \leq i \leq \alpha \\ a_i > 0 & \text{for} & \alpha+1 \leq i \leq n+1 \end{array}$$

We moreover assume that $2 \leq \alpha \leq n-1$. A monomial $z^m = z_1^{m_1} \dots z_{n+1}^{m_{n+1}}$ is $\mathbf{C}^*$ invariant if

$$a_1 m_1 + \cdots + a_{n+1} m_{n+1} = 0$$

Thus we have an immediate result.

Lemma. In the above situation the variety $Z = Spec(\mathbf{C}[z_1, \dots z_{n+1}])^{\mathbf{C}^*}$ is an affine toric variety U_δ whose cone δ, as well as the lattice N, is spanned in $N_{\mathbf{R}} \simeq \mathbf{R}^n$ by vectors $e_1, \dots, e_n, e_{n+1}$ satisfying the relation $a_1 e_1 + \dots + a_{n+1} e_{n+1} = 0$.

We consider $\mathcal{B} = \mathbf{C}^n$ as a toric affine variety $U_{\hat{\delta}}$ associated to a cone $\hat{\delta} = \langle \hat{e}_1, \dots, \hat{e}_{n+1} \rangle$ spanned by generators of the lattice $\hat{N} \simeq \mathbf{Z}^{n+1}$ in $\hat{N}_{\mathbf{R}}$. Then the inclusion of the $\mathbf{C}^*$-invariant polynomials, $\mathbf{C}[z_1, \dots, z_{n+1}]^{\mathbf{C}^*} \subset \mathbf{C}[z_1, \dots, z_{n+1}]$, and the related inclusion in the set of characters $\hat{M} = Hom(\hat{N}, \mathbf{Z})$, can be dually interpreted as the projection $\hat{N}_{\mathbf{R}} \to N_{\mathbf{R}}$ along the line $\mathbf{R} \cdot (\sum_i a_i \hat{e}_i)$. Thus we have a natural toric morphism $\mathcal{B} = \mathbf{C}^{n+1} \to Z$ which in terms of [Mumford65] GIT theory is called *good quotient* or *categorical quotient* and denoted $\mathbf{C}^{n+1} // \mathbf{C}^*$. Note that the fiber of the morphism $\mathcal{B} \to Z$ over the special point (the image of the origin) contains several (geometric) orbits of the $\mathbf{C}^*$ action: apart of the origin 0 it includes orbits which converge to (or diverge from) 0.

Now I introduce three types of objects related to the situation we consider:

(I) First define $\mathbf{C}^*$-invariant open subsets of $\mathcal{B}$: inside $\mathcal{B} = \mathbf{C}^{n+1}$ we consider two subsets:

$$\begin{array}{lll} \mathcal{B}_+ & = & \{z \in \mathbf{C}^{n+1} \text{ such that } \lim_{t \to \infty} t \cdot z \text{ does not exist}\}, \\ \mathcal{B}_- & = & \{z \in \mathbf{C}^{n+1} \text{ such that } \lim_{t \to 0} t \cdot z \text{ does not exist}\} \end{array}$$

we see that

$$\begin{array}{lll} \mathcal{B}_+ & = & \mathbf{C}^{n+1} \setminus \{(z_1, \dots, z_\alpha, 0, \dots, 0)\}, \\ \mathcal{B}_- & = & \mathbf{C}^{n+1} \setminus \{(0, \dots, 0, z_{\alpha+1}, \dots, z_{n+1})\} \end{array}$$

