Extremal contractions

Lecture 2: Vanishing, nonvanishing and slicing

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Appolonius method

Let $X^n \subset \mathbb{P}^N$ be a (smooth) projective variety of dimension n and $X^{n-1} = X^n \cap H \subset H = \mathbb{P}^{N-1}$ its hyperplane section. Then $dim X^{n-1} = n - 1$ and

$$K_{X^{n-1}} = (K_{X^n} + H)_{|X^{n-1}}$$

- Idea: use X^{n-1} to understand X^n, e.g. use divisorial sequence

$$0 \to \mathcal{O}_{X^n}(-X^{n-1}) \to \mathcal{O}_{X^n} \to \mathcal{O}_{X^{n-1}} \to 0$$

- Problem: sometimes the existence of a *good* hyperplane section is not obvious.

relative vanishing

Vanishing Theorem.
(Kawamata, Viehweg, Kollár)
Let X be a variety with at most log terminal singularities and let $\varphi : X \to Z$ be a proper morphism onto a variety Z.
Then $R^i\varphi_*\mathcal{O}_X(K_X) = 0$ for $i > dimX - dimZ$.

If $\mathcal{L}$ is a line bundle on X such that $-K_X + \mathcal{L}$ is φ-ample then $R^i\varphi_*\mathcal{L} = 0$ for $i > 0$.

vanishing w. fract. coeff

Let $\varphi : Y \to Z$ be a projective morphism from smooth Y and let A be φ-ample $\mathbb{Q}$-divisor such that support of fractional divisor $\{A\} = \lceil A \rceil - A$ is simple normal crossing. Then

$$R^i\varphi_*\mathcal{O}_Y(K_Y + \lceil A \rceil) = 0 \quad \text{for} \quad i > 0$$

local set-up

Let $\varphi : X \to Z$ be a local contraction:

- Z normal affine variety, X has log terminal singularities, K_X is $\mathbb{Q}$-Cartier
- $z \in Z$ and $F = \varphi^{-1}(z)$, and

$$\forall_{z' \in Z} \;\; dim(\varphi^{-1}(z')) \leq dim F$$

- $\mathcal{L}$ a φ-ample line bundle and $r \in \mathbb{Q}_{>0}$ such that $K_X + r\mathcal{L}$ is Cartier and φ-trivial, i.e.

$$K_X + r\mathcal{L} \simeq \mathcal{O}_X \text{ or } K_X \simeq -r\mathcal{L}$$

highest cohomology vanishes

Vanishing of cohomology of fiber's subscheme. In the local set-up let F' be a subscheme of X whose support is contained in the fiber F of φ, so that $\varphi(F') = z$. If

- either $t > -r$, or
- $t = -r$ and $dimF > dimX - dimZ$

then

$$H^{dimF}(F', t\mathcal{L}_{|F'}) = 0$$

fiber coh. vanishing, proof

Let $\mathcal{I}_{F'}$ be the ideal of F' in X. Take sequence

$$0 \longrightarrow t\mathcal{L} \otimes \mathcal{I}_{F'} \longrightarrow t\mathcal{L} \longrightarrow t\mathcal{L}_{F'} = t\mathcal{L} \otimes \mathcal{O}_{F'} \longrightarrow 0$$

Take direct image $R^\bullet \varphi_*$. Then

$$R^i \varphi_* (t\mathcal{L} \otimes \mathcal{I}_{F'})_z = 0$$

for $i > d = dim F$. Hence map

$$R^d \varphi_* (t\mathcal{L}_{X'}) \to H^d(F', t\mathcal{L}_{F'})$$

is surjective and we are done by vanishing.

cohomology of conormal

Corollary. Let $\varphi : X \to Z$ be a Fano-Mori or crepant contraction and F' a smooth component of its fiber of dimension d. Then $H^d(F', \mathcal{O}_{F'}) = 0$ and $dim H^d(F', N^*_{F'/X}) \leq dim H^{d-1}(F', \mathcal{O}_{F'})$.

