# Symplectic contractions of 4-folds

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joint project with Marco Andreatta

# symplectic contractions

Local symplectic contraction is a morphism $\pi : X \longrightarrow Y$:

- $X$ is a smooth variety with a closed holomorphic $2$-form, non-degenerate at every point,
- $Y$ is an affine normal variety,
- $\pi$ is a birational projective morphism.

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In dimension 2 symplectic contractions are classical and they are minimal resolutions of Du Val singularities $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. The dual diagram of the $(-2)$-curves in exceptional set is a Dynkin diagram. They are quotients of type $\mathbb{C}^2/H$ with $H < SL(2, \mathbb{C})$ a finite group.

# higher dimensional example

Let $S$ be a smooth surface (proper or not). Denote by $S^{(n)}$ the symmetric product of $S$, that is $S^{(n)} = S^n/\sigma_n$, where $\sigma_n$ is the symmetric group of $n$ elements. Let $Hilb^n(S)$ be the Hilbert scheme of $0$-cycles of degree $n$. A classical result of Fogarty says that $Hilb^n(S)$ is smooth and that $\tau : Hilb^n(S) \to S^{(n)}$ is a crepant resolution of singularities.

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Suppose that $S \to S'$ is a minimal resolution of a Du Val singularity, then $Hilb^n(S) \to S^{(n)} \to (S')^{(n)}$ is a local symplectic contraction. We note that $(S')^{(n)}$ is a quotient singularity with respect to the action of the wreath product $H \wr \sigma_n = (H^n) \rtimes \sigma_n$.

# higher dimensional results

- semi-smallness: for every closed subvariety $F \subset X$ we have $2 codim F \geq codim(\pi(F))$. If the equality holds then $F$ is called a maximal cycle.
- description in codimension 2: locally $\pi$ is the same as for surfaces, possibly modulo automorphism of Dynkin diagram
- McKay correspondence: cohomology is generated by maximal cycles and, if $\pi$ is a resolution of quotient $\mathbb{C}^{2r}/H$, they are related to conjugacy classes of $H$
- small contractions and Mukai flop

# Mori Dream Spaces

By $\mathrm{Nef}(X/Y) \subset N^1(X/Y)$ we understand the closure of the cone spanned by the classes of relatively-ample bundles while by $\mathrm{Mov}(X/Y) \subset N^1(X/Y)$ we understand the cone spanned by the classes of linear systems which have no fixed components.

# Mori Dream Spaces

In the above situation we say that $X$ is a Mori Dream Space (MDS) over $Y$ if in addition

- $\mathrm{Nef}(X/Y)$ is spanned by classes finitely many semi-ample line bundles
- there is a finite collection of small $\mathbb{Q}$-factorial modifications (SQM) over $Y$, $f_i : X - \to X_i$ such that $X_i \longrightarrow Y$ satisfies the above assumptions
- $\mathrm{Mov}(X/Y)$ is the union of the strict transforms of $f_i^*(\mathrm{Nef}(X_i/Y))$ (Mori or nef chambers)

# 4-dim case is MDS

Theorem A: Let $\pi : X \longrightarrow Y$ be a 4-dimensional symplectic contraction:

- $X$ is MDS over $Y$
- all SQM's of $X$ are smooth and connected by sequences of Mukai flops
- $\mathrm{Mov}(X/Y)$ is divided into Mori chambers by hyperplanes

# essential curves

Let $\pi : X \to Y$ be a 4-dimensional local symplectic contraction with the unique 2-dimensional fiber $\pi^{-1}(0)$. By $N_1(X/Y)$ we denote vector space of 1-cycles proper over $Y$. We define $\mathrm{Ess}(X/Y)$ as the convex cone spanned by the classes of curves which are not contained in $\pi^{-1}(0)$.

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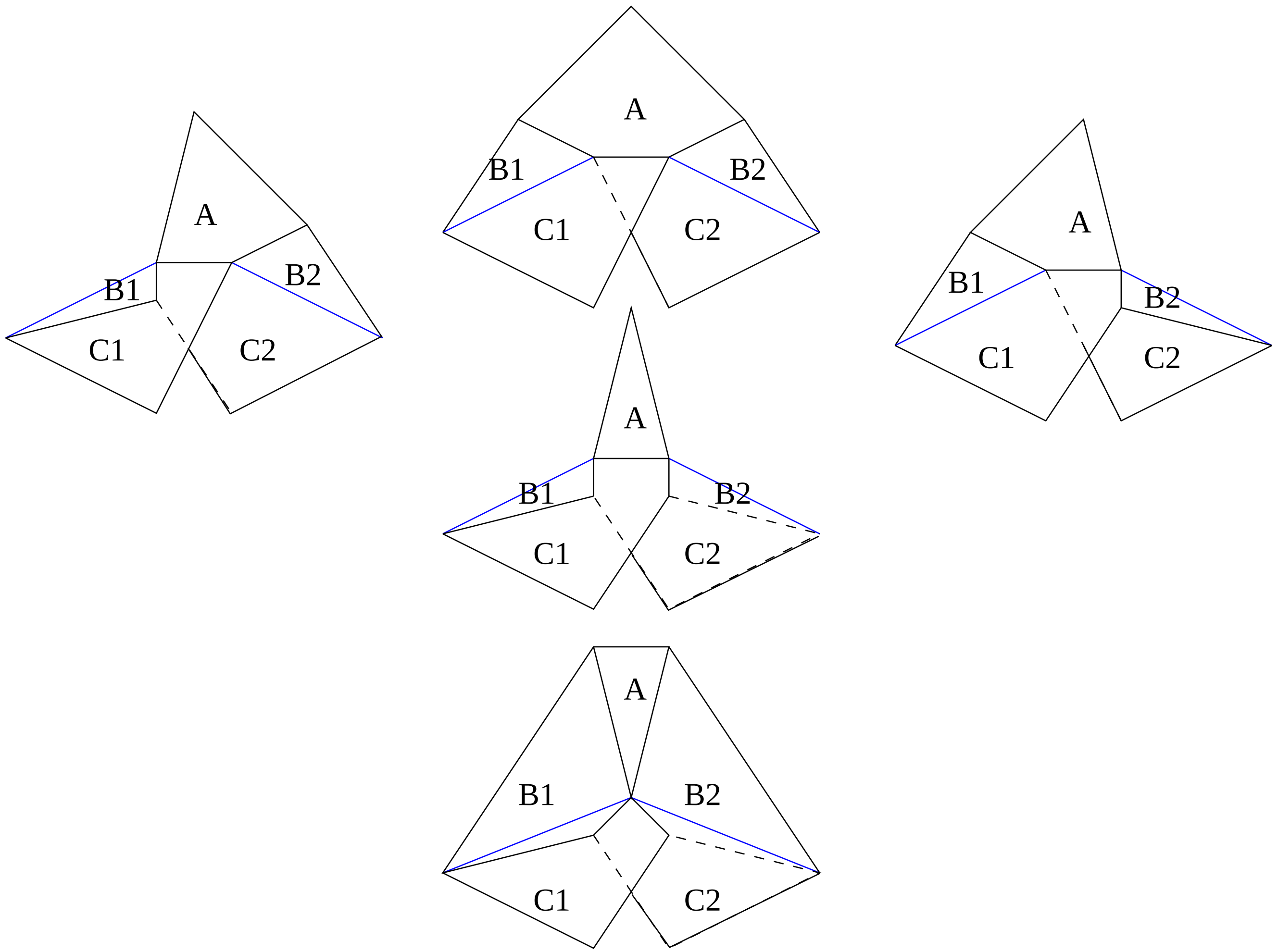
Theorem B: Cones $\mathrm{Mov}(X/Y)$ and $\mathrm{Ess}(X/Y)$ are dual in terms of the intersection product of $N^1(X/Y)$ and $N_1(X/Y)$, that is $\mathrm{Mov}(X/Y) = \mathrm{Ess}(X/Y)^\vee$.

