On Cox rings of resolutions of quotient singularities

According to Facchini, Golzàles-Alonso, Lasoń and Donten-Bury

J. A. Wiśniewski

Uniwersytet Warszawski

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Cox rings

Let X be a $\mathbb{Q}$-factorial normal variety defined over $\mathbb{C}$.
Assume $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$ and X projective over $\operatorname{Spec}\Gamma(X, \mathcal{O}_X)$.
Take a subgroup $\Lambda \subset \mathrm{Div}(X)$ such that the class map gives isomorphism $\Lambda \simeq \mathrm{Cl}(X)$ (or just inclusion $\Lambda \hookrightarrow \mathrm{Cl}(X)$). Take a Λ-graded $\Gamma(\mathcal{O}_X)$-module

$$\mathrm{Cox}_\Lambda(X) = \bigoplus_{D_\lambda \in \Lambda} \Gamma(X, \mathcal{O}_X(D_\lambda))$$

where

$$\Gamma(X, \mathcal{O}_X(D)) = \{f \in \mathbb{C}(X)^* : \mathrm{div}(f) + D \geq 0\} \cup \{0\}$$

and define the multiplication is as in the field of rational functions $\mathbb{C}(X)$ to make the ring structure on $\mathrm{Cox}_\Lambda(X)$.

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group action, quotient

The Λ-grading defines the action of the group $\mathbb{T}_\Lambda = \mathrm{Hom}(\Lambda, \mathbb{C}^*)$ on $\mathrm{Cox}_\Lambda(X)$ by the formula

$$\mathrm{Hom}(\Lambda, \mathbb{C}^*) \times \Gamma(X, \mathcal{O}_X(D_\lambda)) \ni (\mu, \sigma) \longrightarrow \mu(\lambda) \cdot \sigma \in \Gamma(X, \mathcal{O}_X(D_\lambda))$$

If Cox_Λ is finitely generated algebra then this defines the action of $\mathbb{T}_\Lambda$ on $Y_\Lambda = \mathrm{Spec}\, \mathrm{Cox}_\Lambda(X)$.
If moreover Λ contains a divisor ample over $\mathrm{Spec}\, \Gamma(X, \mathcal{O}_X)$ then exists an open $\widehat{Y_\Lambda} \subset Y_\Lambda$, whose complement is of codimension ≥ 2, and X is a $\mathbb{T}_\Lambda$ quotient of $\widehat{Y_\Lambda}$.

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toric varieties

Let $X = X(\Sigma)$ be a toric variety associated to a fan Σ in the space $N_{\mathbb{R}}$ spanned by the lattice N.
Denote $M = \mathrm{Hom}(N, \mathbb{Z})$ and $\mathbb{T}_N = \mathrm{Hom}(M, \mathbb{C}^*)$.
Let $\Sigma^1 = \{v \in N : \mathbb{R}_{\geq 0} \cdot v \in \Sigma\}$ be the set of generators of rays of Σ and $\widehat{N} = \bigoplus_{v \in \Sigma^1} \mathbb{Z} \cdot \widehat{v}$ a free group generated by them.
Take the sequence

$$0 \longrightarrow \Lambda^\vee \longrightarrow \widehat{N} \longrightarrow N$$

where the RHS morphisms maps $\widehat{v} \longrightarrow v$. The dual is

$$0 \longrightarrow M \longrightarrow \widehat{M} \longrightarrow \mathrm{Cl}(X) \longrightarrow 0$$

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For every $D_\lambda \in \mathrm{Cl}(X)$ the torus $\mathbb{T}_N$ acts on the space of sections $\Gamma(X, \mathcal{O}_X(D_\lambda))$ and decomposes it into the eigenspaces associated to linear combinations of $\mathbb{T}_N$-invariant divisors.
Therefore $\mathrm{Cox}(X) = \mathbb{C}[\widehat{M} \cap \widehat{\sigma}^+]$ where $\widehat{\sigma}^+$ denotes the positive orthant in $\widehat{M}_{\mathbb{R}}$ and

$$0 \longrightarrow \Lambda^\vee \longrightarrow \widehat{N} \longrightarrow N$$

determines the quotient structure in codimension 1.
Here are two examples:
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why?

