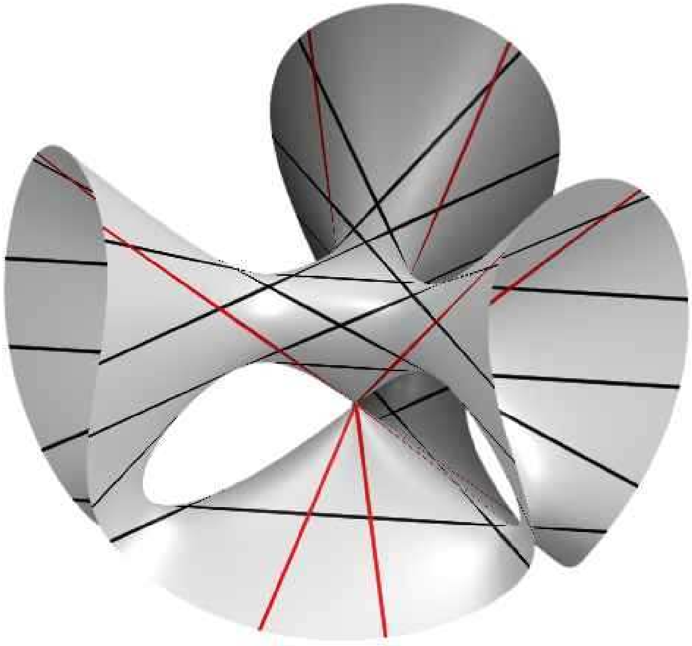
Fano Spaces and Marked Polytopes

Jarosław Wiśniewski, joint project with Klaus Altmann

prologue: a cubic

Celebrated 27 lines on a cubic surface

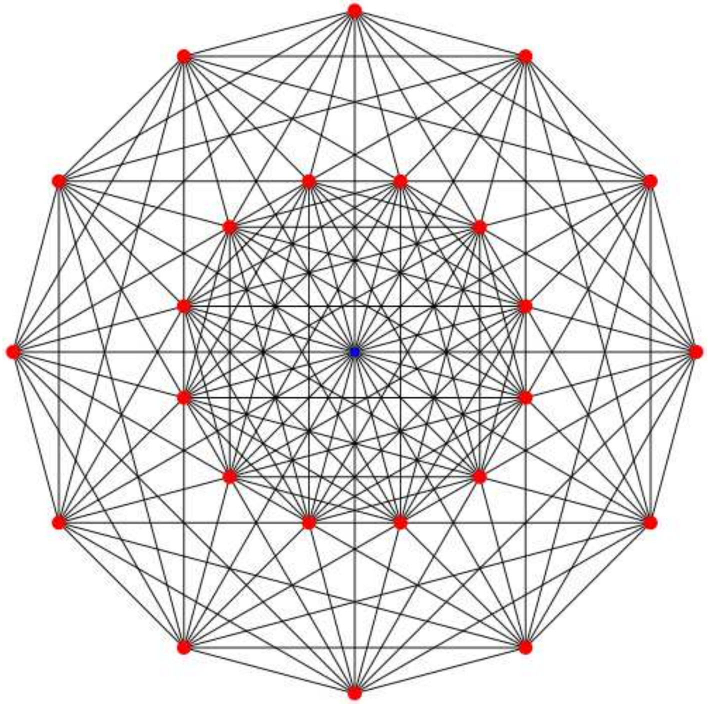
typical view



source:

cubics.algebraicsurface.net

polytopal view



source: wikipedia

Manin-Dolgachev-Mukai table

Root systems associated to $\mathbb{P}^d$ blown-up at r generic pts.

	0	1	2	3	4	5	6	7	8
$\mathbb{P}^1$									
$\mathbb{P}^2$		toric		$A_1 \times A_2$	A_4	D_5	E_6	E_7	E_8
$\mathbb{P}^3$			toric		$A_1 \times A_3$	A_5	D_6	not f.g.	
$\mathbb{P}^4$				toric		$A_1 \times A_4$	A_6	D_7	n.f.g.