# example: quotient by $\mathbb{Z}_3 \wr \mathbb{Z}_2$



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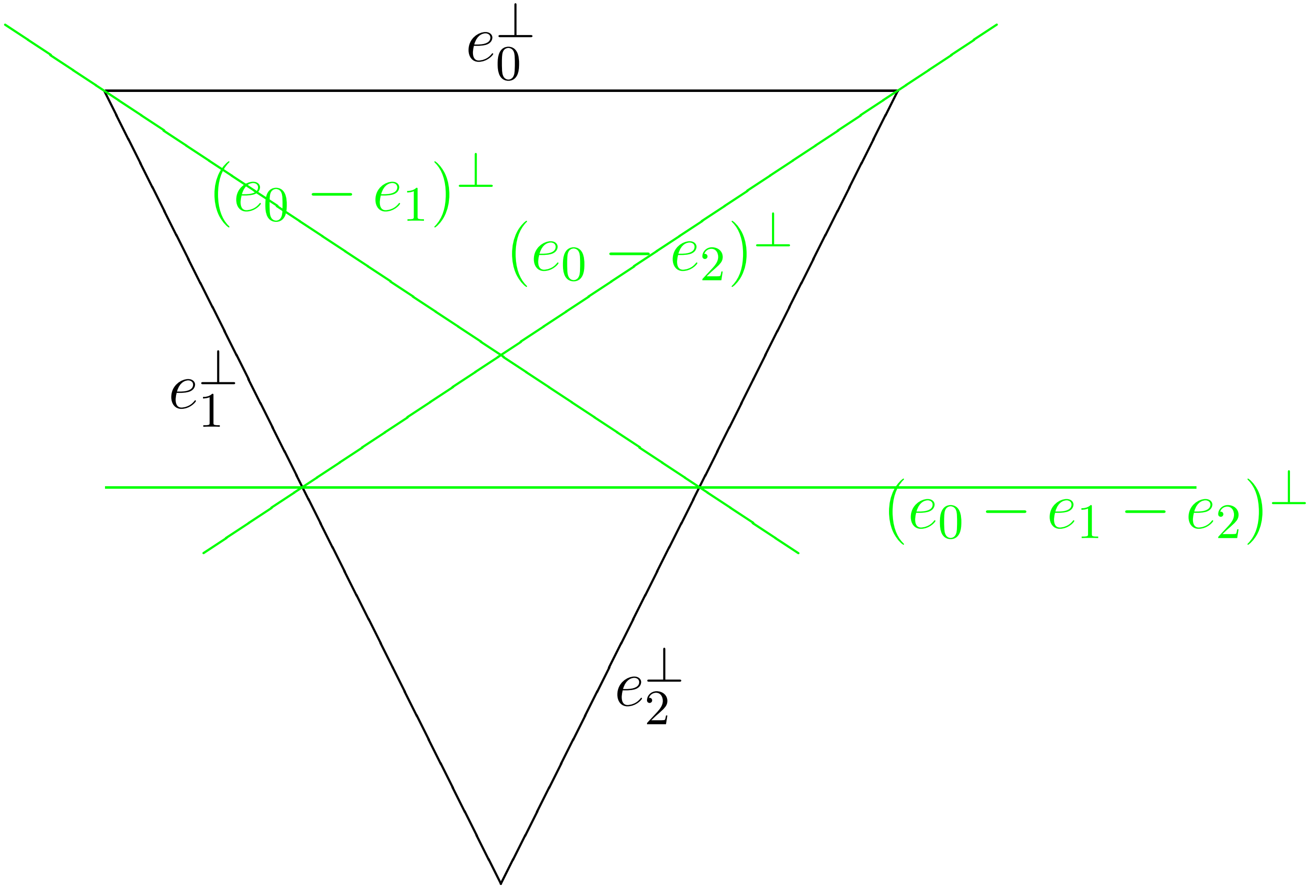


# problems

- $\mathrm{Mov}(X/Y)$ is defined by curves in codimension 1, hence by $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}$ root systems (or their image in $N^1(X/Y)$)
- understand the division of $\mathrm{Mov}(X/Y)$ by flopping hyperplanes
- is this data sufficient to recover the singularity or symplectic contraction?

# set-up for $\mathbb{A}_1 \times \mathbb{A}_n$

Let $\mathbb{A}_1 \times \mathbb{A}_n$ be the root system with minimal positive roots denoted by $e_0, e_1, \ldots, e_n$. Using the intersection form associated to the root system, we can identify the vector space spanned by roots with $N^1(X/Y)$ and its dual $N_1(X/Y)$, where $X \to \mathbb{C}^4/(\mathbb{Z}_{n+1} \wr \mathbb{Z}_2)$ is a symplectic resolution of the quotient singularity. Via this identification exceptional divisors are $E_i = -e_i$ and the cone $\mathrm{Mov}(X)$ is the Weyl chamber of this system.

# theorem

Theorem C: Let $X \to \mathbb{C}^4/(\mathbb{Z}_{n+1} \wr \mathbb{Z}_2)$ be a symplectic resolution. The division of $\mathrm{Mov}(X)$ into Mori chambers is defined by hyperplanes $\lambda_{ij}^{\perp}$ for $1 \le i \le j \le n$, where

$$\lambda_{ij} = e_0 - (e_i + \cdots + e_j)$$

# proof: starting point

- we know the chamber of $\mathrm{Mov}(X)$ associated to the Hilbert resolution of $\mathbb{C}^4/(\mathbb{Z}_{n+1} \wr \mathbb{Z}_2)$
- there is a distinguished divisorial contraction of $X_{Hilb}$ to symmetric product of the minimal resolution of the surface $\mathbb{A}_n$ singularity and it contracts the class of $e_0$ and, in fact, it is associated to the face $\mathrm{Mov}(X) \cap e_0^\perp$
- other faces of this chamber are supported by $-\lambda_{ii} = e_i - e_0$
- thus, if $\lambda \in N_1(X)$ is a flopping class then $\lambda^\perp$ does not meet the relative interior of the face $\mathrm{Mov}(X) \cap e_0^\perp$

# proof: enough to check on a line

On the other hand,

$$\mathrm{Mov}(X) = \mathrm{Mov}(X) \cap e_0^{\perp} + \mathbb{R}_{\geq 0} \cdot (-E_0)$$

Thus, if we take any $D_0$ in the relative interior of $\mathrm{Mov}(X) \cap e_0^{\perp}$ then, by Theorem A, the half-line $D_0 + \mathbb{R}_{\geq 0} \cdot (-E_0)$ must meet hyperplane $\lambda^{\perp}$ for any flopping class $\lambda$. Hence the theorem will be proved if, for a choice of $D_0$, we will show that all hyperplanes $\lambda^{\perp}$ that $D_0 + \mathbb{R}_{\geq 0} \cdot (-E_0)$ meets actually come from the classes $\lambda_{ij}$.

# proof: choose a line

Let us choose a sequence (a vector) of $n$ positive numbers $\overline{\beta} = (\beta_i)$ such that $\beta_1 + \cdots + \beta_{i-1} < \beta_i$, for $i = 2, \ldots, n$. We set $\gamma_{ij} = \beta_i + \cdots + \beta_j$. Then, by our assumption,

$$\gamma_{11} < \gamma_{22} < \gamma_{12} < \gamma_{33} < \gamma_{23} < \gamma_{13} < \gamma_{44} < \gamma_{34} \cdots$$

# proof: choose a line

Let $A$ be the intersection matrix for the root system $\mathbb{A}_n$. The matrix $-A$ is negative definite therefore there exists a unique vector $\overline{\alpha} = (\alpha_i)$ such that $(-A) \cdot \overline{\alpha} = \overline{\beta}$. If we now set $D_0 = \sum_i \alpha_i E_i$ then $D_0 \cdot e_0 = 0$ and $D_0 \cdot e_i = \beta_i > 0$ for $i = 1, \ldots, n$ hence $D$ is in the relative interior of $\mathrm{Mov}(X) \cap e_0^\perp$.

# proof: choose a line

We set

$$D_t = D_0 - (t/2)E_0$$

Then $D_t \cdot \lambda_{ij} = t - \gamma_{ij}$ so that $\gamma_{ij}$ is the threshold value of $t$ for the form $\lambda_{ij}$ on the half-line $\{D_t : t \in \mathbb{R}_{\geq 0}\}$. The SQM model of $X$ on which the divisor $D_t$ is ample will be denoted by $X_t$. Now our theorem is equivalent to saying that the models $X_t$ are in bijection with connected components (open intervals) in $\mathbb{R}_{>0} \setminus \{\gamma_{ij})$. This can be verified by starting from $X_{Hilb}$ associated to interval $(0, \gamma_{11})$ and proceeding inductively.

# proof: choose a line

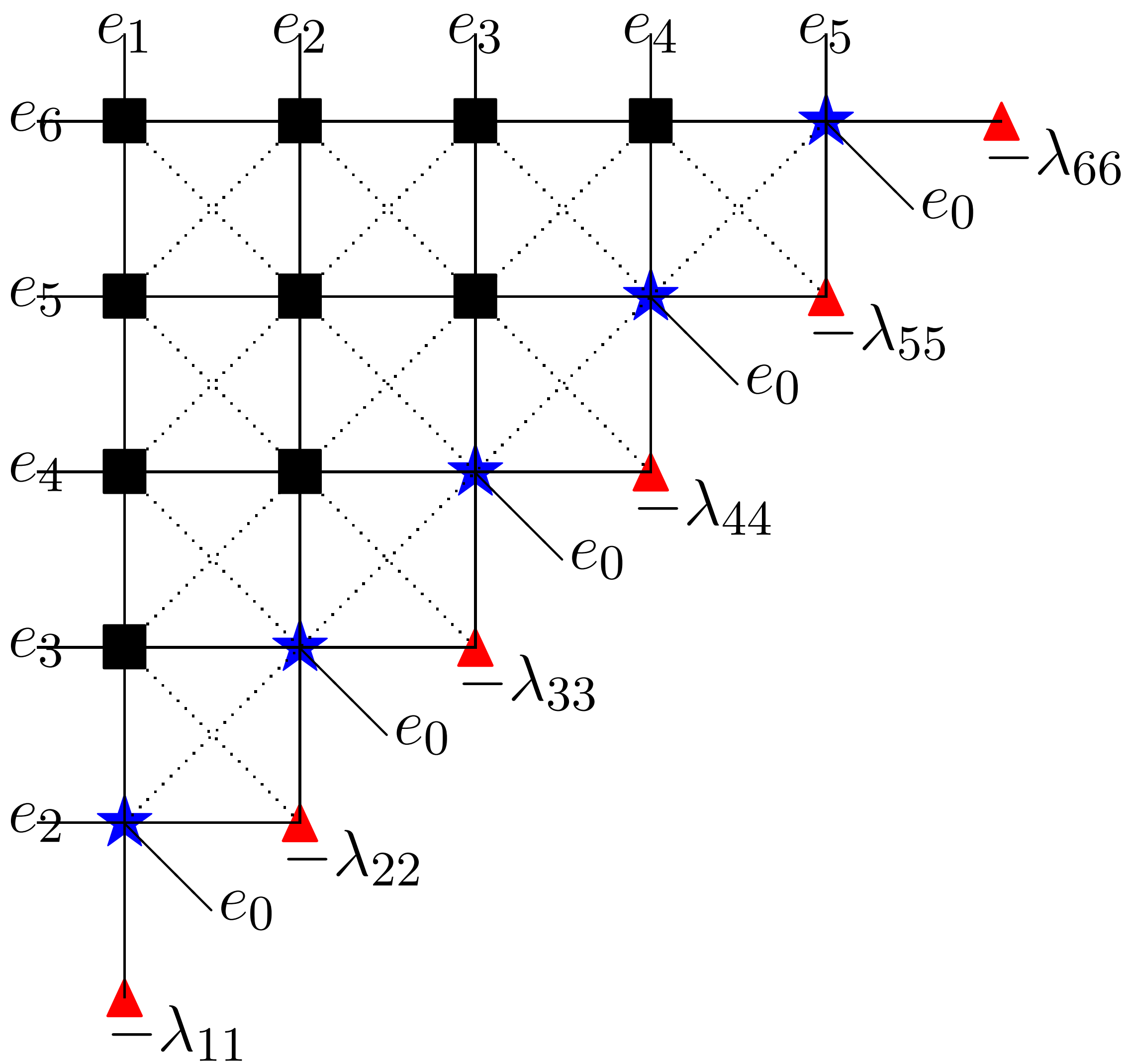
That is, for $t \in (\gamma_{ij}, \gamma_{i'j'})$, where $\gamma_{ij}$ and $\gamma_{i'j'}$ are two consequtive numbers in the sequence of $\gamma$'s, we verify that $\mathbb{P}^2$'s which are in the exceptional locus of $X_t$ have lines whose classes are only of type $\pm\lambda_{rs}$ with pairs $(i, j)$ and $(i', j')$ among those $(r, s)$ which occur on $X_t$. The sign of $\pm\lambda_{rs}$ will depend on the position of $\gamma_{rs}$ with respect to $t$. Hence we can flop the $\mathbb{P}^2$ with lines of type $-\lambda_{i'j'}$ and proceed to the next interval. The argument will stop when $X_t$ contains only one $\mathbb{P}^2$ with lines in the class $+\lambda_{1n}$.

