# On the Kummer construction

JW: joint work with Marco Andreatta,

also reporting work of Maria Donten

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- resolve 16 simple double points,
- get a surface $S$ with $H^1(S, \mathbb{C}) = 0$ and $K_S \simeq 0$

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- Want trivial invariant subspace and $\rho(G) < SL(H^1(A, \mathbb{C}))$
- Take the quotient $Y = A/G$, find a crepant resolution $X \to Y$, get a Calabi-Yau or symplectic manifold, $H^1(X, \mathbb{C}) = 0$ and $K_X \simeq 0$

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- Then representation on $TA^r$ and $H^1(A, \mathbb{C})$ is $r \cdot \rho_{\mathbb{C}}$
- Extended version: $\rho_{\mathbb{Z}} : G \to GL(r, \mathbb{Z}[Aut(A)])$ or $\rho_{\mathbb{Z}} : G \to GL(r, End(A))$

# constrains: representations

- Let $G < GL(r, \mathbb{Z})$ be a finite subgroup. Then, for every prime $p > 2$ its $p$-th reduction $G \hookrightarrow GL(r, \mathbb{Z}) \to GL(r, \mathbb{Z}_p)$ is an embedding.

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- Let $g \in GL(r, \mathbb{Z})$ be of order $m$. Then roots of the characteristic polynomial of $A$, or eigenvalues of $A$, are among (possibly nonprimitive) $m$-th roots of unity. In particular, $\varphi(m) \leq r$, where $\varphi$ denotes the Euler function.

# constrains: Lefschetz theorem

- Let $g : A \rightarrow A$ be an automorphism and let $\rho_g \in GL(TA)$ be its tangent. Then $Fix(g)$ is a subgroup of dimension equal to the multiplicity of $1$ as an eigenvalue of $\rho_g$. If it is zero then $|Fix(g)| = |det(1 - \rho_g)|^2$.

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- The number $|det(1 - \rho_g)|^2$ is integer thus, if $g$ is of order $m$ then $\varphi(m) \leq 2 \cdot dimA$.
- If $dimA = 2$ then $\varphi(m) \leq 2$ hence $m = 2,\ 3,\ 4,\ 6$.

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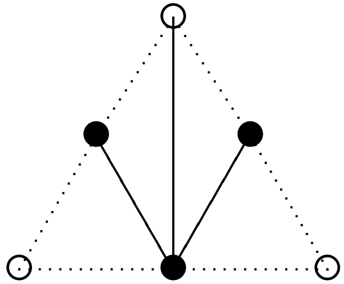
- In dimension 2 and 3 we know a lot about crepant resolutions but in higher dimensions it is hard.
- For solvable groups we can take towers of resolutions of abelian singularities, provided at each step we get an equivariant one.

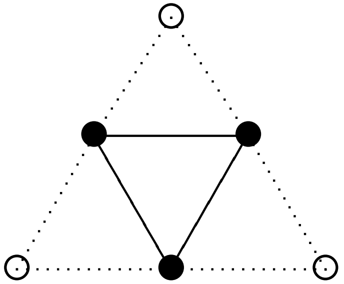
# ex: resolutions for solvable gps

Consider $D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by matrices

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\mathbb{C}^3/D_4$ has different crepant resolutions, one invariant w. resp. to permutations of coordinates.



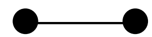
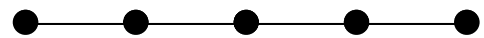
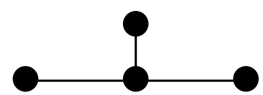
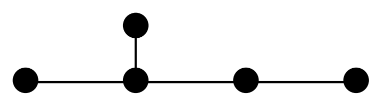
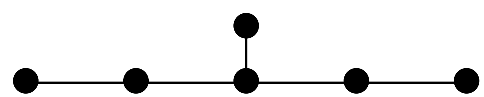


# DuVal singularities

Finite subgroups in $SL(2, \mathbb{C})$ are classified by Dynkin diagrams associated to incidence of $(-2)$ curves in resolution of their quotients.

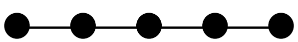
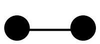
# DuVal singularities

These Du Val groups have elements of order $1,\ 2,\ 3,\ 4,\ 6$:

| group | Dynkin diagram |
|---|---|
| $\mathbb{Z}_2$ |  |
| $\mathbb{Z}_3$ |  |
| $\mathbb{Z}_4$ |  |
| $\mathbb{Z}_6$ |  |
| quaternion $Q_8$ |  |
| binary dihedral $BD_{12}$ |  |
| binary tetrahedral $BT_{24}$ |  |