- $d = 2$, $r \leq 8$: del Pezzo (1885), Manin (1972)
- $d \geq 3$, $r \leq d + 3$: Dolgachev (1978), Mukai (2003), Matsuki (1995)

root systems

Manin-Mukai construction:

- Let $X = X_r^d$ be a blow up of $\mathbb{P}^d$ in r points. In the Picard lattice, or $\mathrm{Cl}(X)$, or $H^2(X, \mathbb{Z})$, we have the pull back of the hyperplane class h, the classes of exceptional divisors e_i and the anti-canonical class $-\omega_X = (d+1)h - (d-1)\sum_i e_i$.
- We define the intersection form in which these classes are orthogonal and moreover $e_i^2 = -1$ and $h^2 = d-1$. We take classes in the lattice orthogonal to $-\omega_X$ whose selfintersection is -2. They give the root system in question.

Cremona transformations

Kantor (1885-1895) studied subgroups of Cremona group of birational transformations of $\mathbb{P}^2$. They admit reflection action in the cohomology of the surface which is the resolution of the map. They are $A_1 \times A_2$, A_4, D_5, E_6, E_7, E_8.

Coble (1917) extended this to higher dimensions. Next generalized by Dolgachev (1978) and Mukai (2001).

physicists: mysterious duality!

Iqbal, Neitzke, Vafa (2001); speculation about mysterious duality of del Pezzo surfaces and M-theory:

”The main observation is that the large diffeomorphisms of del Pezzo surfaces match the Weyl group of the U-duality group of the corresponding compactification of M-theory. The elements of the second homology of the del Pezzo surfaces are mapped to various BPS objects of different dimensions in M-theory.”

physicists: mysterious duality!

Iqbal, Neitzke, Vafa (2001); speculation about mysterious duality of del Pezzo surfaces and M-theory:

”The main observation is that the large diffeomorphisms of del Pezzo surfaces match the Weyl group of the U-duality group of the corresponding compactification of M-theory. The elements of the second homology of the del Pezzo surfaces are mapped to various BPS objects of different dimensions in M-theory.”

??? any idea what this would mean ????

Cox rings and MDS

- Let Z be a $\mathbb{Q}$-factorial projective variety over $\mathbb{C}$ such that $\mathrm{Cl}(Z)$ is a lattice. We define

$$\mathrm{Cox}(Z) = \bigoplus_{D \in \mathrm{Cl}(Z)} \Gamma(Z, \mathcal{O}(D))$$

  with multiplicative structure defined by a choice of divisors whose classes form a basis of $\mathrm{Cl}(Z)$.

- Assume $\mathrm{Cox}(Z)$ finitely generated and call Z a Mori Dream Space (or MDS). The $\mathrm{Cl}(Z)$-grading of $\mathrm{Cox}(Z)$ yields action of torus $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(Z), \mathbb{C}^*) \cong (\mathbb{C}^*)^{\mathrm{rk}(\mathrm{Cl}(Z))}$ on the affine variety $\mathrm{Spec}(\mathrm{Cox}(Z))$.

Batyrev conjecture

Batyrev: for surfaces $\mathrm{Proj}(\mathrm{Cox}(X))$, where $\mathrm{Cox}(X)$ is taken with anti-canonical grading, should admit an embedding into homogeneous variety defined by a representation of the respective simple Lie group.

- Derenthal (2007), Serganova, Skorobogatov (2007): it is the highest weight representation for A_4, D_5, E_6 and E_7,
- for A_{d+2} we have $\mathrm{Proj}(\mathrm{Cox}(X)) = Grass(2, d+3)$: commonly (?) known,
- Sturmfels, Velasco (2009): for D_{d+3} it is half-spinor representation (even-grading Grassmann algebra).

Mori view

Let Z be a $\mathbb{Q}$-factorial projective variety over $\mathbb{C}$ such that $\mathrm{Cl}(Z)$ is a lattice. We define vector space $N^1(Z) = \mathrm{Cl}(Z) \otimes \mathbb{R} = \mathrm{Pic}(Z) \otimes \mathbb{R}$ and inside this space we have closed cones:

- $\mathrm{Eff}(Z)$ spanned by classes of effective divisors
- $\mathrm{Mov}(Z)$ movable divisors, with no base components
- $\mathrm{Nef}(Z)$ nef divisors, closure of ample cone

The dual $N_1(Z)$ is the space of 1-cycles and dual of $\mathrm{Nef}(Z)$ is the cone of effective 1-cycles (Kleiman).