# conclusion of proof: verification

In the subsequent diagrams we verify our statements:

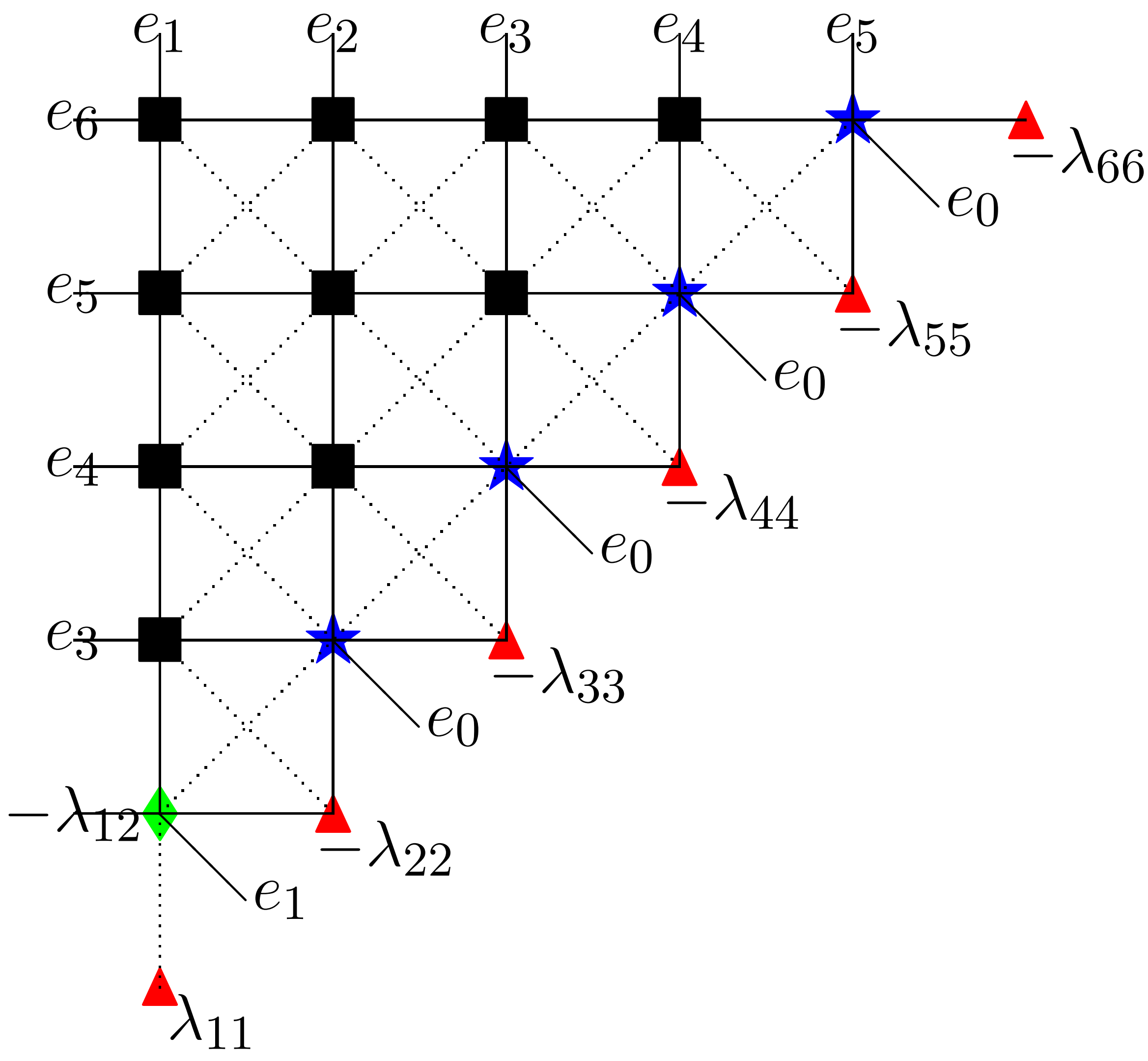
- We ignore the components which are not flopped, that is components which are isomorphic to $F_4$.
- The isomorphisms classes of surfaces: ▲ $= \mathbb{P}^2$, ♦ $= F_1$, ■ $= \mathbb{P}^1 \times \mathbb{P}^1$ and ★ denotes blow-up of $\mathbb{P}^2$ in two points (or $\mathbb{P}^1 \times \mathbb{P}^1$ in one point).
- The incidence curves are described by their class in cohomology: they are $e_0, e_1, \ldots, e_n$ and $\lambda_{ij} = e_0 - (e_i + \cdots + e_j)$.
- The incidence of components in terms of points is denoted by dotted line segments.

# $(0, \gamma_{11})$

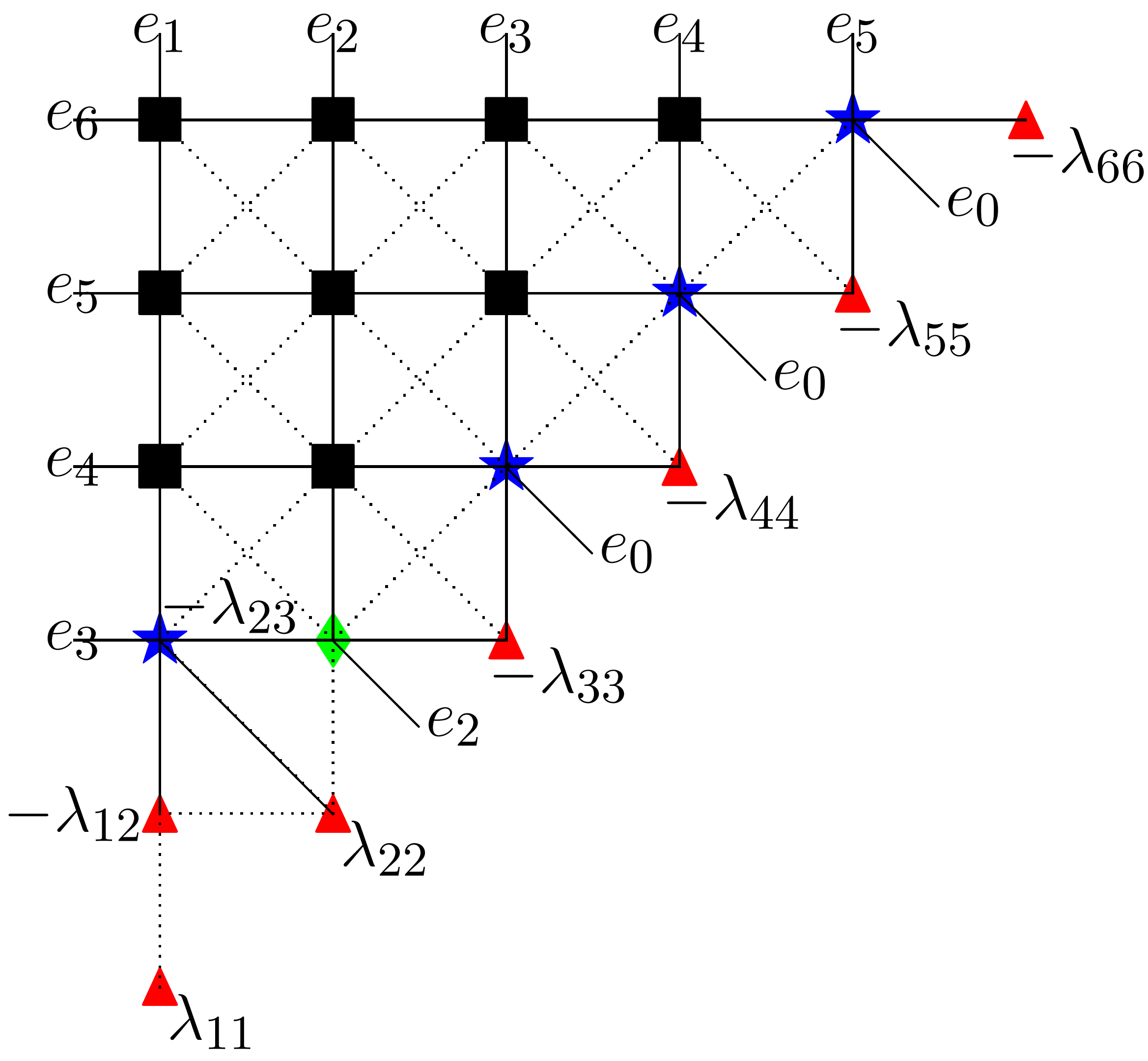


$e_1$
$e_2$
$e_3$
$e_4$
$e_5$
$e_6$
$e_5$
$e_4$
$e_3$
$e_2$
$-\lambda_{66}$
$e_0$
$-\lambda_{55}$
$e_0$
$-\lambda_{44}$
$e_0$
$-\lambda_{33}$
$e_0$
$-\lambda_{22}$
$e_0$
$-\lambda_{11}$