- Q: Why consider Cox rings in the context of singularities?
- A1: Because this is a part of “coxification” attitute towards varieties.
- A2: Because resolutions of singularties are hard and their Cox rings may be simpler.
- A3: Because for quotient singularities we may get an incarnation of McKay correspondence.
- Test case: Du Val surface singularities.

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Du Val singularities

We consider surface singularities

$$A_n: \quad x^{n+1} + y^2 + z^2 = 0 \qquad n \geq 1$$

$$D_n: \quad x^{n-1} + xy^2 + z^2 = 0 \qquad n \geq 4$$

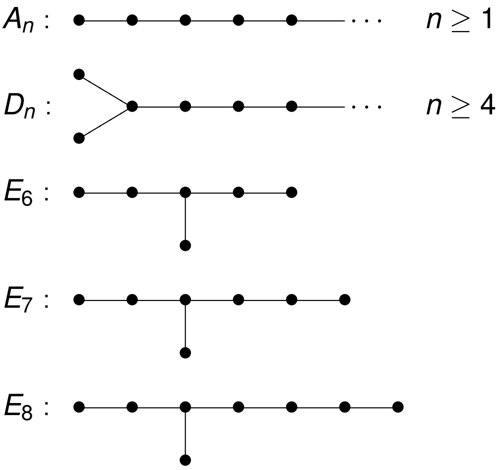
$$E_6: \quad x^4 + y^3 + z^2 = 0$$

$$E_7: \quad x^3y + y^3 + z^2 = 0$$

$$E_8: \quad x^5 + y^3 + z^2 = 0$$

Du Val singularities

Their resolutions give a configuration of (-2)-curves with the following dual Dynkin diagram



Du Val singularities

Important property: the canonical divisor of the resolution is trivial on the exceptional curves.

A_n : $n \geq 1$

D_n : $n \geq 4$

E_6 :

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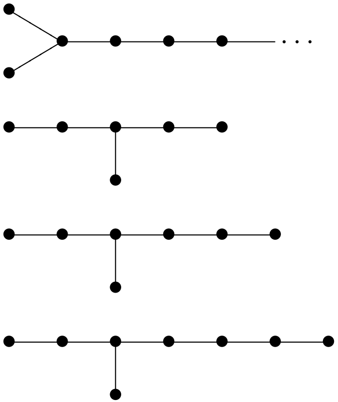
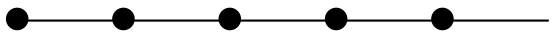


intersection matrix: A_n

The matrix of intersection for A_n:



is as follows

$$\mathbb{A}_n = \begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & \cdots & & \\ & & & & & & & \\ & & & & & & -2 & 1 \\ & & & & & & 1 & -2 \end{pmatrix}$$

intersection matrix: D_n

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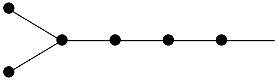

is as follows

$$\mathbb{D}_n = \begin{pmatrix} -2 & 1 & 1 & 1 & & & & \\ 1 & -2 & & & & & & \\ 1 & & -2 & & & & & \\ 1 & & & -2 & 1 & & & \\ & & & 1 & -2 & \cdots & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & -2 & 1 \\ & & & & & & 1 & -2 \end{pmatrix}$$

finite subgroups of $SL(2, \mathbb{C})$

Du Val singularities are quotients of $\mathbb{C}^2$ by finite subgroups of $SL(2, \mathbb{C})$; here ϵ_d is primitive d-root of 1 :

- A_n is the case of cyclic group $\mathbb{Z}_{n+1}$ generated by

$$\begin{pmatrix} \epsilon_{n+1} & 0 \\ 0 & \epsilon_{n+1}^{-1} \end{pmatrix}$$

- D_n is the case of binary dihedral group $\mathbb{BD}_{2n-4}$

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- E_6, E_7, E_8 come from binary tetrahedral, octahedral and isocahedral groups associated to Platonic solids.