Fano varieties

If $-K_Z \in \operatorname{int}\operatorname{Nef}(Z)$ then Z is a Fano variety. Then (assuming good singularities) $\operatorname{Nef}(Z)$ is a rational polyhedral cone and there is a bijection

- contractions $\varphi : Z \to Y_\varphi$ surjective morphisms with connected fibers onto normal varieties
- faces of $\operatorname{Nef}(Z)$ defined by intersection $\operatorname{Nef}(Z) \cap \varphi^* N^1(Y_\varphi) = \varphi^* \operatorname{Nef}(Y_\varphi)$.

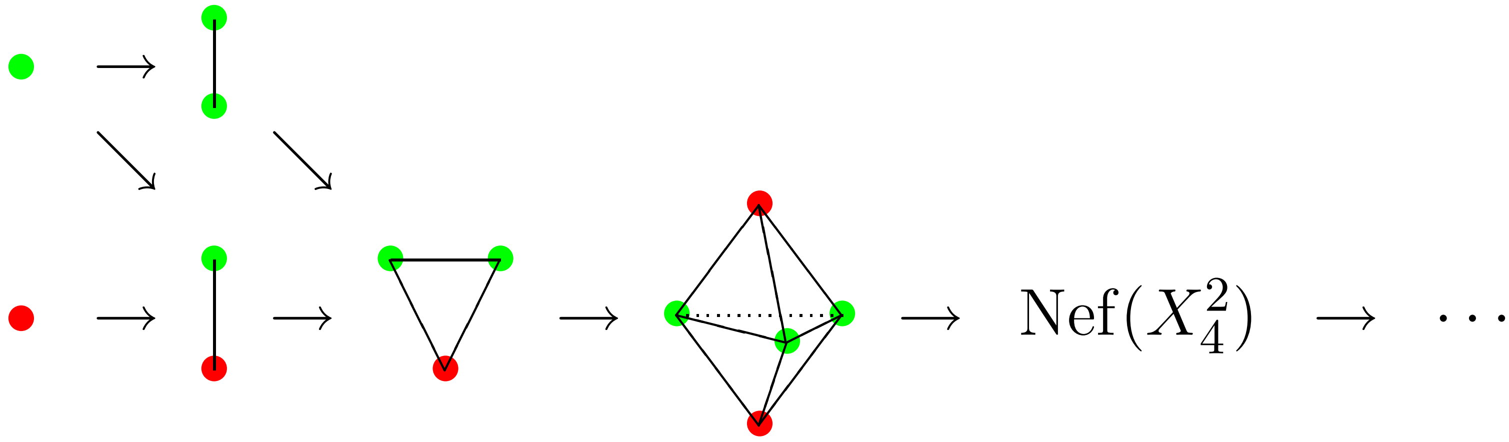
Instead of polyhedral cones we will consider their compact sections, polytopes.

contractions of del Pezzo

Let X^2_r be $\mathbb{P}^2$ blow-up at r points. We have contractions

$$
\begin{array}{ccccccccccc}
\mathbb{P}^1 & \leftarrow & \mathbb{P}^1 \times \mathbb{P}^1 & & & & & & & & \\
 & \nwarrow & & \nwarrow & & & & & & & \\
\mathbb{P}^2 & \leftarrow & X^2_1 & \leftarrow & X^2_2 & \leftarrow & X^2_3 & \leftarrow & X^2_4 & \leftarrow & \cdots
\end{array}
$$

And a sequence of polytopes



marked polytopes

- We mark faces of a polytope Δ of dimension r by non-negative integers. The interior of the polytope is marked by 0 and if β is a face of α then marking of β is not smaller than that of α.
- Marked face polynomial $P_\Delta(x,y) = \sum a_{ij}x^iy^j$ where a_{ij} is the number of faces of codim i marked by j.
- Let Δ_r^2 be a section of $\mathrm{Nef}(X_r^2)$. A face associated to $\varphi : X_r^2 \longrightarrow Y_\varphi$ is marked by $\dim X_r^2 - \dim Y_\varphi$.
- Let P_r^2 be the marked polynomial of Δ_r^2. We see $P_2^2 = 1 + 3x + x^2 + 2x^2y$ and $P_3^2 = 1 + 6x + 9x^2 + 2x^3 + 3x^3y$.