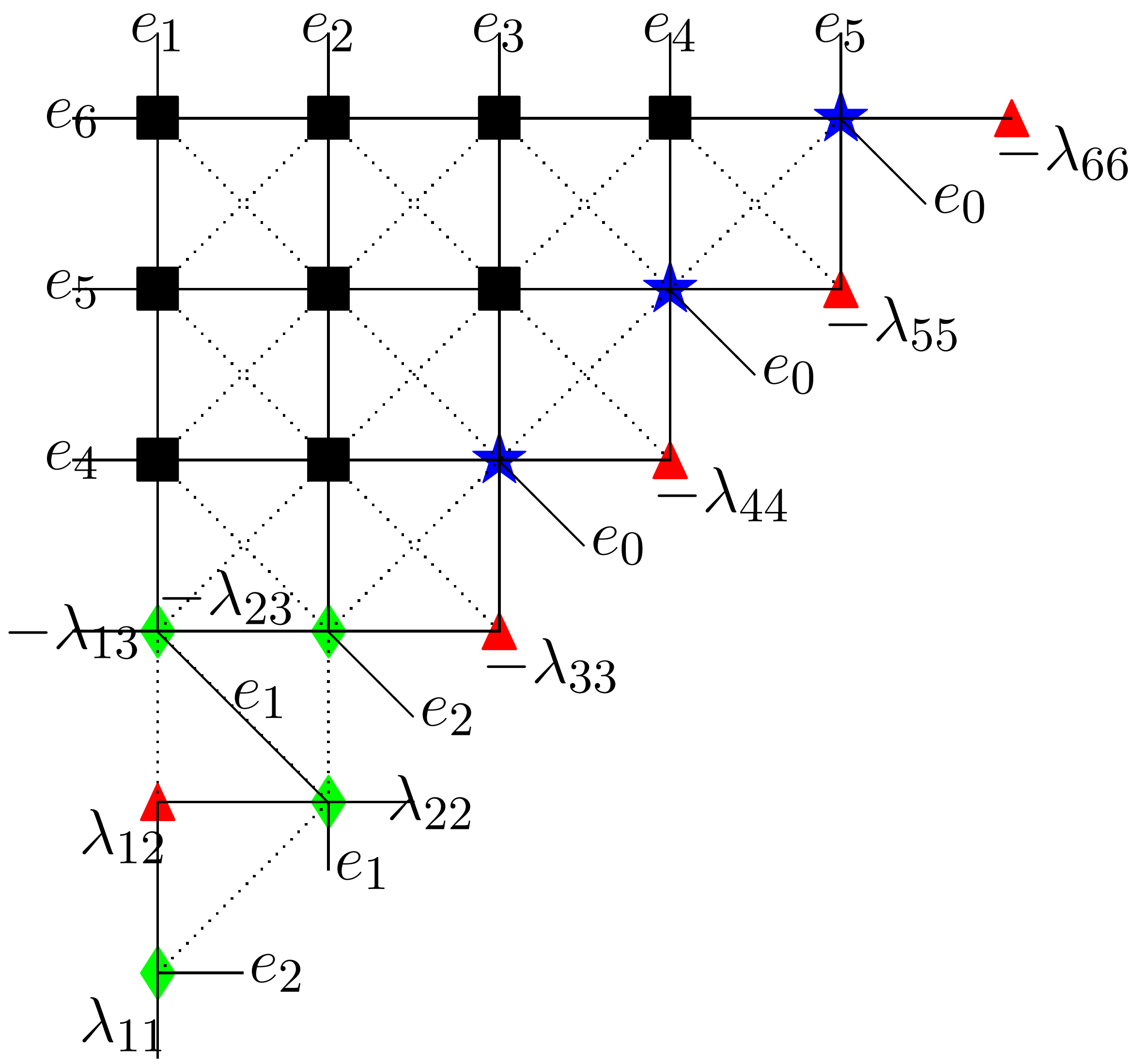
# $(\gamma_{11}, \gamma_{22})$



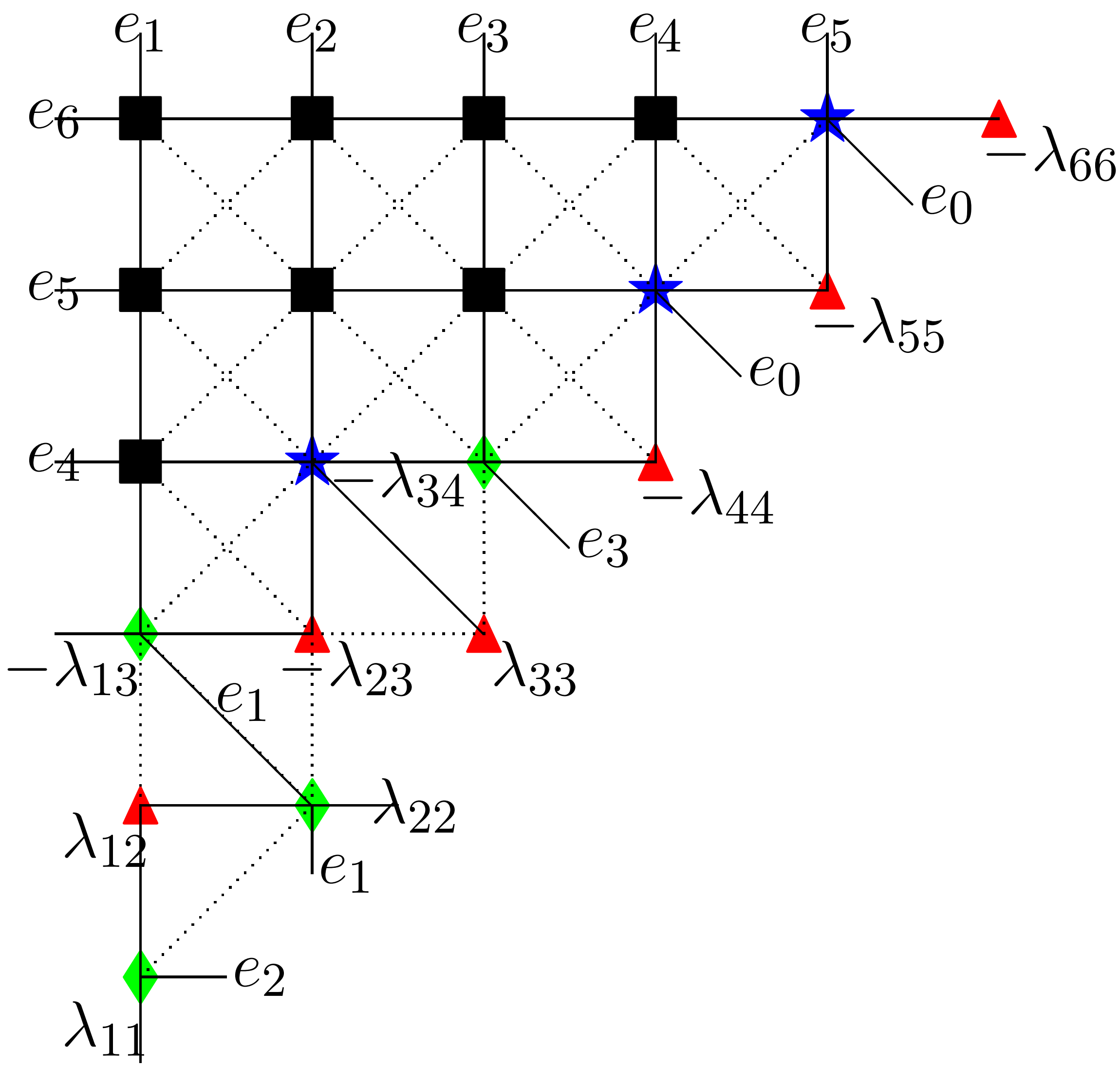
# $(\gamma_{22}, \gamma_{12})$



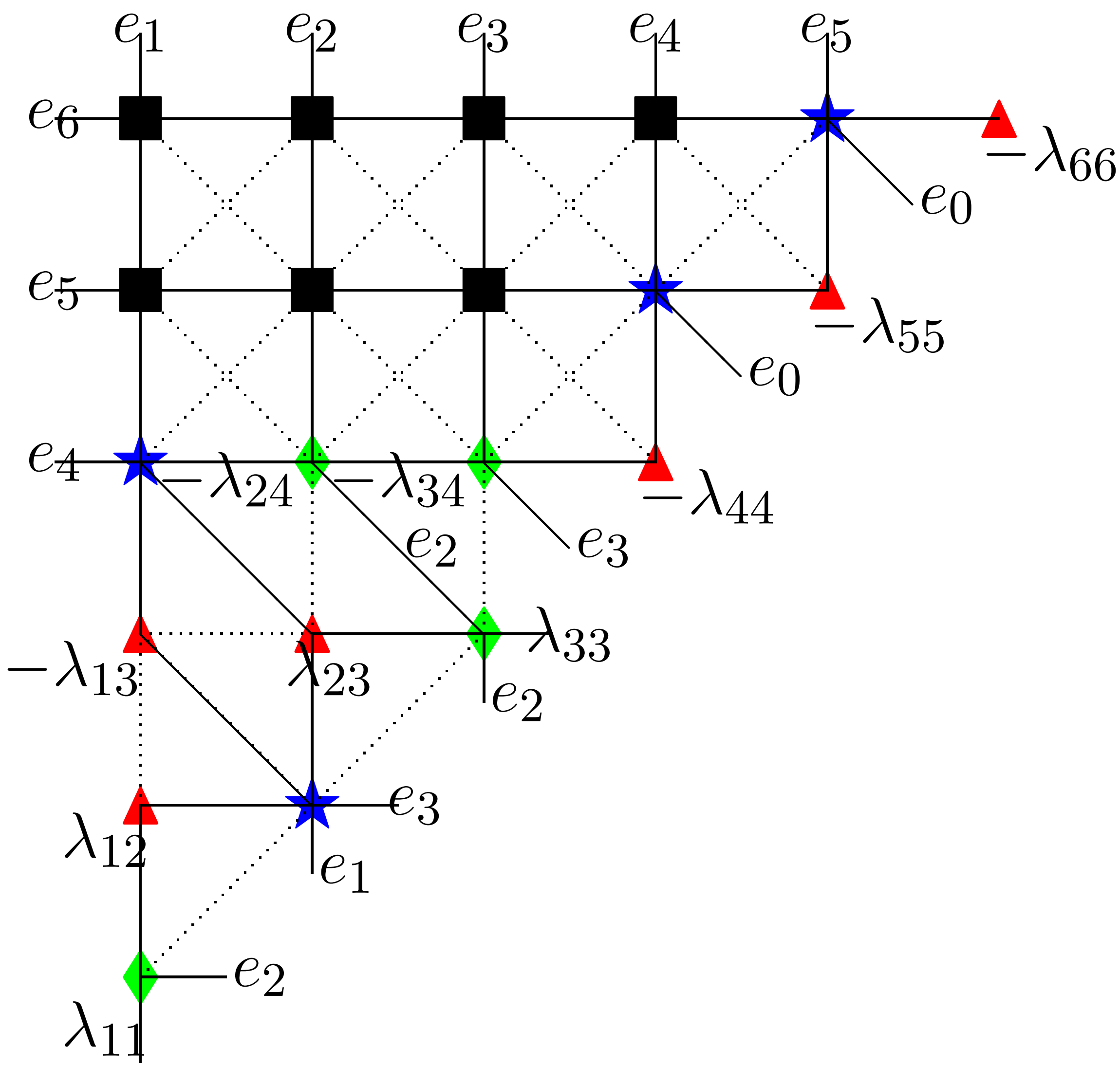