(II) Next define graded algebras: on the polynomial algebra $\mathbf{C}[z_1, \dots, z_{n+1}]$ we introduce a non-standard grading:

$$grad(z_1^{m_1} \dots z_{n+1}^{m_{n+1}}) = a_1 m_1 + \dots + a_{n+1} m_{n+1}$$

We can write $A = \mathbf{C}[z_1, \dots, z_{n+1}]$ as a graded algebra $A = \bigoplus_{m \in \mathbf{Z}} A_m$, where A_m is a $\mathbf{C}$-linear space spanned on monomials of grade m. We easily note that $A_0 = \mathbf{C}[z_1, \dots, z_{n+1}]^{\mathbf{C}^*}$, so that $Z = Spec(A_0)$. If we denote $A^+ = \bigoplus_{m \geq 0} A_m$ and $A^- = \bigoplus_{m \leq 0} A_m$ then both of them are graded finitely generated A_0-algebras and they define naturally sheaves of graded $\mathcal{O}$-algebras over Z.

(III) Finally define toric varieties. Let Δ_+ and Δ_- be two fans coming from two subdivisions of the cone δ (notation as usual, that is $\delta_i = \langle e_1, e_2 \dots \overset{i}{\frown}, \dots, e_n, e_{n+1} \rangle$)

$$\delta = \bigcup_{i=\alpha+1}^{n+1} \delta_i = \bigcup_{i=1}^{\alpha} \delta_i$$

and by $X_\pm = X(\Delta_\pm)$ denote their toric varieties.
Now we can compare the introduced objects.

Lemma. The toric varieties X_+ and X_- are isomorphic to projective relative spectra $Proj_Z A^+$ and $Proj_Z A^-$, respectively. Moreover, there exist morphisms $\mathcal{B}_\pm \to X_\pm$ which make $X_\pm = \mathcal{B}_\pm/\mathbf{C}^*$, geometric quotients in the sense of Mumford; that is, the (closed) points of $X_\pm$ parametrize (closed) orbits of the $\mathbf{C}^*$ action on $\mathcal{B}_\pm$.

The above result can be interpreted as follows. Suppose that you have a contraction of a Mori ray $\varphi_R : X \to X_R$ and you want to construct a flip of φ_R. Then, since $-K_X$ is φ_R ample, we have $X = Proj_{X_R} \bigoplus_{m\geq 0}(\varphi_R)_*\mathcal{O}(-mK_X)$ and you want to get $X_1 = Proj_{X_R} \bigoplus_{m\geq 0}(\varphi_R)_*\mathcal{O}(mK_X)$ as the result of your flip. So the issue is to prove that the $\mathcal{O}_{X_R}$-algebra $\bigoplus_{m\geq 0}(\varphi_R)_*\mathcal{O}(mK_X)$ is finitely generated. On the other hand, chances are that, like in the toric case, the flip can be constructed as the result of some geometric quotient. This line of argument is discussed in [Reid92] and it is related to the notion of cobordism which I want to discuss now.

Namely, the triple $(\mathcal{B}, \mathcal{B}_+, \mathcal{B}_-)$ introduced above, together with the $\mathbf{C}^*$ action, is an example of Morelli—Włodarczyk cobordism of varieties X_+ and X_- (see [Morelli96] and [Włodarczyk99]). Note that although the above structure does not admit anything like a Morse function, some of its properties are similar to the Morse theory features: in particular the vector field coming from the $\mathbf{C}^*$ action can be compared with the gradient field of a Morse function. The quotients X_+ and X_- can be compared to upper/lower boundaries of a Morse cobordism: one can imagine them being glued at ∞ or 0 to orbits $t \cdot z$ which don't have limits at $t \to \infty$ or $t \to 0$, respectively. In fact, in the present example the quotients can be glued together so that the picture is even more convincing (I don't claim however that this can be done as nicely in general).