recursive relations

Theorem. [Stalij, MSc thesis 2001]
For $r = 2, \ldots, 8$ the following holds

- The polytope Δ_r^2 is simple (dual to a simplex) at faces marked by 0 while dual to a cross-polytope (orthoplex or co-cube) at vertices marked by 1.
- The polynomials P_r^2 satisfy equations
  $\partial_x P_r^2(x,0) = \partial_x P_r^2(0,0) \cdot P_{r-1}^2(x,0)$ and
  $2(r-1) \cdot \partial_y P_r^2(1,0) = \partial_x P_r^2(0,0) \cdot \partial_y P_r^2(1,0)$
- If $P_r^2(-1,1) = (-1)^{r+1}$ and $P_{r-1}^2(x,0) < P_r^2(x,0)$ then the equations have unique solution for $r < 8$ and no solution for $r = 9$.

geometric contents

The theorem follows from easy

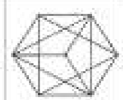
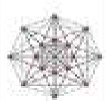
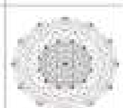
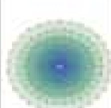
Proposition.

- For $r \geq 2$ every contraction of X_r^2 factors through a blow-down of (-1)-curve, $X_r^2 \to X_{r-1}^2$
- Contractions to $\mathbb{P}^2$ contract r disjoint (-1)-curves.
- Contractions to $\mathbb{P}^1$ have $r-1$ reducible fibers (conics) consisting of two (-1)-curves.
- The number of (-1) curves on X_r^2 is $\partial_x P_r^2(0,0)$

Gosset polytopes (1900)

Elements

Gosset semiregular figures

n-ic	k_{21}	Graph	Name Coxeter-Dynkin diagram	Facets		Elements							
				(n-1)-simplex $\{3^{n-2}\}$	(n-1)-orthoplex $\{3^{n-4,1,1}\}$	Vertices	Edges	Faces	Cells	4-faces	5-faces	6-faces	7-faces
3-ic	-1_{21}		Triangular prism	2 triangles	3 squares	6	9	5					
4-ic	0_{21}		Rectified 5-cell	5 tetrahedron	5 octahedron	10	30	30	10				
5-ic	1_{21}		Demipenteract	16 5-cell	10 16-cell	16	80	160	120	26			
6-ic	2_{21}		2_{21} polytope	72 5-simplexes	27 5-orthoplexes	27	216	720	1080	648	99		
7-ic	3_{21}		3_{21} polytope	576 6-simplexes	126 6-orthoplexes	56	756	4032	10080	12096	6048	702	
8-ic	4_{21}		4_{21} polytope	17280 7-simplexes	2160 7-orthoplexes	240	6720	60480	241920	483840	483840	207360	19440
9-ic	5_{21}		E8 lattice	∞ 8-simplexes	∞ 8-orthoplexes	∞	∞	∞	∞	∞	∞	∞	∞

MDS and SQM's

An MDS Z has finitely many small (iso in codim 1) $\mathbb{Q}$-factorial modifications Z_i (SQM's) [Hu, Keel]

MDS and SQM's

An MDS Z has finitely many small (iso in codim 1) $\mathbb{Q}$-factorial modifications Z_i (SQM's) [Hu, Keel]
Varieties Z_i are exactly the $\mathbb{Q}$-factorial GIT quotients of $\mathrm{Cox}(Z)$ by the Picard torus arising from linearizations of the trivial bundle depending on the choice of a character of the torus [Thaddeus, Reid, Brion, Hu, Dolgachev]

MDS and SQM's

An MDS Z has finitely many small (iso in codim 1) $\mathbb{Q}$-factorial modifications Z_i (SQM's) [Hu, Keel] Z_i share the same Cox ring and, by strict transform, we identify $\mathrm{Div}(Z_i)$ and $\mathrm{Cl}(Z_i)$ with $\mathrm{Div}(Z)$ and $\mathrm{Cl}(Z)$, respectively; same holds for effective and movable cones

$$\mathrm{Eff}(Z_i) = \mathrm{Eff}(Z) \qquad \mathrm{Mov}(Z_i) = \mathrm{Mov}(Z)$$

MDS and SQM's

An MDS Z has finitely many small (iso in codim 1) $\mathbb{Q}$-factorial modifications Z_i (SQM's) [Hu, Keel]
Z_i share the same Cox ring and, by strict transform, we identify $\mathrm{Div}(Z_i)$ and $\mathrm{Cl}(Z_i)$ with $\mathrm{Div}(Z)$ and $\mathrm{Cl}(Z)$, respectively; same holds for effective and movable cones

$$\mathrm{Eff}(Z_i) = \mathrm{Eff}(Z) \qquad \mathrm{Mov}(Z_i) = \mathrm{Mov}(Z)$$

