Extremal contractions

Lecture 1: definitions and examples

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plan of lectures

1. Definitions and examples.
2. Vanishing, nonvanishing, slicing.
3. Rational curves in fibers.
4. Applications.

motivation

Motivation: want to study projective varieties over $\mathbb{C}$ and their morphisms. In fact, understanding morphisms is a part of understanding the varieties themselves. Also, will show how the methods developed in other lectures (vanishings and bend-and-break) work in easy (smooth) case. Examples will follow.

Stein factorization

Let $\psi : X \to Y$ be a projective map of noetherian schemes, X normal. Then ψ admits a factorization

$$X \xrightarrow{\varphi} Z \xrightarrow{\pi} Y$$

where π is finite, Z is normal and φ has connected fibers, i.e. $\varphi_* \mathcal{O}_X = \mathcal{O}_Z$. The scheme Z is normalization of Y in the function field of X or, equivalently, $Z = Spec_Y(\psi_* \mathcal{O}_X)$.

global/local contraction

Standing assumptions

- $\varphi : X \to Z$ projective surjective map of normal algebraic $\mathbb{C}$-varieties, $\varphi_* \mathcal{O}_X = \mathcal{O}_Z$
- X is usually smooth, $dim X = n$
- for local description
  - choose a point $z \in Z$, shrink Z, take $X = \varphi^{-1}(Z)$, restrict φ
  - Z will be an affine variety (or a germ of analytic space)

special fiber

For the fixed point $z \in Z$ with maximal ideal m_z we will consider the fiber $F = \varphi^{-1}(z)$ with, either

- schematic structure $\mathcal{O}_{F_{sch}} = \mathcal{O}_X / f^*(m_z) \cdot \mathcal{O}_X$, i.e. the ideal of F_{sch} is generated by functions from m_z, or
- reduced structure F_{red}

These two structure do not have to coincide.

additional assumptions

A (global/local) contraction $\varphi : X \to Z$ is

- Fano-Mori if $-K_X$ is φ-ample
- crepant if it is birational and $-K_X \simeq \mathcal{O}_X$
- symplectic if $n = 2m$, φ is birational and there exists a closed holomorphic 2-form ω such that $\omega^{\wedge m}$ does not vanish anywhere
- elementary if $dim_{\mathbb{Q}}(Pic(X)/\varphi^* Pic(Z) \otimes_{\mathbb{Z}} \mathbb{Q}) = 1$

locus of contraction

A contraction $\varphi : X \to Z$ is

- either *birational*, or
- *of fiber type*, if $dimX > dimZ$

The locus of a contraction $E(\varphi)$ is either X, if φ is of fiber type, or the smallest subset $A \subset X$ such that

$$X \setminus A \overset{\varphi}{\simeq} Z \setminus \varphi(A)$$

A contraction is called small if its exceptional locus is of codimension ≥ 2.

first examples

These are Fano-Mori contractions:

- Contract Fano manifold to a point $X \to \bullet$
- Project along Fano manifold, $X_1 \times X_2 \to X_2$, where X_1 is Fano.
- Simple blow down $\varphi : X \supset E \to z \in Z$, so that $K_X = \varphi^* K_Z + (n-1)E$.
- Blow down $\varphi : X \to Z$ with exceptional divisor $E \subset X$ contracted to smooth $S \subset Z$, so that $K_X = \varphi^* K_Z + (n-k+1)E$, with k codimension of S in Z.

divisors in projective bundles

Construct X as a complete intersection (e.g. a divisor) in a projective bundle $p : \mathbb{P}(\mathcal{E}) \to Z$, $\varphi = p_{|X}$

Locally: $\mathbb{P}^r$ with coordinates $[z_0, \dots, z_r]$ and $\mathbb{C}^s$ with coordinates $(t_1, \dots t_s)$; then X given by

$$F(z_i, t_j) = 0$$

homogeneous of degree d in z_i.