Let $\hat{e}_1, \ldots, \hat{e}_{n+1}$ be a standard basis of $\mathbf{R}^{n+1}$. Define $\hat{e}_- = -\hat{e}_+ = a_1\hat{e}_1 + \ldots + a_{n+1}\hat{e}_{n+1}$, where a_i, $i = 1, \ldots, n+1$, are as above. Consider a fan $\hat{\Delta}$ which contains the following $(n+1)$-dimensional cones:

$$\begin{array}{ll} \langle \hat{e}_1, \ldots, \hat{e}_{n+1} \rangle & \\ \langle \hat{e}_-, \hat{e}_1, \ldots \overset{\wedge i}{}, \ldots, \hat{e}_{n+1} \rangle & \text{for } \ i = 1, \ldots, \alpha \\ \langle \hat{e}_+, \hat{e}_1, \ldots \overset{\wedge i}{}, \ldots, \hat{e}_{n+1} \rangle & \text{for } \ i = \alpha + 1, \ldots, n+1 \end{array}$$

We note that $\hat{\Delta}$ is a simplicial subdivision of the cone δ^*, non-strictly convex, with vertex along the line the line $L = \mathbf{R} \cdot \hat{e}_\pm$, and $\delta^*/L = \delta$, where δ is as above. Thus we have a morphism $\Phi : X(\hat{\Delta}) \to U_\delta$. On the other hand we have a decomposition

$$X(\hat{\Delta}) = V(e_+) \cup V(e_-) \cup U_{\langle \hat{e}_1, \ldots, \hat{e}_{n+1} \rangle} \simeq X_+ \cup X_- \cup \mathbf{C}^{n+1}$$

and the map Φ over $\mathbf{C}^{n+1}$ is the good quotient of the $\mathbf{C}^*$ action described above, while over the divisor strata Φ is a small contraction of a ray and its opposite.

More information on extensions and applications of the above construction can be found in [AKMW] and in an expository paper [Bonavero00].

Lecture 4: Canonical Divisor.

As before, also in this lecture, we assume that $X = X(\Delta)$ is a complete $\mathbf{Q}$-factorial toric variety, although this is not needed for the first observation which is obtained by simple verification.

Lemma. Let $m_1, \dots, m_n$ be a basis of M and $x_1, \dots, x_n$ the respective functions $T_N \to \mathbf{C}^*$. Then

$$\frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}$$

is a unique (up to a constant) T_N-invariant holomorphic n-form on T_N. As a rational n-form on $X(\Delta)$ it has a simple pole along (the generic point of) any $V(\rho)$, with $\rho \in \Delta(1)$.

Corollary. For any toric variety $X = X(\Delta)$ we have the following linear equivalence

$$K_X = -\sum_{\rho \in \Delta(1)} V(\rho)$$

In particular, if $\omega \in \Delta(n-1)$ is a common border of $\delta_{n+1} = \langle e_1, \dots, e_n \rangle$ and $\delta_n = \langle e_1, \dots, e_{n-1}, e_{n+1} \rangle$, where e_i satisfy the set-up condition, then

$$-K_X \cdot V(\omega) = \frac{mult(\omega)}{mult(\delta_n)} [a_1 + \dots + a_\alpha + a_{\beta+1} + \dots + a_{n+1}]$$

Definition. For any Mori extremal ray R of a variety X we define

$$Length(R) = min\{-K_X \cdot C : \ [C] \in R \text{ and } \ C \text{ is rational}\}$$

We note that if $X(\Delta)$ is smooth and $V(\omega)$ is an extremal rational curve spanning a ray R, and $\omega \in \Delta(n-1)$ is as in our set-up then $a_{\beta+1} = \dots = a_{n+1} = 1$ and $Length(R) = n + 1 - \beta + a_1 + \dots + a_\alpha$.

Exercise. Prove the following statements for smooth toric varieties: (1). Length of any extremal ray is $\leq n+1$. (2). If there exists a ray of length $n+1$ then $X \simeq \mathbf{P}^n$. (3). Classify smooth varieties which have a *long* ray; that is, first consider varieties with a ray of length $\geq n$, next these with a ray of length $\geq n-1$, etc.