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Cox ring of a quotient singularity

Suppose that a finite group G acts effectively and linearly on $\mathbb{C}^r$ with no quasireflection. Let $A = \mathbb{C}[x_1, \ldots, x_r]$ be the ring of coordinates on $\mathbb{C}^r$. Then:

- The ring of invariants A^G is finitely generated and $\mathrm{Spec}(A^G) = \mathbb{C}^r/G$.
- The divisor class group is the abelianization of G: $\mathrm{Cl}(\mathbb{C}^r/G) = G/[G,G] = Ab(G)$
- A decomposes into A^G-modules, associated to irreducible representations of G
- $A^{[G,G]}$ is the sum of A^G-modules of rank 1 associated to characters of G and it is $Ab(G)$-graded A^G-algebra.
- $\mathrm{Cox}(\mathbb{C}^r/G) = A^{[G,G]}$

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Let $\pi : X \longrightarrow \mathbb{C}^2/G$ be a minimal resolution of surface quotient singularities with exceptional divisors $E_1, \ldots, E_n$.

- $E_i \simeq \mathbb{P}^1$ and E_i's meet transversally, dual graph is a tree.
- The matrix $\Delta = (E_i \cdot E_j)$ is negative definite and $E_i^2 \leq -2$.
- $\mathrm{Cl}(X) = \mathrm{Pic}(X)$ is generated by classes of divisors F_i such that $F_i \cdot E_j = 1$ if $i = j$ and $F_i \cdot E_j = 0$ if $i \neq j$.
- We have an exact sequence

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resolutions of singularities

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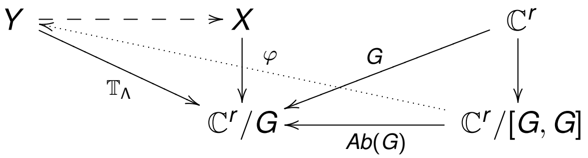
$$0 \longrightarrow \bigoplus_i \mathbb{Z}[E_i] \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(\mathbb{C}^2/G) = Ab(G) \longrightarrow 0$$

  where $[E_i] \longrightarrow \sum_j (E_i \cdot E_j)[F_j]$

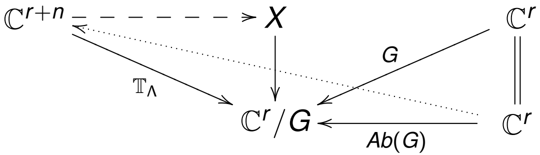
two quotients

Let X be a resolution of the quotient singularity $\mathbb{C}^r/G$ with $Y = \text{Spec}(\text{Cox}(X)) \dashrightarrow X$.
Then we have the following



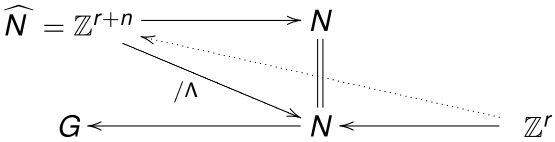
where the dotted arrow φ will be explained later

Let X be a resolution of the quotient singularity $\mathbb{C}^r/G$ with $Y = \mathrm{Spec}(\mathrm{Cox}(X)) \dashrightarrow X$.
If G is abelian



two quotients

Let X be a resolution of the quotient singularity $\mathbb{C}^r/G$ with $Y = \mathrm{Spec}(\mathrm{Cox}(X)) \dashrightarrow X$.
If G is abelian, same picture for tori lattices



where the dotted arrow is the “splitting” homorphism.

the homomorphism of Cox rings

There is a homomorphism of graded rings $\mathrm{Cox}(X) \longrightarrow \mathrm{Cox}(\mathbb{C}^r/G) = \mathbb{C}[x_1, \ldots, x_r]^{[G,G]}$ which comes from the homomorphism of groups $\pi_* : \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(\mathbb{C}^r/G)$:

$$\Gamma(X, \mathcal{O}_X(D)) \ni f \longrightarrow f \in \Gamma(\mathbb{C}^r/G, \mathcal{O}_{\mathbb{C}^r/G}(\pi_*(D)))$$

This yields the regular morphism, the "dotted arrow"

$$\varphi : \mathbb{C}^r/[G,G] \longrightarrow Y = \mathrm{Spec}(\mathrm{Cox}(X))$$

We can extend the above homomorphism to $\mathrm{Cox}(X) \longrightarrow \mathrm{Cox}(\mathbb{C}^r/G)[\mathrm{Cl}(X)]$ which geometrically means that we compose φ with the action of the torus to get:

$$\phi : \mathbb{C}^r/[G,G] \times \mathbb{T}_\Lambda \longrightarrow Y = \mathrm{Spec}(\mathrm{Cox}(X))$$

the latter map is not surjective but it is dominating.