# Kummer surface for $G = \mathbb{Z}_6$

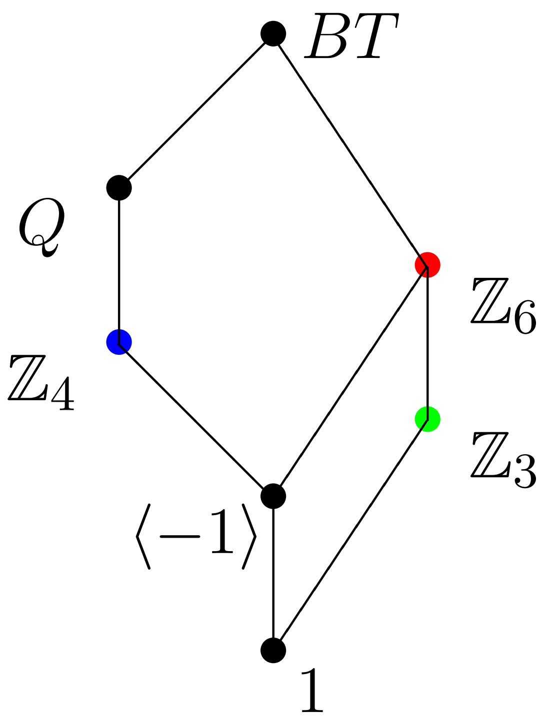
Representation $\frac{1}{6}(1,5)$, $\rho_{\mathbb{C}} = \epsilon_6 + \epsilon_6^{-1}$

| $g$ | # fix pts | # sing pts | resolution |
|---|---|---|---|
| $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ | 1 | 1 |  |
| $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ | 9 | 4 |  |
| $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | 16 | 5 |  |

Number of $(-2)$-curves $= 1 \times 5 + 4 \times 2 + 5 \times 1 = 18$;
dimension of invariant subspace of $\rho_{\mathbb{C}} \otimes \rho_{\mathbb{C}}$ is 2.

# Kummer surface for $BT_{24}$

Lattice of subgroups




$W = \mathbb{Z}_2, \quad W = \mathbb{Z}_2, \quad W = 1$

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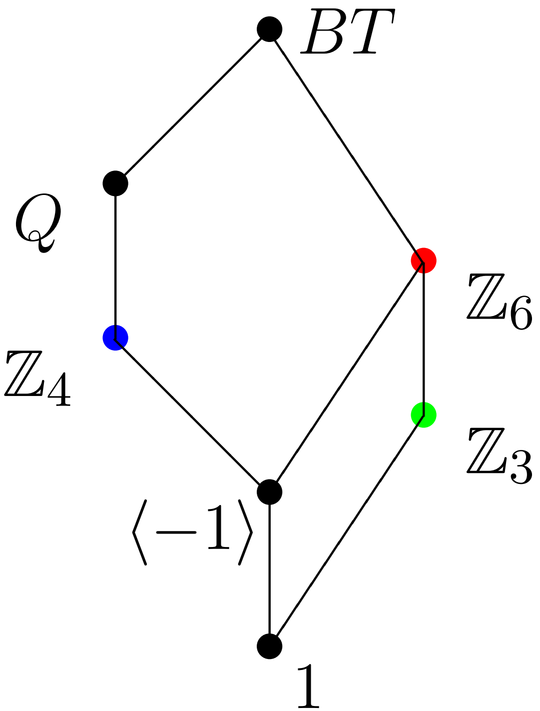
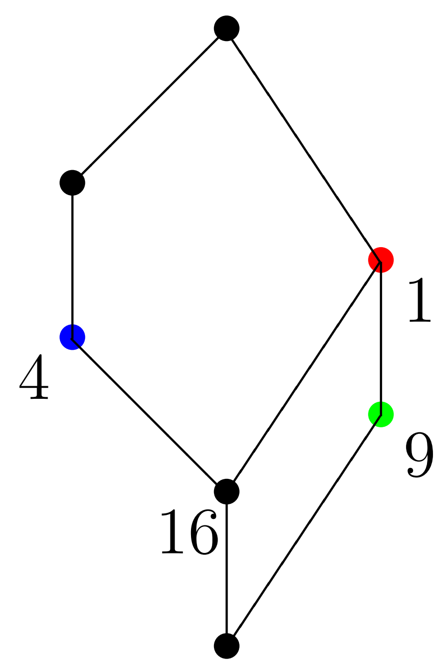



Lefschetz thm fixed pts

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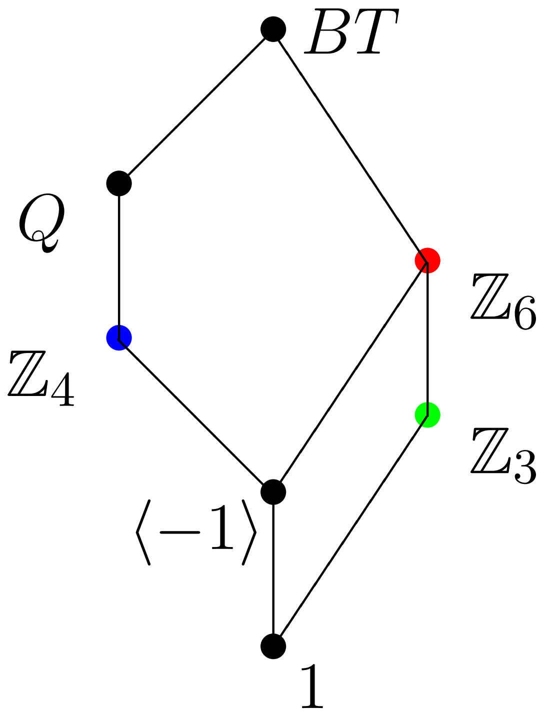
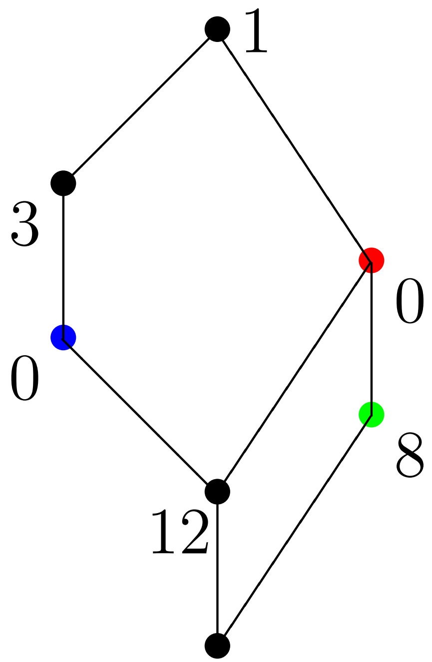