Note: $K_X = \mathcal{O}(-r - 1 + d)$; singular set of X is

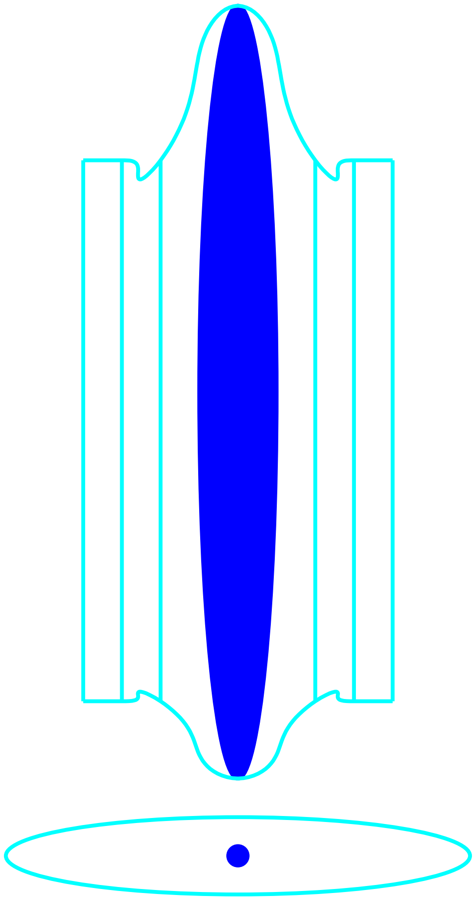
$$F = 0, \quad \frac{\partial F}{\partial z_i} = 0, \quad \frac{\partial F}{\partial t_j} = 0$$

jumping fiber

Consider hyperplane section divisor in $\mathbb{P}^r \times \mathbb{C}^{r+1}$

$$t_0 \cdot z_0 + \cdots + t_r \cdot z_r = 0$$

Note that it is smooth and its contraction to $\mathbb{C}^{r+1}$ has a general fiber $\mathbb{P}^{r-1}$ while the special fiber over $0 \in \mathbb{C}^{r+1}$ is $\mathbb{P}^r$.



quadric bundles

Take a divisor X in $\mathbb{P}^r \times \mathbb{C}^s$

$$f_0 \cdot z_0^2 + \cdots + f_r \cdot z_r^2 = 0$$

where f_i are functions in $(t_1, \ldots, t_s)$.

By adjunction $K_X \simeq \mathcal{O}(r-1)$. If $r = 2$ then X is called conic bundle.

conic bundle, special fiber

X in $\mathbb{P}^2 \times \mathbb{C}$ given by equation

$$z_0^2 + z_1^2 + t \cdot z_2^2 = 0$$

Such X is smooth and has reducible fiber over $0 \in \mathbb{C}$.
X in $\mathbb{P}^2 \times \mathbb{C}^2$ given by equation

$$z_0^2 + t_1 \cdot z_1^2 + t_2 \cdot z_2^2 = 0$$

Such X is smooth and the schematic fiber over $0 \in \mathbb{C}$ is non-reduced line $z_0^2 = 0$.

Lefschetz hyperplane thm

Suppose that Z is projective and $X \subset \mathbb{P}(\mathcal{E}) \to Z$ is an ample divisor.
Then by Letschetz hyperplane section theorem the contraction $X \to Z$ is (globally!) elementary but locally it may be non-elementary in the analytic set up.

vector bundle trivialities

- $\mathcal{E}$ rank r vector bundle over a projective manifold Y
- $Sym\mathcal{E} = \bigoplus_{m\geq 0} S^m\mathcal{E}$ symmetric $\mathcal{O}_Y$-algebra of local sections of $\mathcal{E}$
- $X = Spec_Y(Sym\mathcal{E})$ total space of dual bundle $\mathcal{E}^*$ with $p : X \to Y$
- $Sym\mathcal{E} \to \mathcal{O}_Y$ defines zero section of $\mathcal{E}^*$, denoted $F \subset X$, such that $N^*_{F/X} \simeq \mathcal{E}$.
- $K_X = p^*(K_Y + det\mathcal{E})$ because $\Omega_{X/Y} \simeq p^*(\mathcal{E})$

vector bundle construction

- Assume that $\mathcal{E}$ semiample, i.e. $S^m\mathcal{E}$ generated by global section for $m \gg 0$
- Consider $A = \bigoplus_{m\geq 0} H^0(Y, S^m\mathcal{E})$ with maximal ideal $m_z = \bigoplus_{m>0} H^0(Y, S^m\mathcal{E})$.