Definition. Let $X = X(\Delta)$ be a $\mathbf{Q}$-factorial toric variety. We say that X has terminal singularities if any closed n-dimensional simplex $conv(0, e_1, \dots, e_n)$, spanned on the origin and primitive vectors $e_i \in N$ such that $\langle e_1, \dots, e_n \rangle \in \Delta$, does not contain any point of N except its vertices. We say that X has canonical singularities if $conv(0, e_1 \dots e_n) \setminus conv(e_1, \dots, e_n)$ does not contain any point from N except the origin.

A general definition of terminal and canonical singularities is as follows.

Definition. Let X be a normal variety such that its canonical divisor K_X is $\mathbf{Q}$-Cartier. Suppose that $\pi : Y \to X$ is a resolution of singularities of X such that the exceptional set π is a divisor $\bigcup E_i$ which has only simple normal crossings. We can write

$$K_Y = \pi^* K_X + \sum_i d_i E_i$$

(where the number d_i is called discrepancy of E_i). Then X has terminal (respectively, canonical) singularities if all d_i are positive (respectively, non-negative).

We note that the first of the two definitions above does not depend on the resolution that we choose because, roughly speaking, any two resolutions can be dominated by another one on which we can compare their discrepancies. It is not hard to show that for toric varieties these two definitions are equivalent. Namely, let us consider a cone $\delta = \langle e_2, \ldots, e_{n+1} \rangle \in \Delta(n)$ and a point $e_1 \in \delta \cap N$. Take a blow-up of X which is associated to adding a ray $\rho_1 = \langle e_1 \rangle$ to Δ and sub-dividing δ (and possibly some adjacent cones) accordingly (we may assume that e_1 is primitive on ρ_1); call the resulting fan Δ' and let $X' = X(\Delta')$. Then $\pi : X' \to X$ is a divisorial contraction which we already discussed, so let us use the notation from our set-up, in particular we may write $-a_1 e_1 = a_{\beta+1} e_{\beta+1} + \ldots + a_{n+1} e_{n+1}$ with a_1 negative and the remaining a_i positive. The pull-back of a Cartier divisor preserves the associated piecewise linear function $\psi : N_{\mathbf{R}} \to \mathbf{R}$ (defined as in the first lecture), so in particular $\psi_{K_X} = \psi_{\pi^*(K_X)}$. Thus $\psi_{\pi^*(K_X)}(e_1) = -(a_{\beta+1} + \ldots + a_{n+1})/a_1$ and the discrepancy of E_1 is $d_1 = -1 - (a_{\beta+1} + \ldots + a_{n+1})/a_1$. Now $d_1 > 0$ if and only if $-a_1 < a_{\beta+1} + \ldots + a_{n+1}$ which is equivalent to e_1 being outside $\langle e_{\beta+1}, \ldots, e_{n+1} \rangle$ (similarly for $d_1 \geq 0$). Now, as explained above, any resolution of singularities of X can be compared with such a blow-up and thus we have the equivalence of the two definitions.

The role of the canonical divisor in toric versions of the main results of Mori theory is not apparent but it becomes crucial in the non-toric case. In particular it appears in the class of singularities which are admissible on varieties for which we formulate main theorems of the program. The following two results indicate that varieties with terminal singularities form a convenient class of varieties for which one may run the Minimal Model Program.

Addition to Toric Contraction Theorem. Assumptions as in the contraction theorem. If $\alpha = 1$, and R is a Mori ray, and moreover X has terminal singularities, then X_R has terminal singularities as well.

Addition to Toric Flip Theorem. Assumptions as in the flip theorem. If R is a Mori ray and X has terminal singularities then X_1 has terminal singularities as well.