Pts with given isotropy




$\color{blue}{W = \mathbb{Z}_2}, \quad \color{green}{W = \mathbb{Z}_2}, \quad \color{red}{W = 1}$

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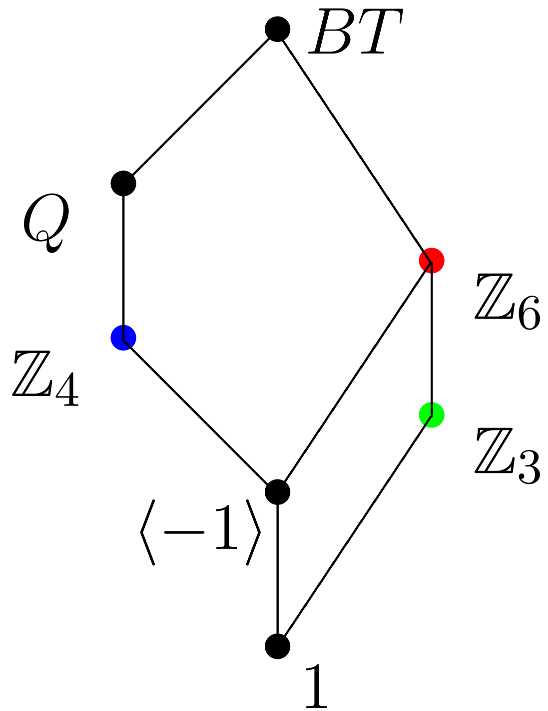


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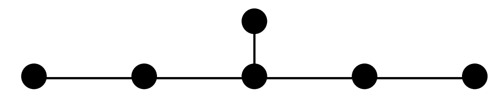
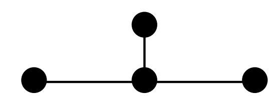
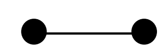
Singular pts & resolutions

| # pts | resolution |
|---|---|
| 1 |  |
| 1 |  |
| 4 |  |
| 1 |  |

Number of $(-2)$-curves $1 \times 6 + 1 \times 4 + 4 \times 2 + 1 \times 1 = 19$; dimension of invariant subspace of $\rho_{\mathbb{C}} \otimes \rho_{\mathbb{C}}$ is 1.

# notation: group action

- Normalizer of $H < G$ is denoted by $N(H)$.
- Weyl group is $W_H = N(H)/H$
- $[H]$ is the conjugacy class of $H$ in $G$, that is the set of subgroups $\{gHg^{-1} : g \in G\}$.
- Note that $\#[H] = [G : N(H)]$ and for $H' \in [H]$ we have $W_{H'} = W_H$.

# notation: group action

- For $G$ acting on $B$, and $H < G$ by $B^H$ we denote the subset of $B$ fixed by $H$ while $B_0^H \subset B^H$ is the set of points whose isotropy (or stabilizer) is exactly $H$.
- The restriction of the action of $G$ to $N(H)$ defines an action of $W_H$ on $B^H$.

# strata of resolution

For understanding resolution

$$f : X \to Y$$

write the quotient $Y = A^r/G$ as disjoint sum of locally closed sets (strata) $Y_{[H]}$ consisting of orbits of points whose isotropy is in the conjugacy class of a subgroup $H < G$. Over $Y_{[H]}$ the singularities of $Y$ are locally quotients of $\mathbb{C}^{rd}$ by action of $H$.

# strata of resolution

Take inverse images of sets $Y_{[H]}$, get a decomposition of $X$ into disjoint sum of locally closed sets $X_{[H]}$ such that the restriction

$$X_{[H]} \to Y_{[H]}$$

is a locally trivial fiber bundle with a fiber $F_H$ depending on the resolution of the $H$-quotient singularity.

# structure of strata

Let $\overline{Y_{[H]}} \subset Y$ denote the closure of $Y_{[H]}$ in $Y = A^r/G$ and $\widehat{Y_{[H]}} \to \overline{Y_{[H]}}$ be its normalization. The morphism

$$\overline{(A^r)_0^H} \to \widehat{Y_{[H]}}$$

is quotient by $W_H$, where the action of $W_H$ on $\overline{(A^r)_0^H}$ is determined by the action of $N(H)$.

# structure of strata

Action of $W_{[H]}$ on $\overline{(A^r)_0^H}$ lifts to $\overline{(A^r)_0^H} \times F_H$, we get commutative diagram

$$\begin{array}{ccccc}
\overline{(A^r)_0^H} \times F_H & \longrightarrow & \left(\overline{(A^r)_0^H} \times F_H\right)/W_{[H]} & \longleftarrow & X_{[H]} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{(A^r)_0^H} & \longrightarrow & \widehat{Y_{[H]}} & \longleftarrow & Y_{[H]}
\end{array}$$

where the horizontal arrows on the left hand side are quotient maps while these on the right hand side are inclusions onto open subsets.