However, the cones $\mathrm{Nef}(Z_i)$ are different, that is $\mathrm{int}\,\mathrm{Nef}(Z_i) \cap \mathrm{int}\,\mathrm{Nef}(Z_j) = \emptyset$ if $Z_i \neq Z_j$ and we have decomposition

$$\mathrm{Mov}(Z) = \bigcup_i \mathrm{Nef}(Z_i)$$

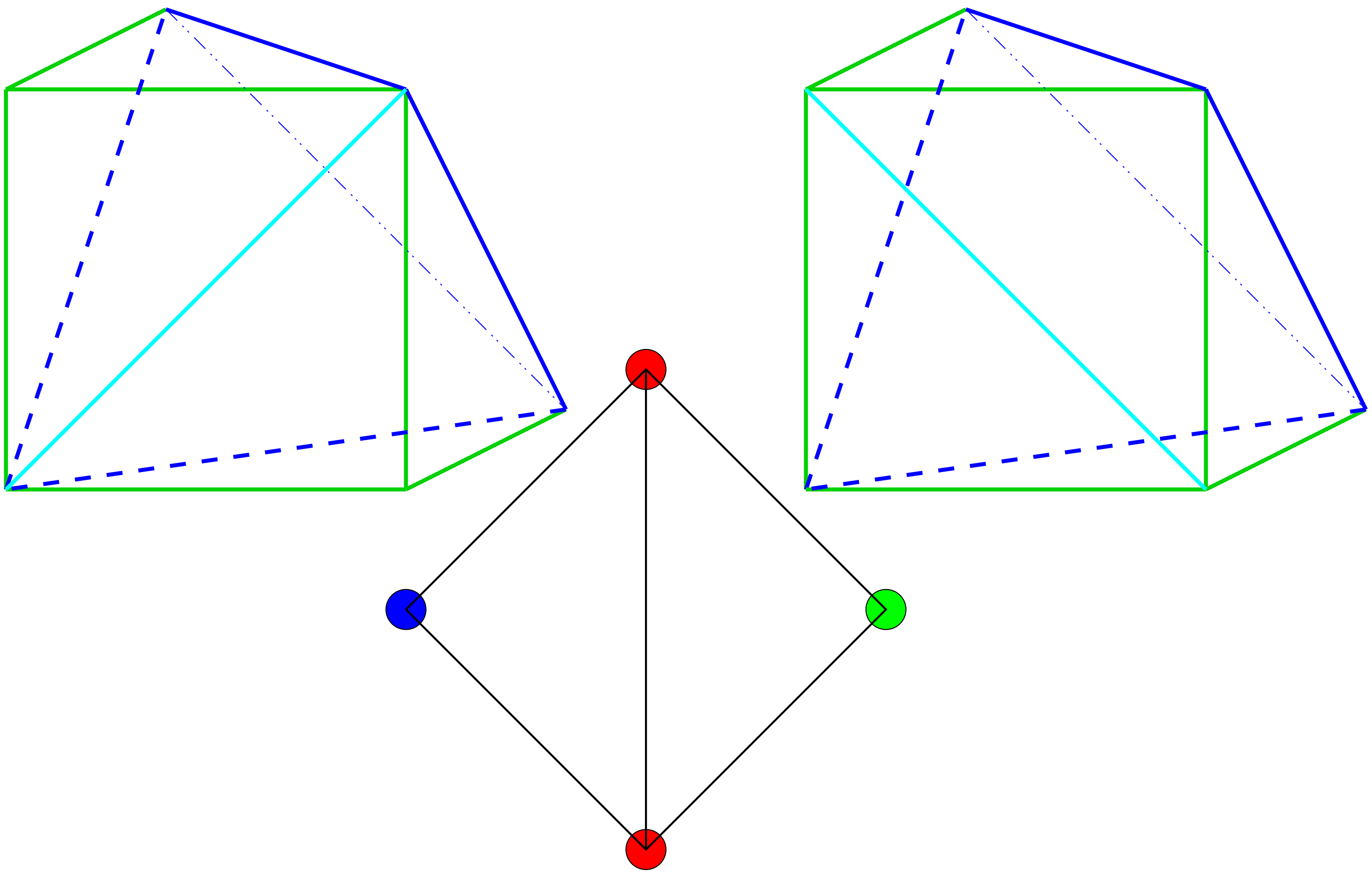
Fano spaces

If $-K_Z \in \operatorname{int} \operatorname{Mov}(Z)$ then an MDS Z is called a Fano space. The set of all its SQM's will be denoted by $\mathcal{Z}$ There is a bijection