Proof. Let $\mathcal{I}_{F'}$ be ideal of F' in X; consider cohomology of sequence

$$0 \to \mathcal{I}_{F'}/\mathcal{I}^2_{F'} \to \mathcal{O}_X/\mathcal{I}^2_{F'} \to \mathcal{O}_X/\mathcal{I}_{F'} \to 0$$

1 dim fiber

Corollary. Let $\varphi : X \to Z$ be F-M or crepant contraction. The reduced 1-dimensional fiber of φ is a tree of $\mathbb{P}^1$'s and conormal of each component F' satisfies

$$H^1(F', N^*_{F'/X}) = 0$$

thm: relative freeness

Local base-point-free theorem. Let $\varphi : X \to Z$ be a local contraction. Assume moreover that

- either $dim F < r+1$ if $dim Z < dim X$
- or $dim F \leq r+1$ if φ is birational

Then the evaluation morphism

$$\varphi^* \varphi_* \mathcal{L} \to \mathcal{L}$$

is surjective at every point of F.

application: slicing

Let X' be a general divisor from $|\mathcal{L}|$. Then

- outside base point locus of $|\mathcal{L}|$ singularities of X' are not worse than these of X
- any section of $\mathcal{L}$ on X' extends to X
- if $\varphi' := \varphi_{|X'}$ and $\mathcal{L}' = \mathcal{L}_{|X'}$ then $K_{X'} + (r-1)\mathcal{L}'$ is φ'-trivial
- the map φ' is a contraction if
  - either $r > 1$ or
  - $r \geq 1$ and φ is birational

nonvanishing theorem

Theorem. Let $\varphi : X \to Z$, F, $\mathcal{L}$ and r be as in the local set-up. Assume moreover that

- either $dimF < r + 1$ if $dimZ < dimX$, or
- $dimF \leq r + 1$ if φ is birational.

Then the base point locus of $\mathcal{L}$ does not contain any component of F.

proof, set-up

Take divisor B on X (pull back from Z) s.t.(X, B) is log canonical outside F and not log canonical at generic point of every component of F.
Let $f : Y \to X$ be a log resolution of B. Then

- $K_Y = f^*K_X + \sum e_i E_i$ where $e_i > -1$
- $f^*B = \sum b_i E_i$ where $b_i \geq 0$ and
  - $f^*(\epsilon\mathcal{L}) = A + \sum p_i E_i$ and $\epsilon > 0$, if $dimZ < dimX$
  - $f^*\mathcal{O}_X = A + \sum p_i E_i$, if φ is birational
- where A is $\varphi \circ f$-ample $\mathbb{Q}$-div. $0 \leq p_i \ll 1$

proof, sorting coefficients

Let $F' \subset F$ be an irreducible component, define

$$c := \text{ min} \left\{ \frac{e_i + 1 - p_i}{b_i} : F' \subset f(E_i),\ b_i > 0 \right\}$$

May assume the minimum is taken for one i, denote the corresponding divisor by E_0. Then

- $0 < c < 1$ and $f(E_0) = F'$,
- if $cb_i - e_i + p_i < 0$ then E_i is f-exceptional,
- if $cb_i - e_i + p_i \geq 1$ and $i \neq 0$ then $F' \not\subset f(E_i)$.

proof, sorting divisors

$$K_Y + A + \sum (cb_i - e_i + p_i) E_i + f^*(t\mathcal{L})$$

- is equivalent to $f^*((t-r)\mathcal{L})$ if φ birational, or
- equivalent to $f^*((t-r+\epsilon)\mathcal{L})$ if $dim Z < dim X$

Write

$$\sum (cb_i - e_i + p_i) E_i = E_0 + H'' - H' + Fr$$

with E_0, H', H'' eff. div. no comm. comp, H' f-except, $f(H'') \not\supset F'$, $Fr = \sum \{cb_i - e_i + p_i\} E_i$ fract.

