Extremal contractions

Lecture 4: applications

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global set-up

In this lecture X is a complex projective manifold of dimension n. We tackle two questions:

- why a surfaces can have ∞ of (-1)-curves but at most two fiber type contractions?
- a contraction (-2)-curve is a section of contraction of isolated curve in a 3-fold, why this is not true for a (-1)-curve?

We begin with some general comments on topology of contractions.

let's recall

- $N^1(X) \subset H^2(X,\mathbb{R})$ and $N_1(X) \subset H_1(X,\mathbb{R})$ are subspaces spanned by cohomology and homology classes of divisors and curves, resp.
- topological intersection of cycles and cocycles restricts to $N_1(X)$ and $N^1(X)$ and coincides with the intersection product
- intersection product gives nondegenerate pairing on $H^2(X,\mathbb{R}) \times H_2(X,\mathbb{R})$ and $N^1(X) \times N_1(X)$

extremal rays

- given $\alpha \in H_2(X, \mathbb{R})$ define
  $\alpha^\perp := \{\chi \in H^2(X, \mathbb{R}) : \chi \cdot \alpha = 0\}$
- dual cones: of curves $\mathcal{C} \subset N_1(X)$ and of nef divisors $\mathcal{P} \subset N^1(X)$
- $R \subset \mathcal{C}$ Mori (extremal) ray if $R \cdot K_X < 0$

Theorem If $\varphi_R : X \to Z$ contraction of R then

$$R^\perp = \varphi_R^*(H^2(Z, \mathbb{R}))$$

topological vanishings

Lemma Assume: $\varphi : X \to Z$ contraction of projective manifold, $-K_X$ is φ-big and nef.

- $\forall z \in Z$ and $\forall 0 \neq \mathcal{L} \in (R^1\varphi_*\mathcal{O}_X^*)_z$ exists $C \subset \varphi^{-1}(z)$ such that $\mathcal{L} \cdot C \neq 0$
- $(R^1\varphi_*\mathcal{O}_X^*)_z$ is a f.g. torsionfree ab gp
- $R^1\varphi_*\mathbb{Z}_X = 0$
- $\varphi^* : H^2(Z, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is injective
- $coker(Pic(X)/\varphi^* Pic(Z) \hookrightarrow H^2(X, \mathbb{Z})/\varphi^* H^2(Z, \mathbb{Z}))$ is torsion group

proof

First point follows because $\forall m \gg 0$ bundle $m\mathcal{L}$ is φ-spanned.
For second use intersection
$(R^1\varphi_*\mathcal{O}_X^*)_z \times H_2(\varphi^{-1}(z),\mathbb{Z}) \to \mathbb{Z}$ to get
$R^1\varphi_*\mathcal{O}_X^* \hookrightarrow Hom(H_2(\varphi^{-1}(z),\mathbb{Z}),\mathbb{Z})$
Next use push-forward

$$\begin{array}{ccccccc} 0 & \to & R^1\varphi_*\mathbb{Z}_X & \to & R^1\varphi_*\mathcal{O}_X & \to & R^1\varphi_*\mathcal{O}_X^* \\ & \to & R^2\varphi_*\mathbb{Z}_X & \to & R^2\varphi_*\mathcal{O}_X & \to & \end{array}$$

to get $R^1\varphi_*\mathbb{Z}_X = 0$ and $R^1\varphi_*\mathcal{O}_X^* \simeq R^2\varphi_*\mathbb{Z}_X$

proof, Leray

use Leray spectral

$$\begin{array}{ccccccc}
0 & \to & PicZ & \to & PicX & \xrightarrow{u} & H^0(R^1\varphi_*\mathcal{O}^*_X) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^2(Z,\mathbb{Z}) & \to & H^2(X,\mathbb{Z}) & \xrightarrow{v} & H^0(R^2\varphi_*\mathbb{Z}_X)
\end{array}$$

- vertical arrows are 1st Chern class
- rhs vertical arrow is an isomorphism
- $U := im(u)$, $V := im(v)$ torfree ab, $U \hookrightarrow V$.

proof, conclusion

- $\mathcal{Z}_1(\varphi) := \{\Gamma = \sum_{fin} a_i C_i : a_i \in \mathbb{Z},\ \varphi(C_i) = \bullet\}$
- pairing $H^0(Z, R^1\varphi_* \mathcal{O}^*_X) \times \mathcal{Z}_1(\varphi) \longrightarrow \mathbb{Z}$ yields (because of 1st part of thm) $coker(\mathcal{Z}_1(\varphi) \to Hom(V, \mathbb{Z}))$ is torsion
- if a cycle $\Gamma \in \mathcal{Z}_1(\varphi)$ has trivial intersection with $PicX$ then it is trivial on $H^2(X, \mathbb{Z})$

hard Lefschetz theorem

Definition $\eta \in H^2(X, \mathbb{R})$ satisfies Lefschetz condition if the k-th cup product map

$$\begin{aligned} L_k(\eta): \ H^{n-k}(X,\mathbb{R}) &\longrightarrow H^{n+k}(X,\mathbb{R}) \\ \chi &\longrightarrow \chi \cup \eta^{\cup k} \end{aligned}$$

is an isomorphism for $k = 1, \ldots n$

Hard Lefschetz Theorem Class of ample line bundle satisfies Lefschetz condition.