# $(\gamma_{12}, \gamma_{33})$



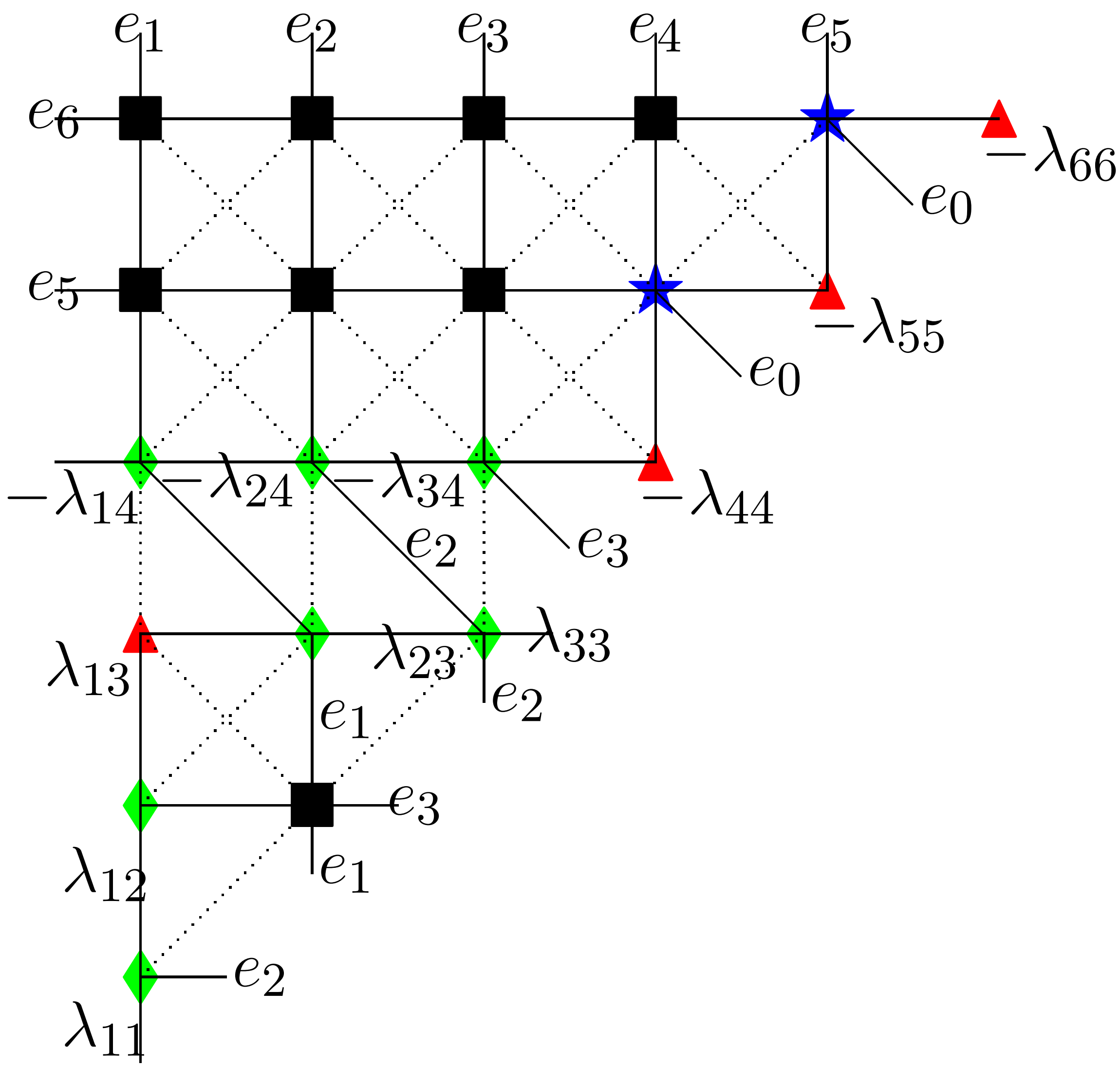
# $(\gamma_{33}, \gamma_{23})$



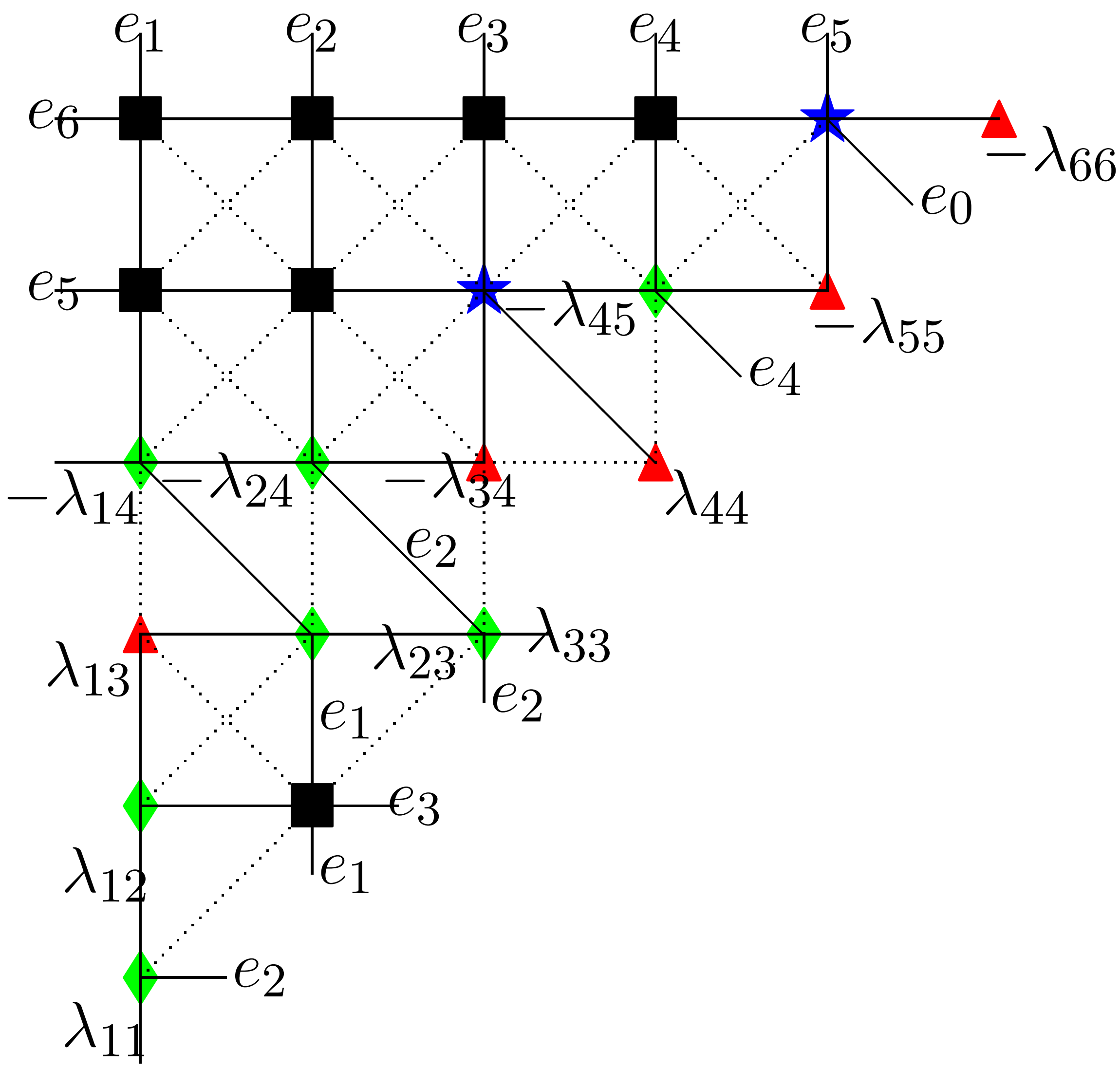
# $(\gamma_{23}, \gamma_{13})$



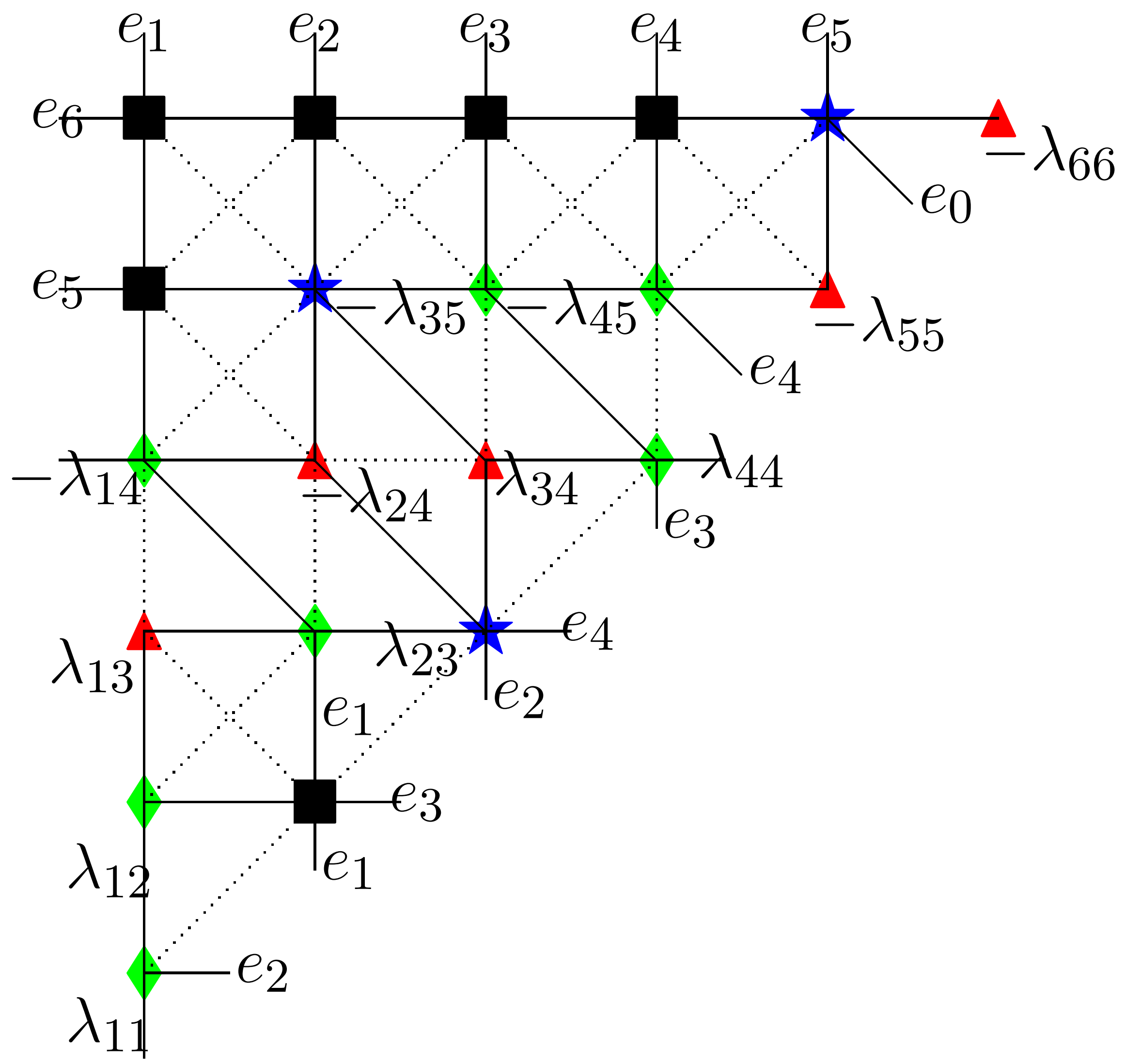