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- By $d \cdot \rho$ denote the sum of $d$ copies of representation while by $\rho^{\otimes m}$ and $\rho^{\wedge m}$ we denote $m$-th tensor and, respectively, alternating power of $\rho$.
- We have a map $\mu_0 : R(G) \to \mathbb{Z}$ which to a representation $\rho$ assigns the rank of its maximal trivial subrepresentation.

# virtual Poincaré

- For compact manifold $X$ of dim $n$ its Poincaré polynomial $P_X(t) = \sum_{i=0}^{2n} b_i(X)\, t^i \in \mathbb{Z}[t]$, with $t$ formal variable, $b_i(X) = dim H^i_{DR}(X)$.

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- Virtual Poincaré polynomials are defined also for differences of compact varieties, that is for $U = X \setminus Z$ set $P_U = P_X - P_Z$.
- Coefficients of virtual Poincaré $P_X(t)$ are equal to (standard) Betti numbers if $X$ is compact and has quotient singularities.

# cohomology of quotients

- Given action of $G$ on variety $Z$ define $G$-Poincaré polynomial $P_{Z,G}(t) \in R(G)[t]$ whose coefficient with $t^i$ is the vector space $H^i(Z, \mathbb{C})$ with induced $G$ action.

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- In our set-up

$$P_{A^r,G}(t) = \sum_{i=0}^{2rd} (2d \cdot \rho_{\mathbb{C}})^{\wedge i} \cdot t^i$$

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$$P_{A^r,G}(t) = \sum_{i=0}^{2rd} (2d \cdot \rho_{\mathbb{C}})^{\wedge i} \cdot t^i$$

- For $Y = A^r/\rho_A$ we have $P_Y(t) = \mu_0(P_{A^r,G}(t))$ .

# strata of $Y$

Let $K \subset Y_{[H]}$ irreducible component with normalized closure $\widehat{K}$. Then $\widehat{K} \simeq A_K/W_K$ where $W_K < W_H$ is the subgroup which preserves $A_K \simeq A^{r_0}$, a component of the closure of $(A^r)_0^H$ which dominates $K$.

$$P_{A_K,W_K}(t) = \sum_{i=0}^{2r_0 d} (2d \cdot \nu_K)^{\wedge i} \cdot t^i$$

where $\nu_K : W_K \to GL(r_0, \mathbb{C})$ is a representation of $W_K$ induced from $\rho_{\mathbb{C}}$.

# strata of $X$

McKay correspondence postulates a *canonical* relation of conjugacy classes of a group $H$ with cohomology of a crepant resolution of its quotient singularity.

# strata of $X$

$W_H$ acts on the cohomology of $F_H$ as it acts on the conjugacy classes of $H$. So, $W_K$-Poincaré polynomial $P_{F_H,W_K}$ is determined by the adjoint action of $W_K$ on conjugacy classes of $H$, which is $w([h]) \mapsto [whw^{-1}]$, where $w \in N(H)$ represents an element of $W_H = N(H)/H$ and $h \in H$. Thus

$$P_{(A_K \times F_H)/W_K} = \mu_0(P_{A_K,W_K} \cdot P_{F_H,W_K})$$

# diagram of strata, again

Thus we have cohomology of some entries in the diagram:

$$\begin{array}{ccccc}
\overline{(A^r)_0^H} \times F_H & \longrightarrow & \left(\overline{(A^r)_0^H} \times F_H\right)/W_{[H]} & \longleftarrow & X_{[H]} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{(A^r)_0^H} & \longrightarrow & \widehat{Y_{[H]}} & \longleftarrow & Y_{[H]}
\end{array}$$

The difference $\left(\overline{(A^r)_0^H} \times F_H\right)/W_{[H]} \setminus X_{[H]}$ lives in strata associated to supergroups of $H$.