Proof. I do only the first case: the second case is somewhat more complicated but similar, see [Reid83]. Thus we take $\varphi_R : X \to X_R$, a divisorial contraction. We write $-a_1 e_1 = a_2 e_2 + \ldots + a_{n+1} e_{n+1}$ and if we divide this equality by $a_2 + \ldots + a_{n+1}$ then we see that the ray ρ_1 meets the affine hyperplane spanned on $e_2, \ldots, e_{n+1}$ at $(-a_1/(a_2 + \ldots + a_{n+1})) \cdot e_1$. Because $-K_X \cdot V(\omega) > 0$, which means $a_1 + \ldots + a_{n+1} > 0$, it follows that $-a_1/(a_2 + \ldots + a_{n+1}) < 1$ hence e_1 is outside the simplex $conv(0, e_2, \ldots, e_{n+1})$. Moreover we have an inclusion

$$conv(0, e_2, \ldots, e_{n+1}) \subset \bigcup_{i>1} conv(0, e_1, \ldots \hat{}^{\,i} \ldots e_{n+1})$$

and since none of the simplices on the right hand side contains any lattice point apart of e_j, the same holds for the simplex on the left hand side.

The numerical properties of the (anti)canonical divisor of a toric variety can be described by using an associated polytope.

Definition. Let Δ be a simplicial fan. We define a polytope in $N_{\mathbf{R}}$

$$P(\Delta) = \bigcup_{\langle e_1,\dots,e_k\rangle\in\Delta} conv(0, e_1, \dots, e_k)$$

where, as usually, e_i denote primitive elements spanning rays in Δ.

In view of exercises characterizing ampleness and nefness we get

Lemma. The divisor $-K_{X(\Delta)}$ is nef and, respectively, ample if and only if $P(\Delta)$ is convex and, respectively, strictly convex, where the latter means that each face of $P(\Delta)$ is of the form $conv(e_{i_i}, \dots, e_{i_r})$ where $\langle e_{i_1}, \dots e_{i_r}\rangle \in \Delta$.

Complements: Euler sequence.
For any complete non-singular variety X we consider a vector space $\mathcal{H} := H^1(X, \Omega_X)$ and the related trivial sheaf $\mathcal{H}_X := \mathcal{H} \otimes \mathcal{O}_X$. Let us note that

$$Ext^1(\mathcal{H}_X, \Omega_X) = Hom(\mathcal{H}, \mathcal{H})$$

Definition. The extension sequence

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{L}_X^\vee \longrightarrow \mathcal{H}_X \longrightarrow 0$$

associated, under the above identification, to the identity in $Hom(\mathcal{H}, \mathcal{H})$ is called the Euler sequence of X and the (locally free) sheaf $\mathcal{L}_X$ is called the potential sheaf.

Theorem. [Batyrev-Mel'nikov, Jaczewski] A complete non-singular variety X is isomorphic to a toric variety if and only if there exists a simple normal crossings divisor $D = \bigcup_i D_i$, with D_i denoting its irreducible components, such that $\mathcal{L}_X \simeq \bigoplus_i \mathcal{O}_X(D_i)$.

Actually, if $X = X(\Delta)$ then $\mathcal{L}_X \simeq \bigoplus_{\rho\in\Delta(1)} \mathcal{O}_X(V(\rho))$ and the toric Euler sequence can be related to a sequence of vector spaces which we discussed already. Namely, $PicX \otimes \mathbf{C} = H^0(X, \Omega_X)$ and therefore we can identify $N_1(X) \otimes \mathcal{O}_X$ with the dual sheaf $\mathcal{H}_X^\vee$. We have a commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{H}_X^\vee & \longrightarrow & \bigoplus_{\rho\in\Delta(1)} \mathcal{O}_X \cdot \tilde{e}_\rho & \longrightarrow & N \otimes \mathcal{O}_X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H}_X^\vee & \longrightarrow & \bigoplus_{\rho\in\Delta(1)} \mathcal{O}_X(V(\rho)) & \longrightarrow & TX & \longrightarrow & 0
\end{array}$$

in which the first row is coming from an exact sequence of vector spaces which we discussed in the first lecture and the second row is the dual of the Euler sequence. The vertical arrow in the middle evaluates the section of $\mathcal{O}_X(V(\rho))$ which vanishes exactly at $V(\rho)$ while the right hand arrow associates to any 1-parameter subgroup in the torus T_N its tangent vector field, see [Jaczewski94] for details.