proof, setting vanishings

For $t \in \mathbb{Z}$ such that

- $t \geq -r$, if φ is birational
- $t \geq -r + \epsilon$, if φ is of fiber type

$$f^*(t\mathcal{L}) - E_0 + H' - H'' - K_Y - Fr$$

is $\varphi \circ f$-ample so, by vanishings, for $i > 0$

$$\begin{aligned} H^i(Y, f^*(t\mathcal{L}) - E_0 + H' - H'') &= 0 \\ H^i(E_0, (f^*(t\mathcal{L}) + H' - H'')_{|E_0}) &= 0 \end{aligned}$$

proof, extending sections

If we set

$$\mathcal{N}(t) := f^*(t\mathcal{L}) + H' - H''$$

then restriction

$$H^0(Y, \mathcal{N}(t)) \to H^0(E_0, \mathcal{N}(t)_{|E_0})$$

is surjective for $t \geq -r$ in the birational and for $t > -r$ in the fiber case.

So, if s is a non-zero section of $\mathcal{N}(t)_{|E_0}$ then it extends to a section of $\mathcal{N}(t)$ on Y.

proof, descending sections

Divisor E_0 is not in the support of H'' so we get a section $s \in H^0(Y, f^*(t\mathcal{L}) + H')$ which is not zero on E_0. Since H' is f-exceptional

$$H^0(Y, f^*(t\mathcal{L}) + H') = H^0(X, t\mathcal{L})$$

and s descends to a section of $t\mathcal{L}$ which is not zero on $F' = f(E_0)$.
Therefore, if $dimF' > 0$, then $H^0(E_0, \mathcal{N}(t)) = 0$ for

- $-r \leq t < 0$, if φ is birational
- $-r < t < 0$, in the fiber case

proof, conclusion

Set $\chi(t) := \chi(E_0, \mathcal{N}(t))$ then it is a polynomial of degree $\leq dim f(E_0) = dim F'$ and $\chi(t) = 0$ for $-1 \geq t \geq -r$ (resp. $-1 \geq t > -r$), if φ is birational (resp. if φ is of fiber type.).
Thus $dim F' \geq \lfloor r \rfloor$ (resp. $dim F' \geq (r-1)$) and since $\chi(0) \geq 0$ and $\chi(t) > 0$ for $t \gg 0$ the assumption on $dim F$ implies $H^0(E_0, \mathcal{N}(1)) \neq 0$.

Finally, by extending and descending part we obtain a section of $\mathcal{L}$ in a neighborhood of F which does not vanish along F'.

application: bounding dimension

Corollary. As in the previous theorem, let $\varphi : X \to Z$ be a local contraction of a variety with terminal singularities. Assume that $K_X = -r\mathcal{L}$ for some positive rational number r and φ ample line bundle $\mathcal{L}$. Then:

- $dimF \geq \lfloor r \rfloor$, if φ is birational
- $dimF \geq r - 1$, if $dimZ < dimX$.

application: boundary case

Assume $\varphi : X \to Z$ be a local contraction, X smooth, $K_X = -r\mathcal{L}$ where $dimF \leq r \in \mathbb{Z}$. Then Z is smooth and one of the following occurs.

- $dimF = r$ and φ blow-down of divisor to smooth subvariety of Z
- $dimZ < dimX$ and one of the following holds
  - $dimF = r - 1$ and φ is projective bundle
  - $dimF = r = dimX - dimZ$, φ quadr. bund.
  - $dimF = r = dimX - dimZ - 1$ and loc. anal. φ is jump fiber contr.

sketch of pf, birational case

- F' component of F, $\hat{F}' \to F'$ normalization, $\hat{\mathcal{L}}'$ pullback of $\mathcal{L}$; by Kobayashi-Ochiai

$$(\hat{F}', \hat{\mathcal{L}}') = (\mathbb{P}^r, \mathcal{O}(1))$$

- by base-point-freeness $(F', \mathcal{L}_{F'}) = (\mathbb{P}^r, \mathcal{O}(1))$
- by vanishing of cohomology and slicing

$$N_{F'/X} = \mathcal{O}(-1) \oplus \mathcal{O}^{\oplus(n-r-1)}$$

- by deformations φ is divisorial and blow-down