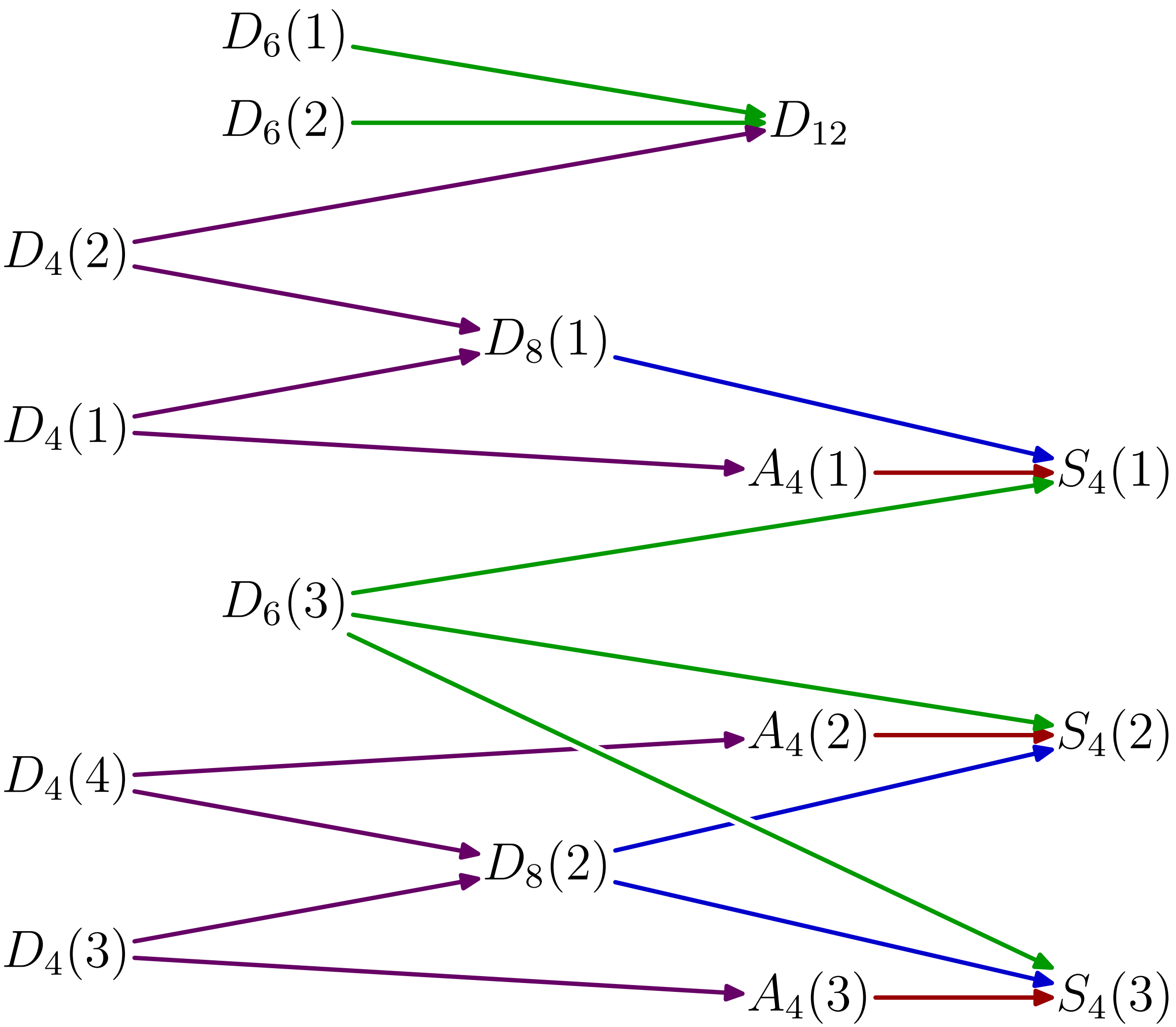
# subgroups of $SL(3,\mathbb{Z})$

The following are, up to isomorphism, (non-trivial) finite subgroups of $SL(3,\mathbb{Z})$:

- cyclic groups $\mathbb{Z}_a$, for $a = 2,\ 3,\ 4$ and $6$ (they are not interesting since $H^1$ of quotient is $\neq 0$),
- dihedral groups $D_{2a}$, for $a = 2,\ 3,\ 4$ and $6$,
- alternating group $A_4$ (e.g. tetrahedral group $\mathbf{T}$ of isometries of tetrahedron)
- symmetric group $S_4$ (e.g. octahedral group $\mathbf{O}$ of isometries of a cube)

We are interested in their conjugacy classes in $GL(3,\mathbb{Z})$.

# subgroups of $SL(3, \mathbb{Z})$



# subgroups of $SL(3, \mathbb{Z})$

Note that some of these groups are conjugate in $GL(3, \mathbb{C})$!

# case O $= S_4$, 1-dim sing

| $g$ | $W(\langle g\rangle)$ | $\widehat{Y(\langle g\rangle)}$ | $P_{F_{\langle g\rangle}}$ |
|---|---|---|---|
| $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $6 \times \mathbb{P}^1$ | $1 + t$ |
| $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\mathbb{Z}_2$ | $4 \times \mathbb{P}^1$ | $1 + (2 + \epsilon)t$ |

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|---|---|---|---|
| $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ | $\mathbb{Z}_2$ | $4 \times \mathbb{P}^1$ | $1+t$ |
| $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ | $\mathbb{Z}_2$ | $1 \times \mathbb{P}^1$ | $1+(1+\epsilon)t$ |