Lecture 5: Fano manifolds.

I start by recalling the following

Definition. A normal variety X is Fano if and only if some multiple of the anticanonical divisor $-K_X$ is an ample Cartier divisor.

As explained in the introduction, Fano varieties play a special role in the Minimal Model Program: they form a class opposite to varieties with nef canonical divisor and they are fibers of Fano-Mori fibrations.

In the present lecture I will deal with the smooth case only.

Definition. A convex polytope $P \subset N_{\mathbf{R}}$ whose vertices are in N is called Fano if each $(n-1)$-dimensional face of P is spanned by vectors which form a basis of the lattice N and moreover the origin 0 is the only point of N in the interior of P.

To any Fano polytope P we can associate a simplicial fan $\Delta(P)$, rays of which are generated by the vertices of P. If we take now $P(\Delta(P))$, where $P(\Delta)$ is the polytope defined in my previous lecture, then the result is P again. Now it is clear that the associated toric variety $X(\Delta(P))$ is a Fano manifold and this construction defines a bijective relation between Fano polytopes and Fano manifolds. Thus, in order to study Fano manifolds one should understand Fano polytopes. We have the following observation.

Lemma. The number of vertices of n-dimensional Fano polytope is bounded by n^2+1.

Proof. [Voskresenskii-Klyachko] see [Batyrev99]

The following result is more complicated

Proposition. [Batyrev99 and references therein] The number of n-dimensional toric Fano manifolds is bounded.

In fact we have a more general result.

Theorem. [Kollár-Miyaoka-Mori, Campana] The number of deformation types of n-dimensional Fano manifolds is bounded.

This may encourage us to classify Fano n-folds. The case $n = 1$ is trivial. The case $n = 2$ is a bit harder but classical. Fano manifolds in dimension 2 are called del Pezzo surfaces. We have the following classical

Theorem. There are 10 deformation types of del Pezzo surfaces: $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{P}^2$ blown-up in $0, \ldots, 8$ "sufficiently general" points.

The following observation is easier and just requires analysing Fano polygons (do it as an exercise).

Proposition. There are 5 Fano polygons and they correspond to $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{P}^2$ blown-up at $0, \ldots, 3$ fixed points of the torus action.

Now let me explain the ideas related to the classification of higher dimensional Fano manifolds. In toric case we have an easy.

Proposition. $\mathbf{P}^n$ is the only toric (Fano) n-fold with $b_2 = 1$.

The corresponding non-toric version of this result is incomparably harder and, in fact, the main obstacle to the classification of Fano n-folds. Actually, the story of the subsequent classification result (which I present in a very vague form) goes back to early '30 while its rigorous proof was completed in mid '80 (in between there were several false claims and omissions).

Theorem. [Fano, Roth, Iskovskih, Shokurov] There exists a classification of Fano 3-folds with $b_2 = 1$: there are 17 deformation types of them.

Now let me explain an approach leading to the classification of toric 3-folds. I will not deal with a combinatorial approach, due to Batyrev, Oda, Watanabe-Watanabe and Sato — which works in dimension 4 too — instead I will explain a method of Mori-Mukai which led to the following

Theorem. [Mori-Mukai] There exists a classification of Fano 3-folds with $b_2 \geq 2$: there are 89 deformation types of them.

Actually, the enumeration of all possible Fano polytopes and associated toric Fano 3-folds becomes rather easy once we prove the following

Theorem. Any toric Fano 3-folds is either isomorphic to $\mathbf{P}^3$ or to a $\mathbf{P}^1$-bundle over a del Pezzo surface, or is obtained by blowing them up along 1-strata of the torus action.