# $(\gamma_{13}, \gamma_{44})$



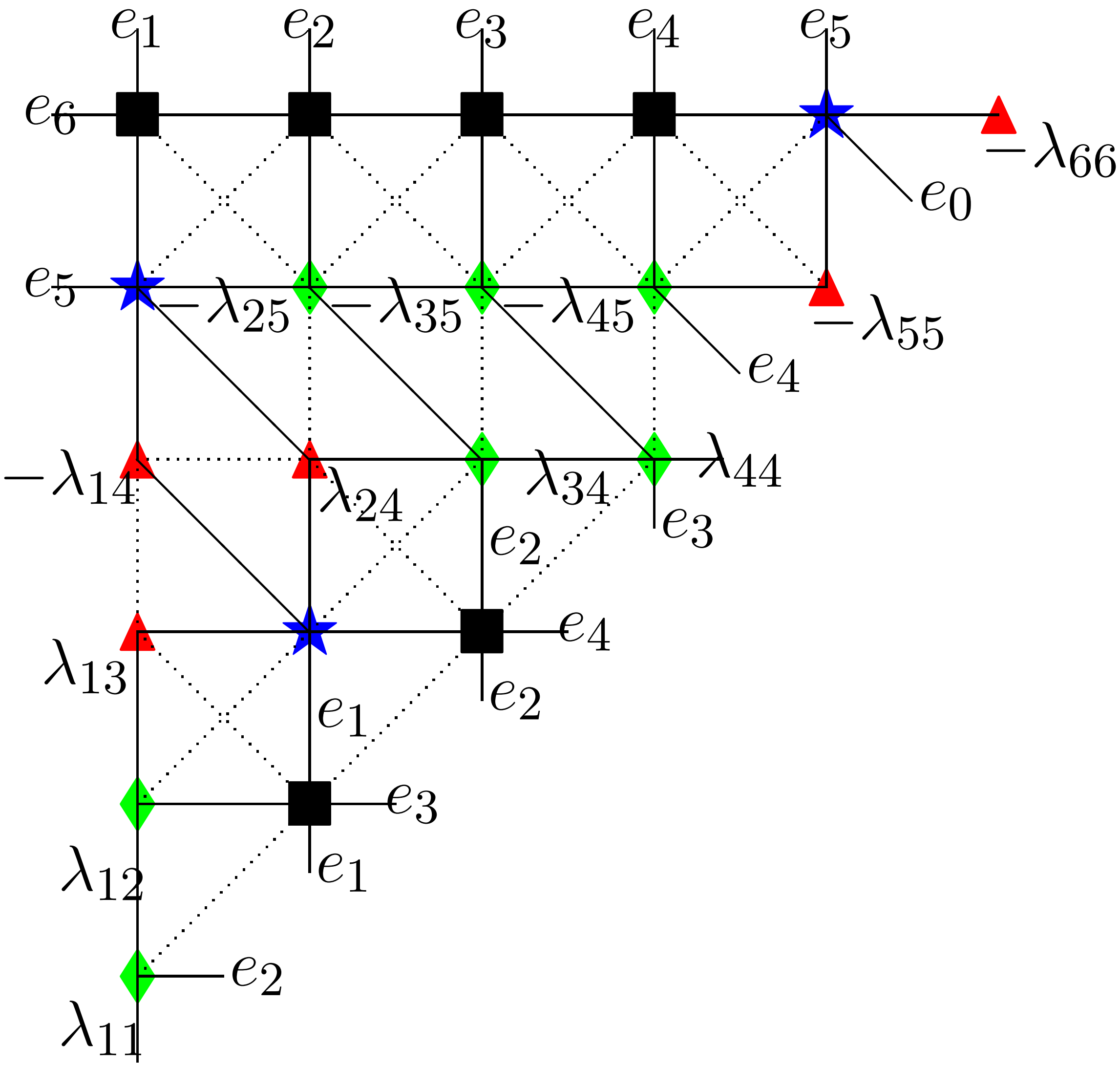
# $(\gamma_{44}, \gamma_{34})$



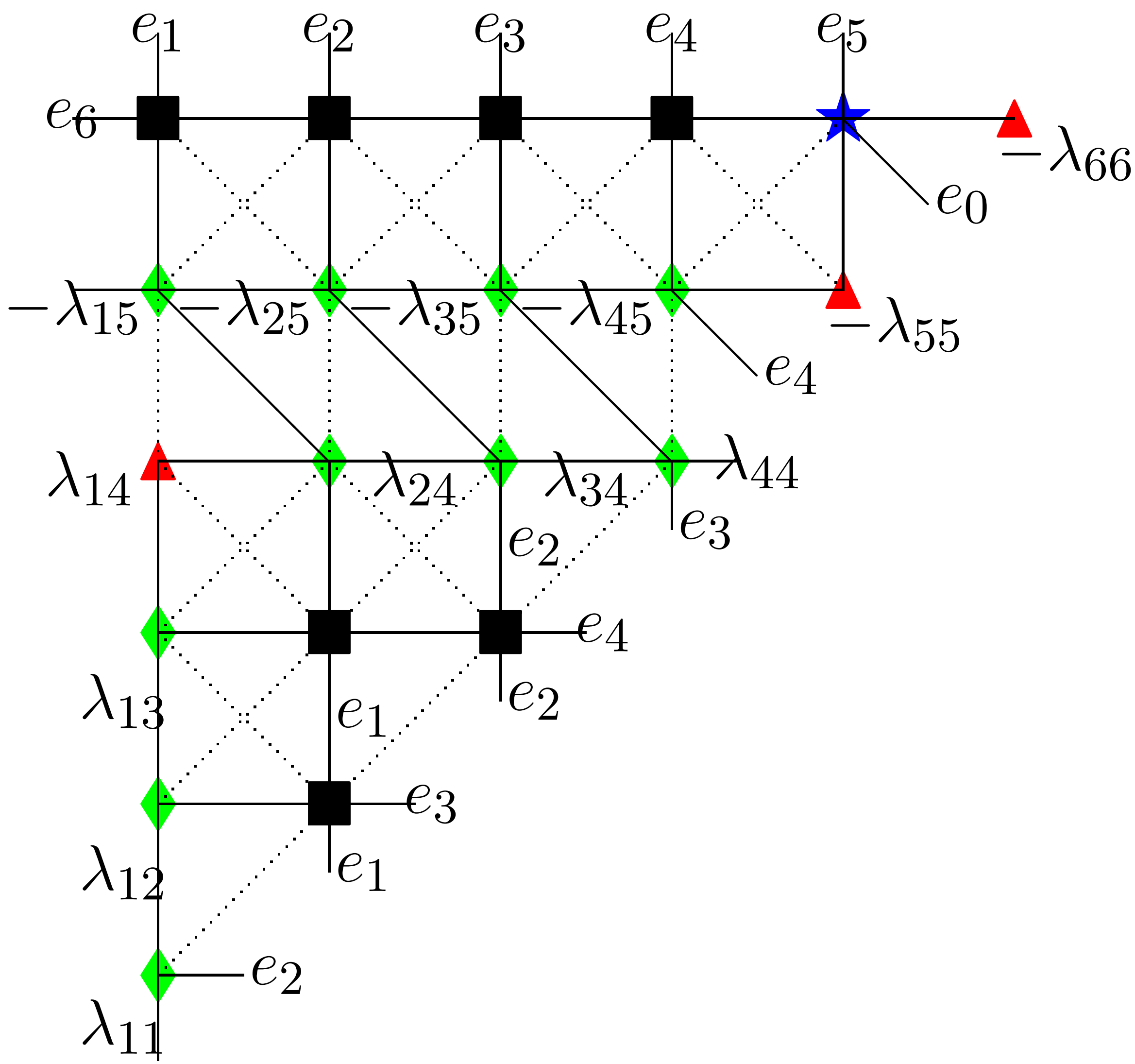
# $(\gamma_{34}, \gamma_{24})$