# case O $= S_4$, 0-dim sing

1-dim non-free point sets meet in at

$$\{p \in A^3 : 2p = 0\}$$

cardinality $4^3 = 64$

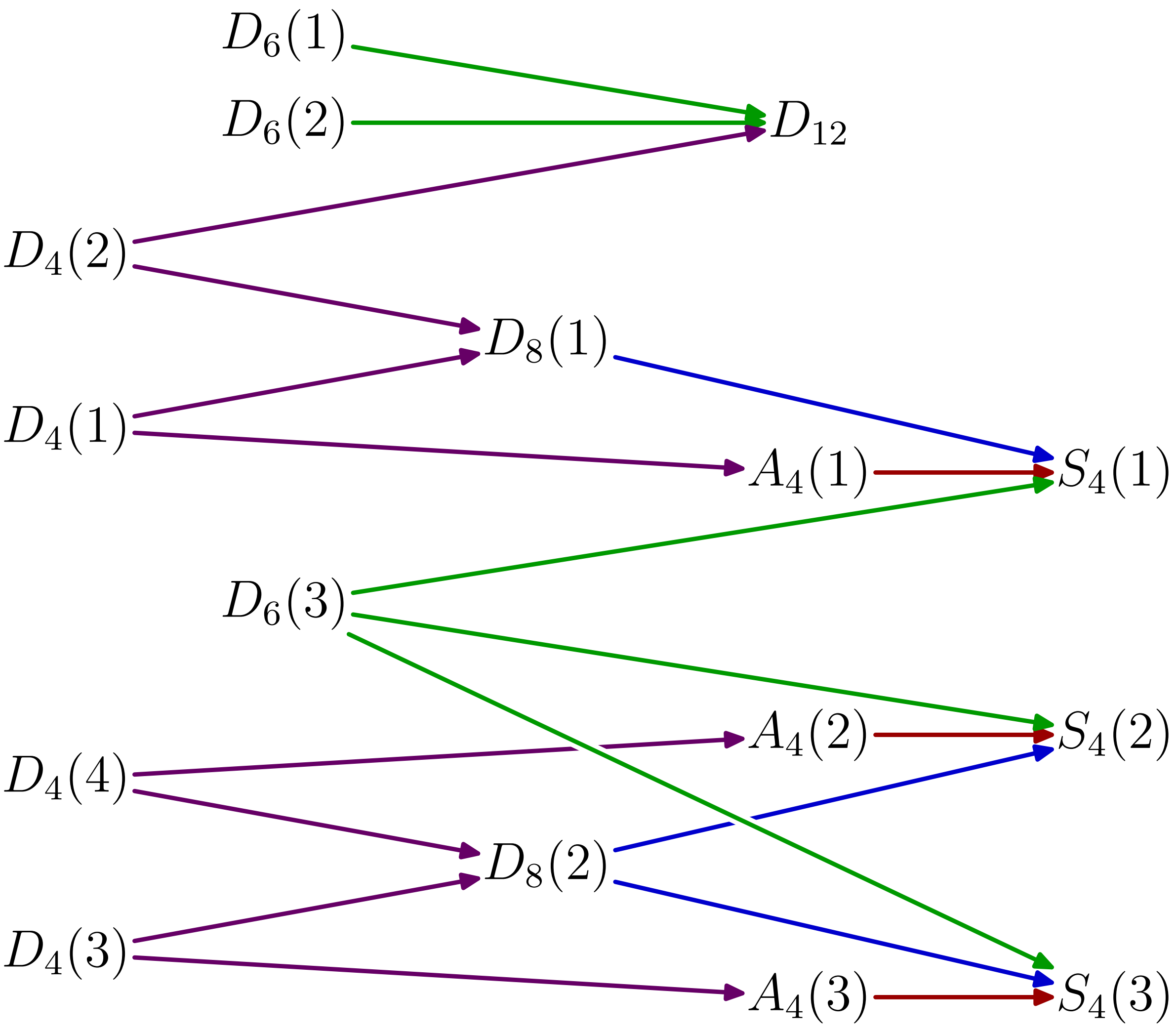
| $H$ | # fixed pts | # sing pts | $P_{F_H}$ |
|---|---|---|---|
| $D_4$ | 24 | 4 | $1+3t$ |
| $3 \times D_8$ | 36 | 12 | $1+4t$ |
| $G = S_4$ | 4 | 4 | $1+4t$ |

# calculation in `maxima`

```
S3(t):=1+t^2+4*t^3+t^4+t^6
      -(15*(1+t^2-4)+20);
S12(t):=10*((1+t^2)*(1+t^2)
       -4*(1+t^2));
S13(t):=((t^4+2*t^3+2*t^2+1)
       -4*(1+t^2));
S14(t):=4*((2*t^4+2*t^3+3*t^2+1)
       -4*(1+2*t^2));
S0(t):=4*(1+3*t^2)+(12+4)*(1+4*t^2);
P(t):=S3(t)+S12(t)+S13(t)+S14(t)+S0(t);
```

$$P_X(t) = t^6 + 20\,t^4 + 14\,t^3 + 20\,t^2 + 1$$

# Poincaré in $dim = 3$



# Poincaré in $dim = 3$

| | |
|---|---|
| $D_4$ | $t^6 + 51t^4 + 8t^3 + 51t^2 + 1$ |
| $D_4$ | $t^6 + 21t^4 + 20t^3 + 21t^2 + 1$ |
| $D_4$ | $t^6 + 15t^4 + 8t^3 + 15t^2 + 1$ |
| $D_4$ | $t^6 + 15t^4 + 8t^3 + 15t^2 + 1$ |
| $D_6$ | $t^6 + 15t^4 + 32t^3 + 15t^2 + 1$ |
| $D_6$ | $t^6 + 15t^4 + 32t^3 + 15t^2 + 1$ |
| $D_6$ | $t^6 + 7t^4 + 16t^3 + 7t^2 + 1$ |
| $D_8$ | $t^6 + 36t^4 + 14t^3 + 36t^2 + 1$ |
| $D_8$ | $t^6 + 15t^4 + 8t^3 + 15t^2 + 1$ |

# Poincaré in $dim = 3$

| | |
|---|---|
| $D_{12}$ | $t^6 + 21t^4 + 20t^3 + 21t^2 + 1$ |
| $A_4$ | $t^6 + 19t^4 + 8t^3 + 19t^2 + 1$ |
| $A_4$ | $t^6 + 7t^4 + 8t^3 + 7t^2 + 1$ |
| $A_4$ | $t^6 + 7t^4 + 8t^3 + 7t^2 + 1$ |
| $S_4$ | $t^6 + 20t^4 + 14t^3 + 20t^2 + 1$ |
| $S_4$ | $t^6 + 11t^4 + 8t^3 + 11t^2 + 1$ |
| $S_4$ | $t^6 + 11t^4 + 8t^3 + 11t^2 + 1$ |