Then the ring A is finitely generated (Zariski, [Reid, YPG p. 354]) so $SpecA$ is an affine scheme and m_z defines its closed point z.

contraction of zero section

Evaluations

$$H^0(Y, S^m\mathcal{E}) \otimes \mathcal{O}_Y \to S^m\mathcal{E}$$

define a map

$$\psi : X \to SpecA \times Y \to SpecA$$

Its Stein factorization $\varphi : X \to Z$ is a contraction, $F \mapsto z$.

example: contraction to cone

Let $\mathcal{L}$ be an ample line bundle over a projective manifold Y. Define $X = Spec_Y(\bigoplus_{m\geq 0} m\mathcal{L})$ and $Z = Spec(\bigoplus_{m\geq 0} H^0(Y, m\mathcal{L}))$ with the contraction morphism $\varphi : X \to Z$, then $E(\varphi) = F$.

If the pair $(Y, \mathcal{L})$ is projectively normal, i.e. the graded ring $\bigoplus_{m\geq 0} H^0(Y, m\mathcal{L})$ is generated by its first gradation, then Z is an affine cone over Y embedded in the projective space via complete linear system $|\mathcal{L}|$.

small contractions

Take Y a Fano manifold with ample line bundle $\mathcal{L}$ such that $r\mathcal{L} \simeq -K_Y$, for $r \in \mathbb{Z}_{>0}$. Define

$$X = Spec_Y \left(\bigoplus_{m \geq 0} S^m(\mathcal{L}^{\oplus s}) \right)$$

Then the contraction of zero section $\varphi : X \to Z$ has exceptional locus of codimension s and it is Fano-Mori if $s < r$ or crepant if $s = r$.

Kawamata ex: disconnected locus

- V projective 4-fold, K_V ample, $PicV \simeq \mathbb{Z}$
- $C,\ S \subset V$ smooth curve and surface
- C meets S transversally in $p_1, \ldots p_m$
- take blow-ups, $\hat{S}$ strict transform of S wrt α

$$V \xleftarrow{\alpha} V_C \xleftarrow{\beta} V_{C,\hat{S}}$$

- strict transforms of $\alpha^{-1}(p_i)$ wrt β are $\mathbb{P}^2$'s with normal $\mathcal{O}(-1)^{\oplus 2}$ and contain numerically equivalent lines

symplectic contraction

Let F be homogeneous variety with TF spanned, e.g. $F = \mathbb{P}^r$. There is a universal 1-form σ over cotangent bundle $T^*F = \Omega F$, that is, take projection $p : T^*F \to F$ then section

$$\sigma : T^*F \to p^*\Omega F \to \Omega(T^*F)$$

comes from identity $T^*F = \Omega F$.

$\omega = d\sigma$ is a symplectic form and contraction of zero section of T^*F is a symplectic contraction

mix'n match, intersect bundle

- $\mathbb{Q}_3$ 3-dim quadric, $p : \mathcal{S} \to \mathbb{Q}_3$ spinor bundle, $c_1 = 1$, $c_2 = 2$, spanned
- $Y = Spec_{\mathbb{Q}_3}(Sym\mathcal{S}) \to \mathbb{C}^4$ contraction, generically $\mathbb{P}^1$-bundle
- $\exists$ smooth divisor $X \subset Y$ such that $\varphi : X \to \mathbb{C}^4$ is generically blow-down
- special fiber is either smooth quadric, or quadric cone, or two planes

$\mathbb{C}^*$ quotient

$\mathbb{C}^*$ action on $\mathbb{C}^4$ with weights $(-1,-1,1,1)$:

$$t \cdot (x_1, x_2, x_3, x_4) = (t^{-1}x_1, t^{-1}x_2, tx_3, tx_4)$$

quotient is quadric cone; remove orbits which have limits at 0 or ∞, get quotients, two sides of Atiyah flop.

Compactify $\mathbb{C}^4$, add quotients at both sides of $\mathbb{C}^*$ orbits, get smooth X with proper map onto quadric cone, gen. $\mathbb{P}^1$ bundle, special fiber two $\mathbb{P}^2$ meeting at pt.

toric picture