I begin the proof of the theorem by recalling possible types of contractions $\varphi_R : X \to X_R$ of Mori extremal rays of smooth toric 3-folds (proving this is an exercise):
(1) contraction of $\mathbf{P}^3$ to a point,
(2) $\varphi_R : X \to \mathbf{P}^1$ is a $\mathbf{P}^2$ bundle,
(3) X_R is a smooth toric surface and $\varphi_R : X \to X_R$ is a $\mathbf{P}^1$-bundle,
(4) $\varphi_R : X \to X_R$ is a simple blow-down of an invariant divisor E_R to a 1-strata in smooth X_R,
(5) $\varphi_R : X \to X_R$ is a blow-down of an invariant divisor $E_R \simeq \mathbf{P}^2$ to a fixed point in X_R, which may be non-smooth.

We will say that two rays R_1 and R_2 are twins if their contractions are of type (4) and moreover $E_{R_1} = E_{R_2}$. Let us note that $E_{R_i} = \mathbf{P}^1 \times \mathbf{P}^1$ and twin rays span a 2-dimensional face in $NE(X)$ supported by $-K_X + E_{R_i}$.

Now we assume that X is a Fano 3-fold. Let me go through the steps of the proof of the theorem. We are supposed to prove that, except the case (1), X admits a contraction of type (3) or (4) and the target X_R is a Fano manifold.
(a) If X is a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ (contraction of type (2)) then it is either $\mathbf{P}^1 \times \mathbf{P}^2$ or the blow up of $\mathbf{P}^3$ along a line (easy verification).
(b) If X admits a contraction of type (5) then it admits also a contraction of type (3) or (4). Indeed, take a ray R' such that $E_R \cdot R' > 0$ then the contraction of R' can not have fiber of dimension 2 because, taking the intersection of the loci of these rays, we would get a curve in both R and R'.
(c) If X admits a $\mathbf{P}^1$ bundle structure over a surface X_R (contraction of type (3)) then X_R is a del Pezzo surface. This follows from a more general

Lemma. [Szurek-Wiśniewski, Kollár-Miyaoka-Mori] Let $\varphi : X \to Y$ be a projective bundle (or, more generally, a smooth map). If X is a Fano manifold then Y is a Fano manifold too.

(d) If X admits a contraction φ_R of type (4) then the target X_R is Fano unless the ray R has a twin ray R'. Indeed, by adjunction formula $\varphi_R(C) \cdot (-K_{X_R}) \geq (C \cdot (-K_X))$, unless $C \subset E_R$.

(e) If X has twin rays R_1 and R_2 then it has also a ray R whose contraction is of type (3) or (4), and in the latter case R has no twin. The proof of this statement is similar to the one of (b): this time we consider the contraction of the face spanned by R_1 and R_2, whose exceptional divisor is $E = E_{R_1} = E_{R_2}$, then we choose a ray R' such that $R \cdot E > 0$.

This concludes the proof of the theorem. Now you can do the classification of Fano 3-folds along the following lines. (0) Blow-up 1-dimensional strata of $\mathbf{P}^3$ as long as you still get Fano manifolds. (1) Prove that any toric $\mathbf{P}^1$ bundle is decomposable hence of type $\mathbf{P}(L \oplus \mathcal{O})$ and its fan can be obtained from the fan of $\mathbf{V}(L)$ (constructed as in the exercise to Lecture 1) by adding a ray opposite to $\langle e_0 \rangle$ (notation as in the exercise). (2) Find Fano polytopes associated to possible $\mathbf{P}^1$ bundles over toric del Pezzo. (3) Find out which edges of the obtained polytopes can be blown up to obtain other Fano polytopes. The result should be as follows:

Theorem. There are 18 toric Fano 3-folds.

Readings.

The following list contains mostly papers to which I refer directly in my lectures; additionally, references to the sources of Mori Theory you can find in [CKM88] while references to articles on general Fano manifolds can be found in [Batyrev99].

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