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| | |
|---|---|
| $D_{12}$ | $t^6 + 21t^4 + 20t^3 + 21t^2 + 1$ |
| $A_4$ | $t^6 + 19t^4 + 8t^3 + 19t^2 + 1$ |
| $A_4$ | $t^6 + 7t^4 + 8t^3 + 7t^2 + 1$ |
| $A_4$ | $t^6 + 7t^4 + 8t^3 + 7t^2 + 1$ |
| $S_4$ | $t^6 + 20t^4 + 14t^3 + 20t^2 + 1$ |
| $S_4$ | $t^6 + 11t^4 + 8t^3 + 11t^2 + 1$ |
| $S_4$ | $t^6 + 11t^4 + 8t^3 + 11t^2 + 1$ |

Note that groups conjugate in $GL(3, \mathbb{C})$ may give non-isomorphic Kummer manifolds. Dual representations yield the same Poincaré; they are related to (local) symmetries of diagram of inclusions.

# symplectic Kummer

- Let $G$ be a Weyl group of a simple Lie algebra acting on its lattice of roots $\Gamma$ with $\Gamma \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h}$ its Cartan algebra. The action of $G$ preserves the Killing form on $\mathfrak{h}$, hence a symplectic form on $\mathfrak{h} \oplus \mathfrak{h}^*$.

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- Both series are obtained by integral Kummer construction with $A$ an abelian surface.
- Two (out of four known) topological types of irreducible symplectic manifolds are Kummer. May be there are more of them?

# binary tetrahedral, again

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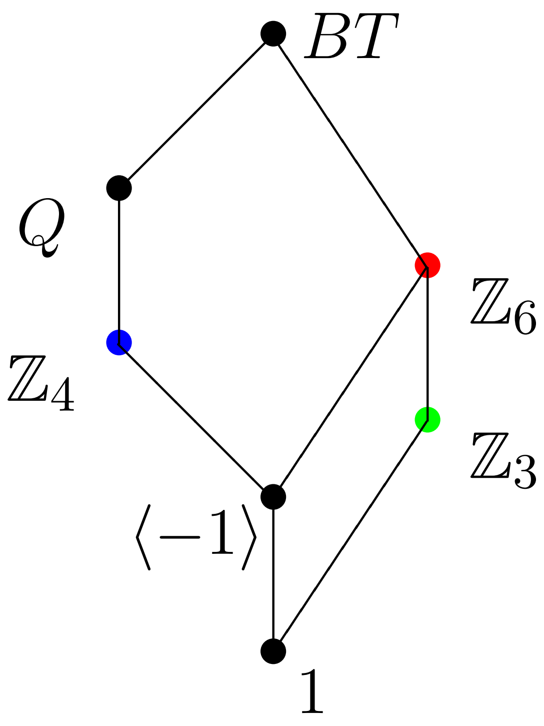
- Type $E_6$ with $G = BT_{24}$ also admits a (local!) symplectic resolution but it does not allow a global resolution.
- Key fact: symplectic resolutions are semi-small so isolated quotient singularities do not have a symplectic resolution in dimension $> 2$

# binary tetrahedral, again

Lattice of subgroups



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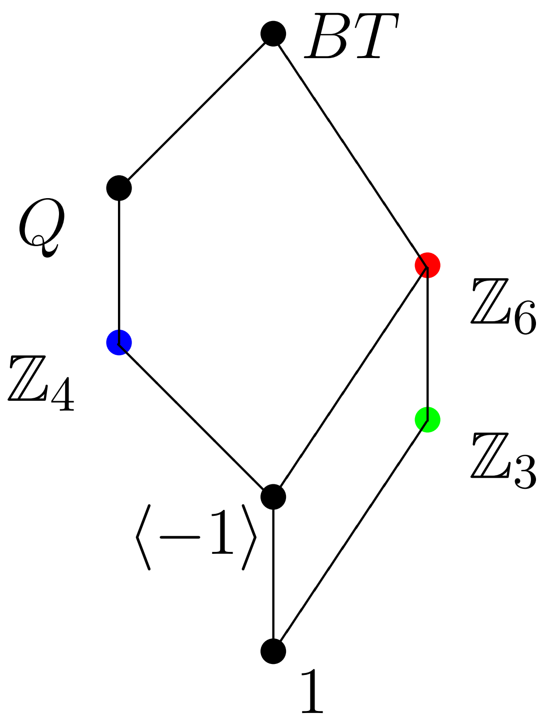
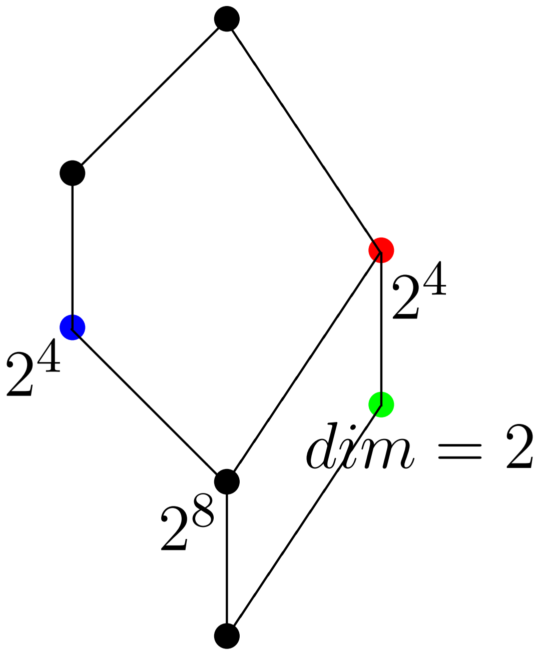
Lattice of subgroups