Lefschetz pairing

Alternatively, take $(-1)^{n-k}$-symmetric bilinear pairing

$$\begin{aligned} H^{n-k}(X,\mathbb{R}) \times H^{n-k}(X,\mathbb{R}) &\longrightarrow \mathbb{R} \\ (\chi,\nu) &\longrightarrow \chi \cup \nu \cup \eta^{\cup k} \end{aligned}$$

so η gives a 2-form $A_k(\eta)$ which lives in $S^2(H^{n-k}(X,\mathbb{R})^*)$ or $\bigwedge^2(H^{n-k}(X,\mathbb{R})^*)$

Lefschetz locus

Let $\Delta_k = \Delta_k(X) := \{\eta \in H^2(X, \mathbb{R})$: $L_k(\eta)$ is not an isomorphism$\}$. Then

- Δ_k is the zero set of a homogeneous polynomial $\delta_k = det A_k$ of degree kb_{n-k}
- if $n-k$ is odd then δ_k is the square of the pfaffian of A_k
- identifying rational homology $H_2(X, \mathbb{Q})$ with rational linear forms on $H^2(X, \mathbb{R})$ we have $\delta_k \in S^{kb_{n-k}}(H_2(X, \mathbb{Q}))$

Lefschetz line (or ray)

Lefschetz line $\mathbb{R} \cdot \alpha \subset H_2(X, \mathbb{R})$ divides δ_k for some $k = 1, \ldots n$, or, equivalently, $R^\perp \subset \Delta_k$. The ray R is of type k and multiplicity m if it divides δ_k with multiplicity exactly m.

If $\#(k, m)$ is the number of Lefschetz lines of type k and multiplicity m then

$$\sum_{m \geq 1} m \cdot \#(k, m) \leq k \cdot b_{n-k}$$

Lefschetz theorem revisited

Let $R = \mathbb{R} \cdot \alpha$ be rational Lefschetz line in $H_2(X, \mathbb{R})$. Then

- the cone $\mathcal{P}$ is not cut by the hyperplane $R^\perp$, or equivalently, either $\alpha \cdot \mathcal{P} \geq 0$ or $\alpha \cdot \mathcal{P} \leq 0$;
- $(R + (N^1)^\perp) \cap \bar{\mathcal{C}} \neq \{0\}$, that is, if $\beta \in N_1(X)$ is the projection of α along $N^1(X)^\perp$ (i.e. $\beta \in N_1(X)$ such that $(\beta - \alpha) \cdot N^1(X) = 0$) then either $\beta \in \bar{\mathcal{C}}$ or $-\beta \in \bar{\mathcal{C}}$.

L-condition for ray contraction

Proposition Let $\varphi_R : X \to Z$ be contraction of ray R. If there exists a subset $S \subset X$ such that $2dim_{\mathbb{C}}S - dim_{\mathbb{C}}\varphi(S) \geq n + m$ for some positive integer m then R is a Lefschetz ray of type $k := 2dim_{\mathbb{C}}S - n$ and multiplicity $\geq m$.

proof

- set $dim_{\mathbb{C}}S = a$, $dim_{\mathbb{C}}\varphi(S) = b-1$ so that $k = 2a - n$ and $m = k - b + 1$
- consider embeddings $i : S \to X$, $j : \varphi(S) \to Z$ and homology class $[S]_X \in H_{2a}(X, \mathbb{R})$
- take cocycle $\eta_Z \in H^2(Z, \mathbb{R})$ then $[S]_X \cap \varphi^*(\eta_Z)^{\cup b} = i_*([S] \cap i^*(\varphi^*(\eta_Z))^{\cup b}) = i_*([S] \cap \varphi_S^*(j^*(\eta_Z)^{\cup b}))$
- but $j^*(\eta_Z)^{\cup b} \in H^{2b}(\varphi(S), \mathbb{R}) = 0$ hence $\nu_S \cup \varphi^*(\eta_Z)^{\cup b} = 0$ where ν_S cohomology class of S

semismall maps

Definition A map $\varphi : X \to Z$ is called semismall if for any subset $S \subset X$ it holds

$$2dim_{\mathbb{C}}S - dim_{\mathbb{C}}\varphi(S) \leq n$$

A ray is semismall if its contraction is semismall.

Corollary If a ray is not semismall then it is Lefschetz.

properties of semismall rays

Let R be a semismall ray

- R has length 1
- every component of $Hom(\mathbb{P}^1, X)$ parametrizing minimal rational curves in R has expected dimension
- if $\mathcal{X} \to \mathcal{B}$ is a smooth deformation of $X = \mathcal{X}_0$ then homology class of a minimal rational curve in R is holomorphic in a general $\mathcal{X}_b$

nef value/threshold

Assume K_X not nef and H ample divisor on X. Define

$$\tau(H) = min\{t \in \mathbb{R} : K_X + tH \text{ nef } \}$$

Theorem Let $\mathcal{X} \to \mathcal{B}$ be a smooth deformation of $X = \mathcal{X}_0$ with an ample divisor $\mathcal{H}$, $\mathcal{H}_0 = H$. Then the function $\mathcal{B} \ni b \to \tau(\mathcal{H}_b)$ is locally constant.