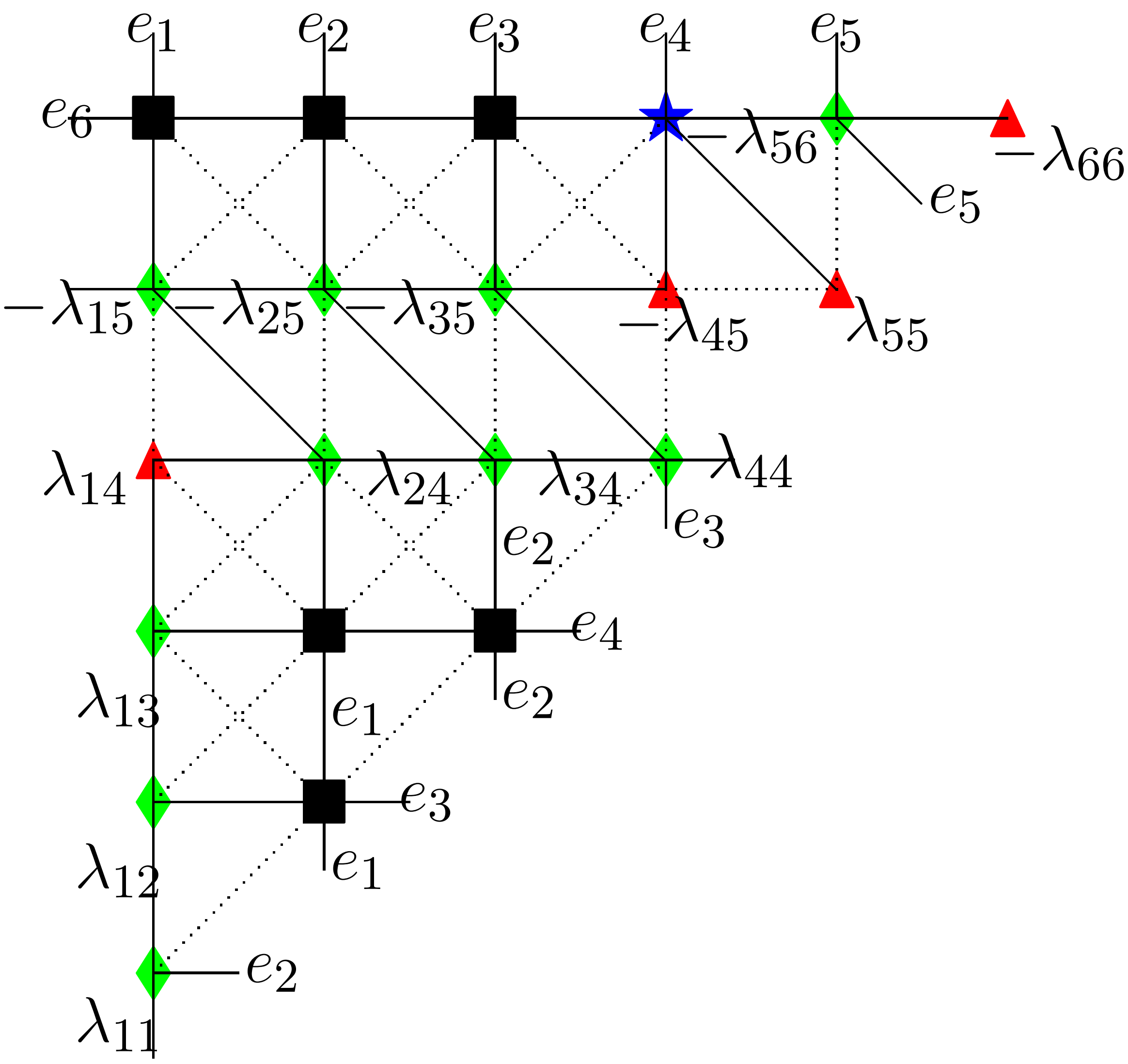
# $(\gamma_{24}, \gamma_{14})$



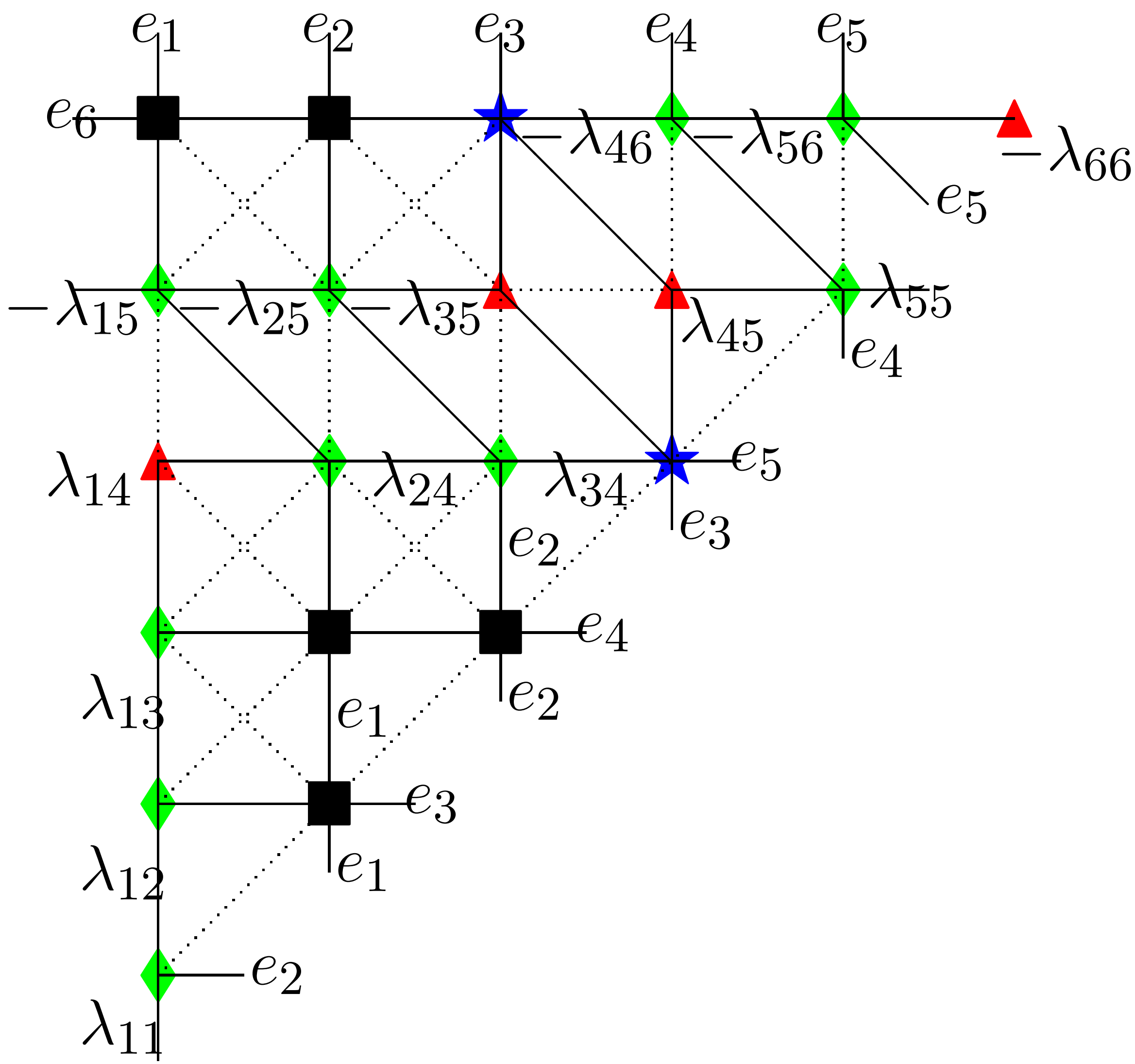
# $(\gamma_{14}, \gamma_{55})$



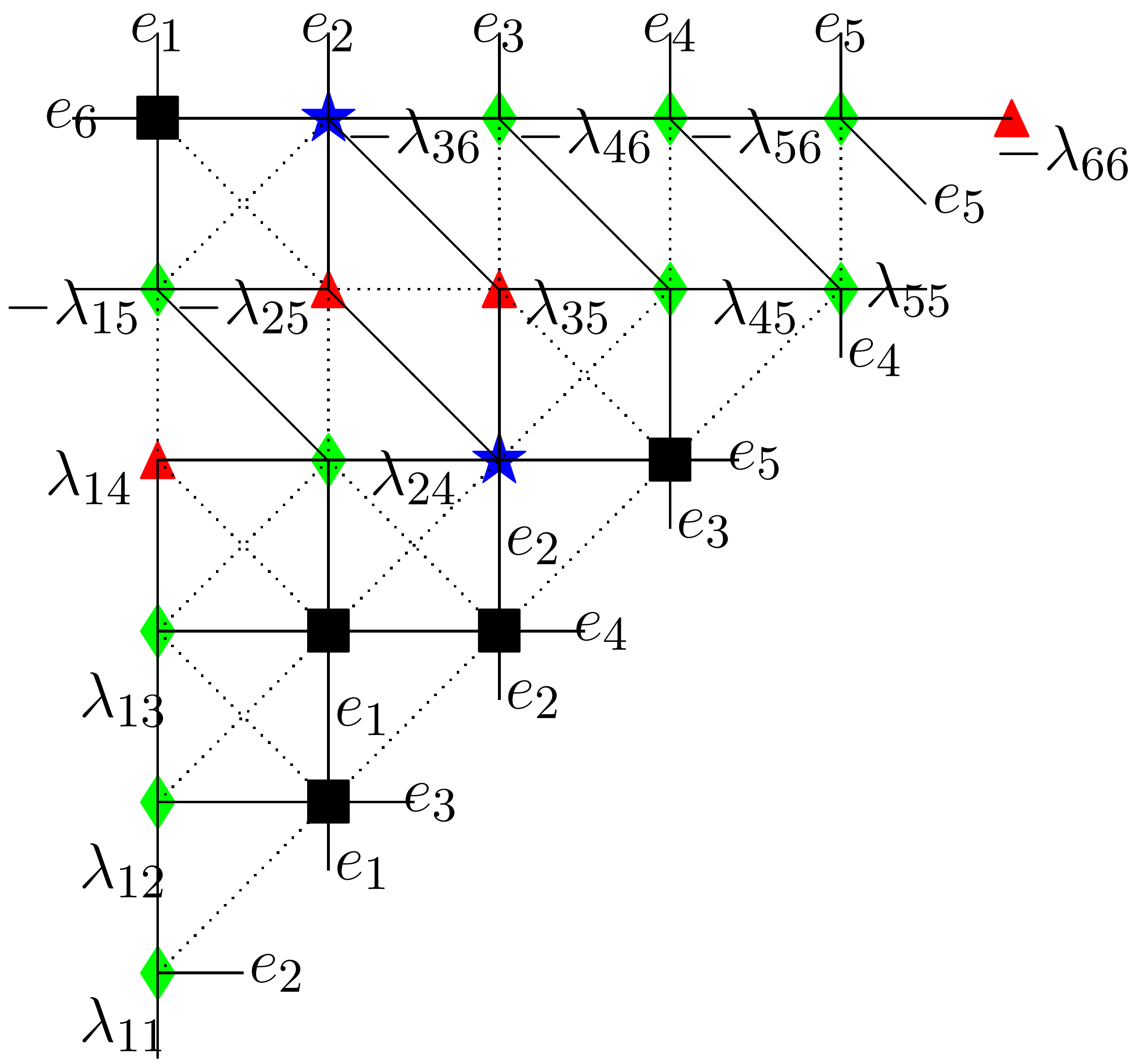
