Extremal contractions

Lecture 3: rational curves in fibers

J.A. Wiśniewski

J.Wisniewski@mimuw.edu.pl

Warsaw University, Poland

Mori's method

In 1979 and 1982 Mori published two papers in Ann. Math. (110 and 116) about the existence and applications of rational curves.

Theorem.
(Mori, Miyaoka-Mori, Kawamata)
Let $\varphi : X \to Z$ be a Fano-Mori or crepant contraction of variety with log-terminal singularities. Then exceptional locus $E(\varphi)$ is covered by rational curves contracted by φ.

many rational curves

Assume: X a projective manifold of dimension n and $f : \mathbb{P}^1 \to C \subset X$ birational, then

- $dim_f Hom(\mathbb{P}^1, X) \geq n + deg f^*(TX) = n - K_X \cdot C$
- this estimate holds for every component of Hom containing f
- analytically, around f, $Hom(\mathbb{P}^1, X)$ is defined in a disc of dimension $h^0(\mathbb{P}^1, f^*(TX))$ by at most $h^1(\mathbb{P}^1, f^*(TX))$ equations

connectedness

Definition (Flenner et al) Analytic space Z is connected in dimension d at $p \in Z$ if

- any component of Z containing p is at least of dimension d
- for any subspace $T \subset Z$, $T \ni p$, such that $\dim T < d$, and any open $V \ni p$ there exists $U \ni p$, $U \subset V$ such that $U \setminus T$ is connected

Theorem $Hom(\mathbb{P}^1, X)$ is connected in dim $(n + \deg(f^* T_X) - 1)$ at f.

length of a ray

Let $\varphi : X \to Y$ be an elementary Fano-Mori contraction, or extremal ray contraction. Define the length of φ:

$$l(\varphi) = \min\{-K_X \cdot C : \varphi(C) = \bullet\}$$

Theorem. Suppose that X is smooth and F a component of a fiber of φ. Then

$$dimF + dimE(\varphi) \geq l(\varphi) + dimX - 1$$

(Same works for crepant contractions, $l(\varphi) = 0$.)

proof of fiber-locus ineq.

- take $f : \mathbb{P}^1 \to C \subset F$ with min. $deg(f^*(-K_X))$
- $V \ni f$ irr. component of $Hom(\mathbb{P}^1, X)$
- by bend'n break $dim\{g \in V : g(0) = f(0), g(\infty) = f(\infty)\} \leq 1$
- evaluation $\epsilon : (V \times \mathbb{P}^1)/Aut(\mathbb{P}^1) \to X$ gives

$$\begin{aligned} dimE &+ dimF(\varphi) \geq \\ dim(im(\epsilon)) &+ dim(\epsilon^{-1}(f(0))) + 1 \\ &\geq dimV - 1 \end{aligned}$$

- but $dimV \geq dimX + l(\varphi)$, we're done!

applications

If X is smooth and $\varphi : X \to Z$ F-M contraction

- $dim E(\varphi) \geq dim X/2$
- no small contraction for $n \leq 3$

Theorem. (Kawamata) A small contraction of a 4-fold is (locally analytically) contraction of zero section of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^2$.

a lemma on surfaces

Assume

- S normal surface with line bundle $\mathcal{L}_S$.
- $f : \mathbb{P}^1 \to C \subset S$ of minimal degree with respect to $\mathcal{L}_S$ and $f(0) = p \notin SingS$
- $dim_f Hom(\mathbb{P}^1, X; 0 \mapsto p) \geq 3$

Then $(S, \mathcal{L}_S) = (\mathbb{P}^2, \mathcal{O}(1))$.

Mind an example: $S = \mathbb{P}^2/\mathbb{Z}_3$ with $\mathbb{Z}_3$ action $[z_0, z_1, z_2] \mapsto [z_0, \varepsilon z_1, \varepsilon^2 z_2]$.

proof of thm: fiber

- F component of fiber, $S \to F$ normalization, $\mathcal{L} = -K_X$ and $\mathcal{L}_F$, $\mathcal{L}_S$ its respective pullback
- by lemma: $(S, \mathcal{L}_S) = (\mathbb{P}^2, \mathcal{O}(1))$
- base-point-freeness: $(F, -K_{X|F}) = (\mathbb{P}^2, \mathcal{O}(1))$
- slicing: components meet along line
- connectedness in dim 3 of Hom: fiber irreducible

extending sections

Recall slicing: if $\varphi : X \to Z$ is F-M birational contraction with fiber F, $\mathcal{L} = -K_X$ and $X' \in |\mathcal{L}|$ then $\varphi' = \varphi_{|X'}$ is a crepant contraction. Moreover

$$H^0(X, \mathcal{O}_X) \longrightarrow H^0(X', \mathcal{O}_{X'}) \longrightarrow 0$$

In particular if the schematic fiber of φ' is reduced (hence its conormal generated by sections) then the same holds for F along $F \cap X'$.

proof of thm: normal bundle

Idea: consider splitting of $N^*_{F/X}$ on a line, do slicing, use information on crepant contractions of 3-folds

- possible splitting types $(1,1)$, $(0,2)$, $(-1,3)$
- by extending sections $(-1,3)$ is not possible
- by extending sections $N^*_{F/X}$ is spanned
- use classification of v.b. over $\mathbb{P}^2$:
  - $\mathcal{E}$ of rank 2 spanned and $c_1\mathcal{E} \leq 2$.

v.b. on plane

Spanned rank 2 vector bundles over $\mathbb{P}^2$, $c_1 \leq 2$

c_1	c_2	contraction $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{E})^*)$
0	0	$\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$
1	0	blow-down to point in $\mathbb{P}^3$
1	1	$\mathbb{P}^1$-bundle over $\mathbb{P}^2$
2	0	blow-down to cone over $\mathbb{P}^2 \subset \mathbb{P}^5$
2	1	Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^5$
2	2	blow-down to line in smooth quadric
2	3	blow-down to twisted cubic in $\mathbb{P}^3$
2	4	conic bundle over $\mathbb{P}^2$

$\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ if $(c_1, c_2) \in \{(0,0), (1,0), (2,0), (2,1)\}$

classification: general strategy

- describe geometric fiber F, first look at its normalization
- understand conormal sheaf $\mathcal{I}_F/\mathcal{I}_F^2$, and scheme theoretic fiber $\tilde{F}$ defined by $\varphi^{-1}(m_z) \cdot \mathcal{O}_X$
- describe completion of local ring $\mathcal{O}_{Z,z}$ and formal or analytic nbhd of F in X

unicity

(Grauert, Hironaka-Rossi, Mori)
Assume

- $\varphi : X \to Z$ is F-M or crepant
- F smooth and $N^* = N^*_{F/X}$ is nef
- $H^1(F, T_F \otimes S^i(N^*)) = H^1(F, N \otimes S^i(N^*)) = 0$ for $i \geq 1$

Then formal nbhd of F in X is unique and isomorphic to nbhd of zero section in bundle N.