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a diagram for toric quotients

For toric varieties: combine previous diagrams for abelian quotients of $\mathbb{C}^r/G$ to get the following diagram

$$\begin{array}{ccccc}
M & \longrightarrow & \widehat{M} & \longrightarrow & \mathrm{Cl}(X) \\
\downarrow & & \downarrow & & \| \\
\mathbb{Z}^r & \longrightarrow & \mathbb{Z}^r \oplus \mathrm{Cl}(X) & \longrightarrow & Cl(X) \\
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G & = & G & &
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This yields the description of the morphism ϕ.

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matrix $e\mathbb{A}_n$

Let us define the extended intersection matrix

$$e\mathbb{A}_n = \begin{pmatrix} 1 & & -2 & 1 & & & & & & \\ & & 1 & -2 & 1 & & & & & \\ & & & 1 & -2 & 1 & & & & \\ & & & & 1 & -2 & 1 & & & \\ & & & & & 1 & -2 & \cdots & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & -2 & 1 \\ & 1 & & & & & & & 1 & -2 \end{pmatrix}$$

Cox ring of the resolution of A_n

Theorem

The Cox ring $\mathrm{Cox}(X_{A_n})$ *of the resolution of the singularity* A_n *is isomorphic to the polynomial ring* $\mathbb{C}[u_1, u_n, y_1, \ldots, y_n]$ *with the multidegree of variables* $u_1, u_n, y_1, \ldots, y_n$ *given by the matrix* $e\mathbb{A}_n$.

Theorem

The matrix $e\mathbb{A}_n$ *determines the morphism*

$$\phi : \mathbb{C}^2 \times (\mathbb{C}^*)^n \longrightarrow \mathbb{C}^{n+2} = \mathrm{Spec}(\mathrm{Cox}(X_{A_n}))$$

given by monomials

$$(x_1 t_1,\ x_2 t_n,\ t_1^{-2} t_2,\ t_1 t_2^{-2} t_3, t_2 t_3^{-1} t_4,\ \ldots)$$

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 & & & 1 & & & -2 & 1 & & & \\
 & & & & & & 1 & -2 & \cdots & & \\
 & & & & & & & & & & \\
 & & & & & & & & & & \\
 & & & & & & & & & & \\
 & & & & & & & & & -2 & 1 \\
 & & 1 & & & & & & & 1 & -2
\end{pmatrix}
$$

Cox ring of resolution of D_n

Theorem (Facchini, Gonzàles-Alonso, Lasoń)

The Cox ring of the resolution of singularity D_n is isomorphic to

$$\mathbb{C}[u_1, u_2, u_{n-1}, y_0, y_1, \ldots, y_{n-1}]/(y_1 u_1^2 + y_2 u_2^2 + y_3 y_4^2 \cdots y_{n-1}^{n-3} u_{n-1}^{n-2})$$

where the multidegree of variables $u_1, u_2, u_{n-1}, y_0, y_1, \ldots, y_{n-1}$ are given by the matrix $e\mathbb{D}_n$.

Set $y_i = 1$ and note that you get the relation in $\mathrm{Cox}(X_{D_n})$:

$$u_1^2 + u_2^2 + u_{n-1}^{n-2} = 0$$

so u_i are invariants of the cyclic group

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such that $u_i = g_i(x_1, x_2)t_i$ *for* $i = 1,\ 2,\ n-1$ *and some explicit* G*-invariant* g_i *and* y_i*'s for* $i = 0, \dots n-1$ *given by monomials*

$$t_1 t_2 t_3 t_0^{-2},\ t_0 t_1^{-2},\ t_0 t_2^{-2},\ t_0 t_4 t_3^{-2},\ t_3 t_4^{-2} t_5,\ t_4 t_5^{-2} t_6, \dots t_{n-2} t_{n-1}^{-2}$$

Similar holds for singularities of type E.

parametrization for D_n

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