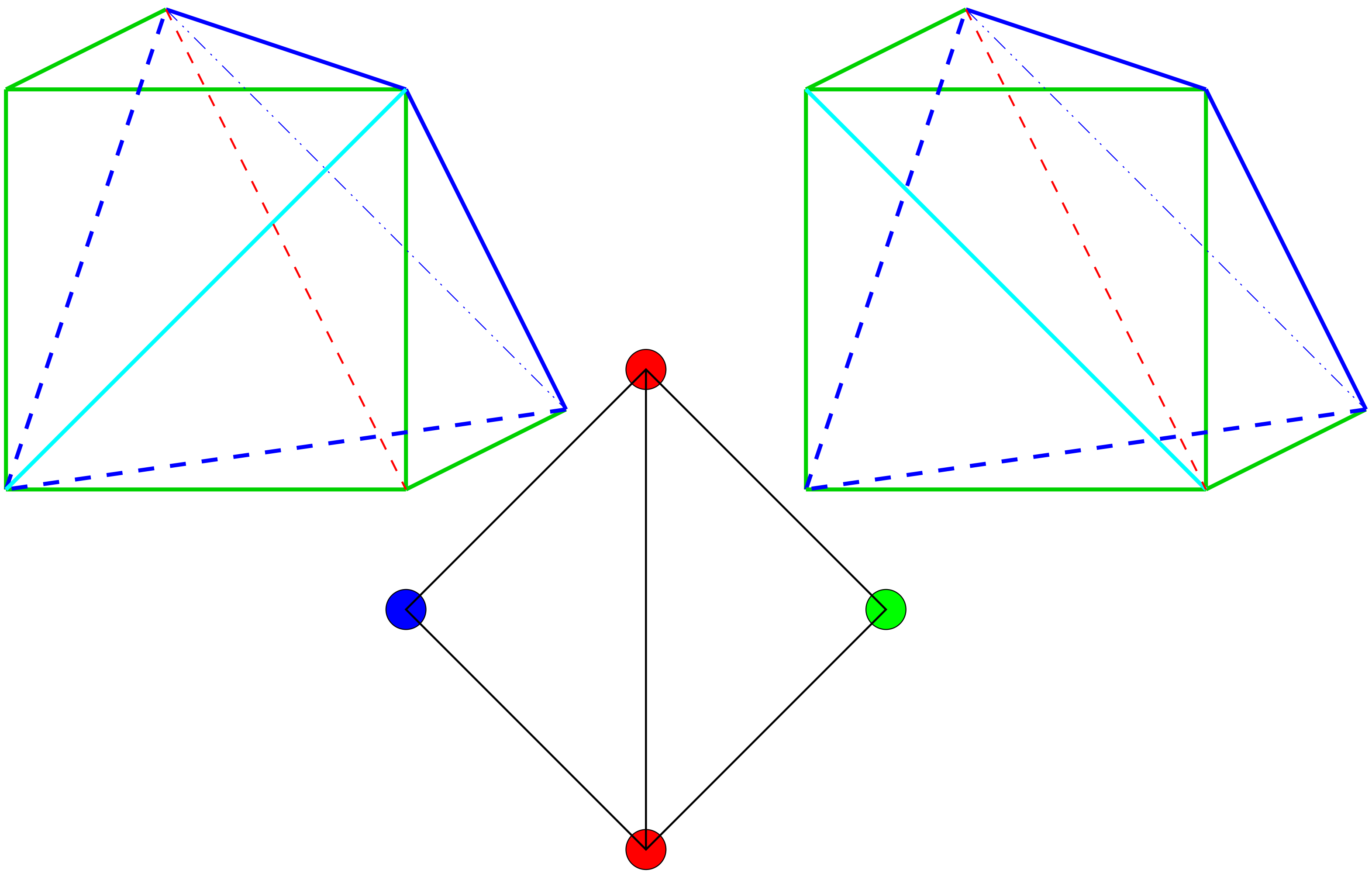
- classes (!) of divisorial or fiber type contractions $\varphi : Z_i \to Y_{\varphi,i}$, where $Z_i \in \mathcal{Z}$ and $Y_{\varphi,i} \in \mathcal{Y}_\varphi$ (call them essential contractions)
- faces of $\operatorname{Mov}(Z)$ defined by intersection $\operatorname{Mov}(Z) \cap \varphi^* N^1(Y_\varphi) = \varphi^* \operatorname{Mov}(Y_\varphi)$.