# $(\gamma_{55}, \gamma_{45})$



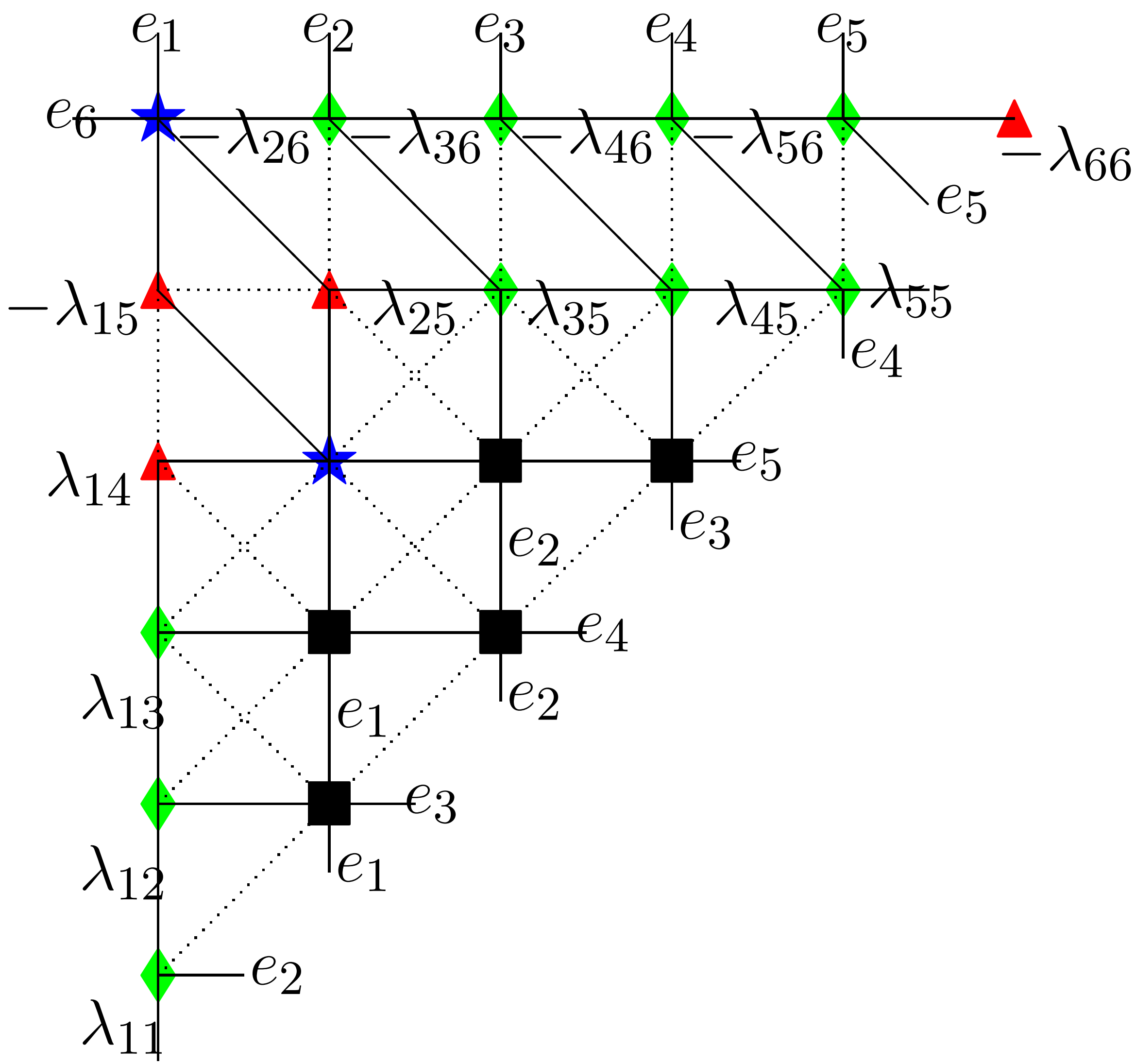
# $(\gamma_{45}, \gamma_{35})$



# $(\gamma_{35}, \gamma_{25})$



# $(\gamma_{25}, \gamma_{15})$



# $(\gamma_{15}, \gamma_{66})$