Lefschetz thm fixed pts



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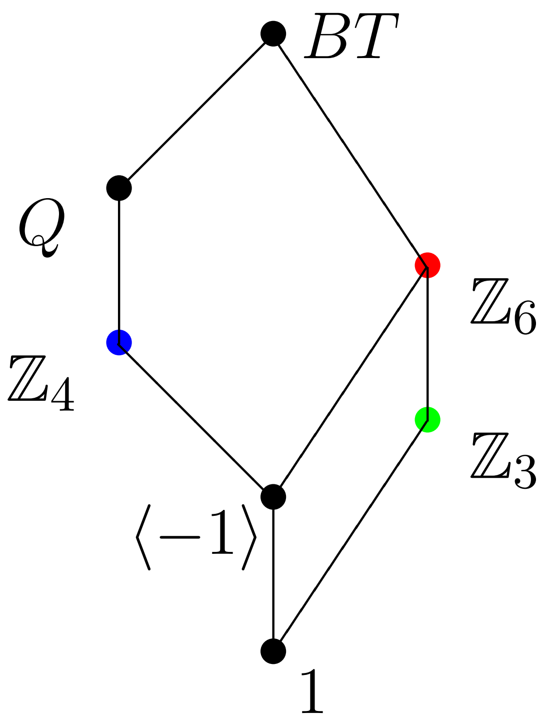
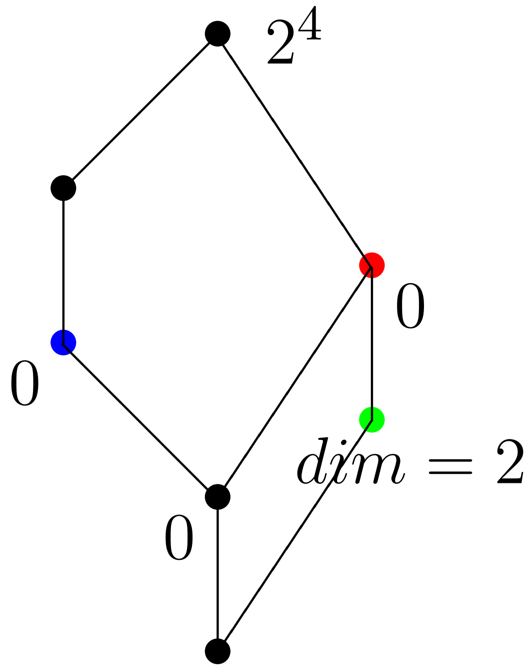
Lattice of subgroups

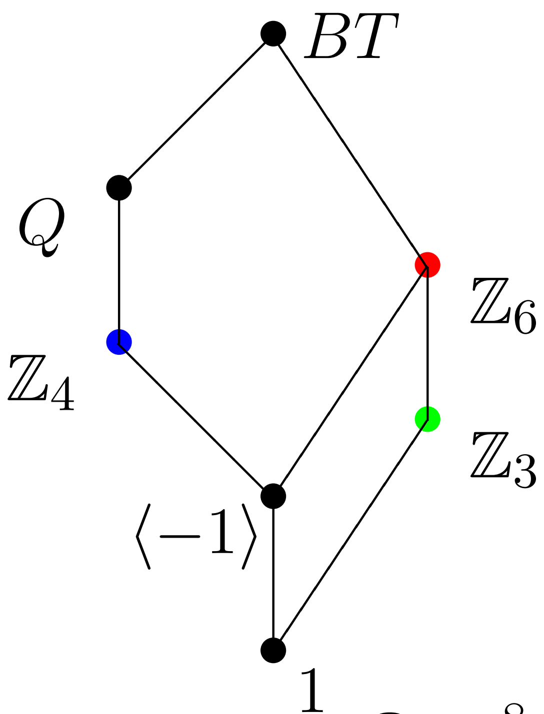



Pts with given isotropy



# binary tetrahedral, again

Lattice of subgroups

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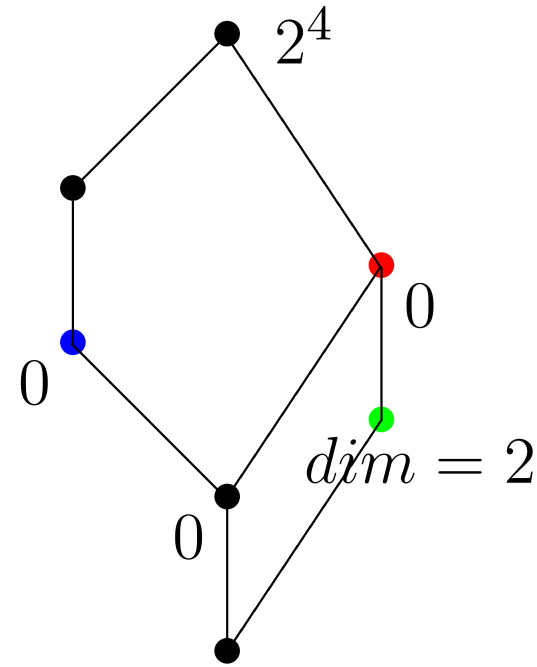


So $2^8 = 2^4$, contradiction!