Instead of polyhedral cones we will consider, as usually, their compact sections, polytopes. We color vertices by targets of the contractions $\bullet = \mathbb{P}^1$, $\bullet = \mathbb{P}^2$, $\bullet = \mathbb{P}^3$.

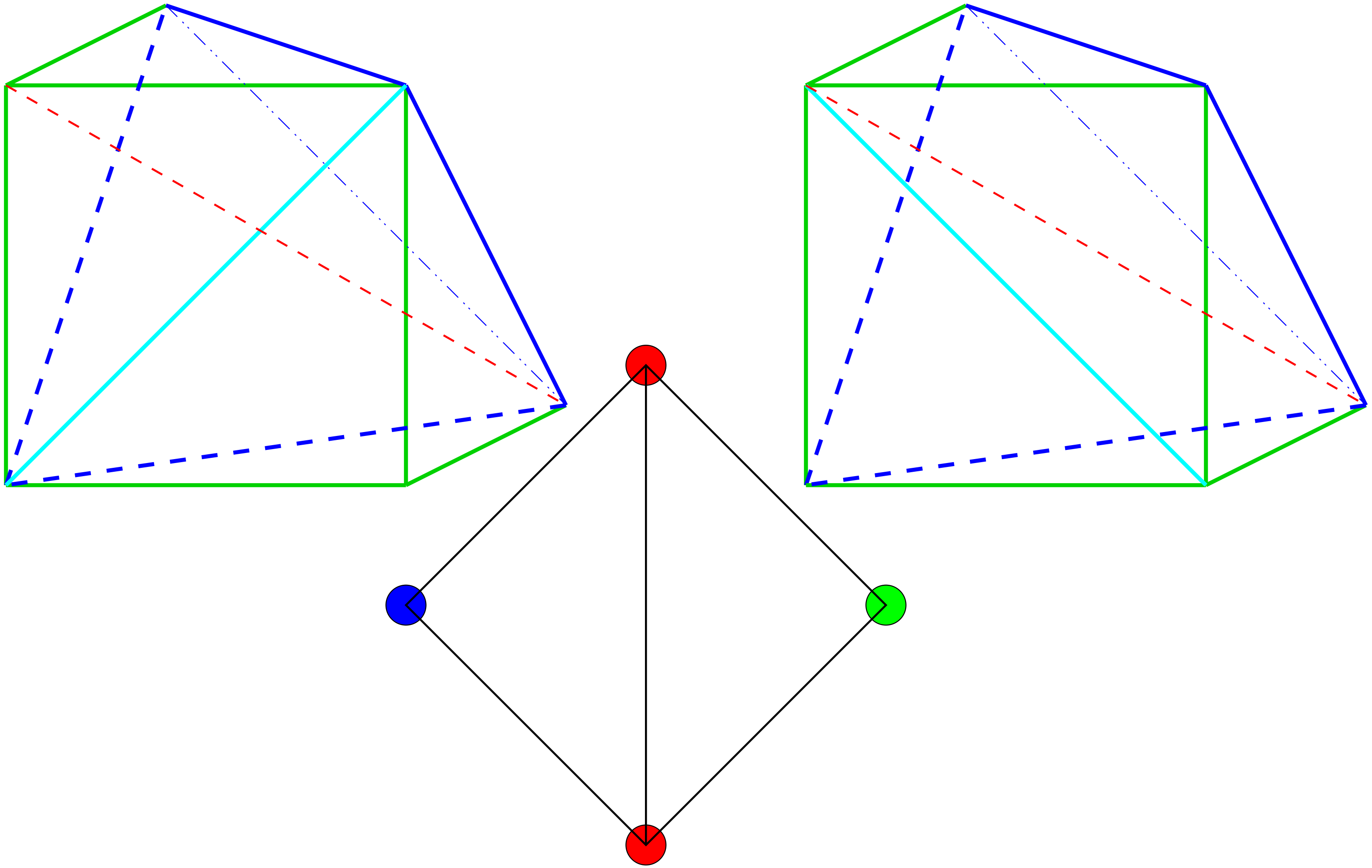
blow-up of $\mathbb{P}^3$ in 2 points



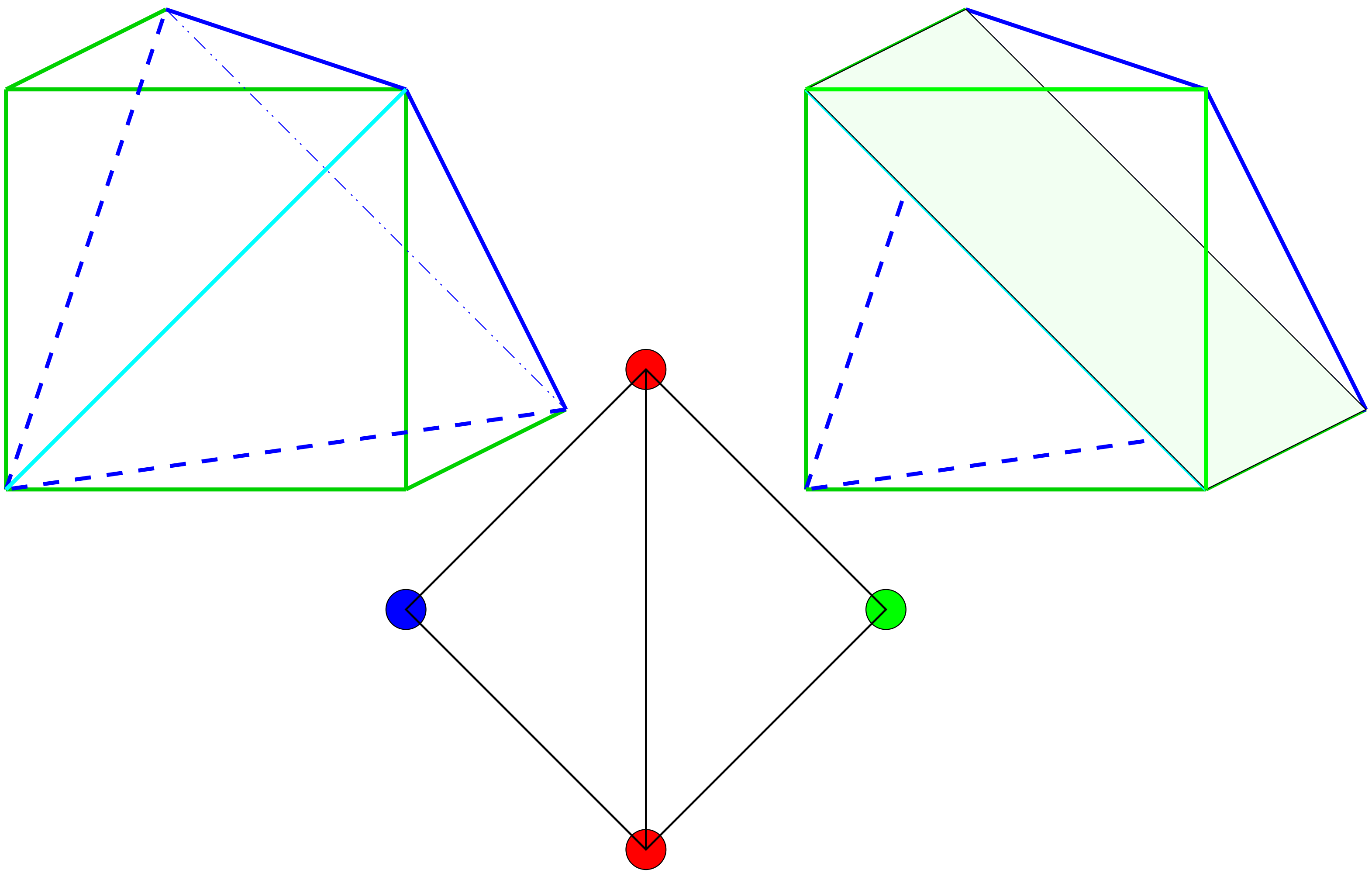
blow-up of $\mathbb{P}^3$ in 2 points



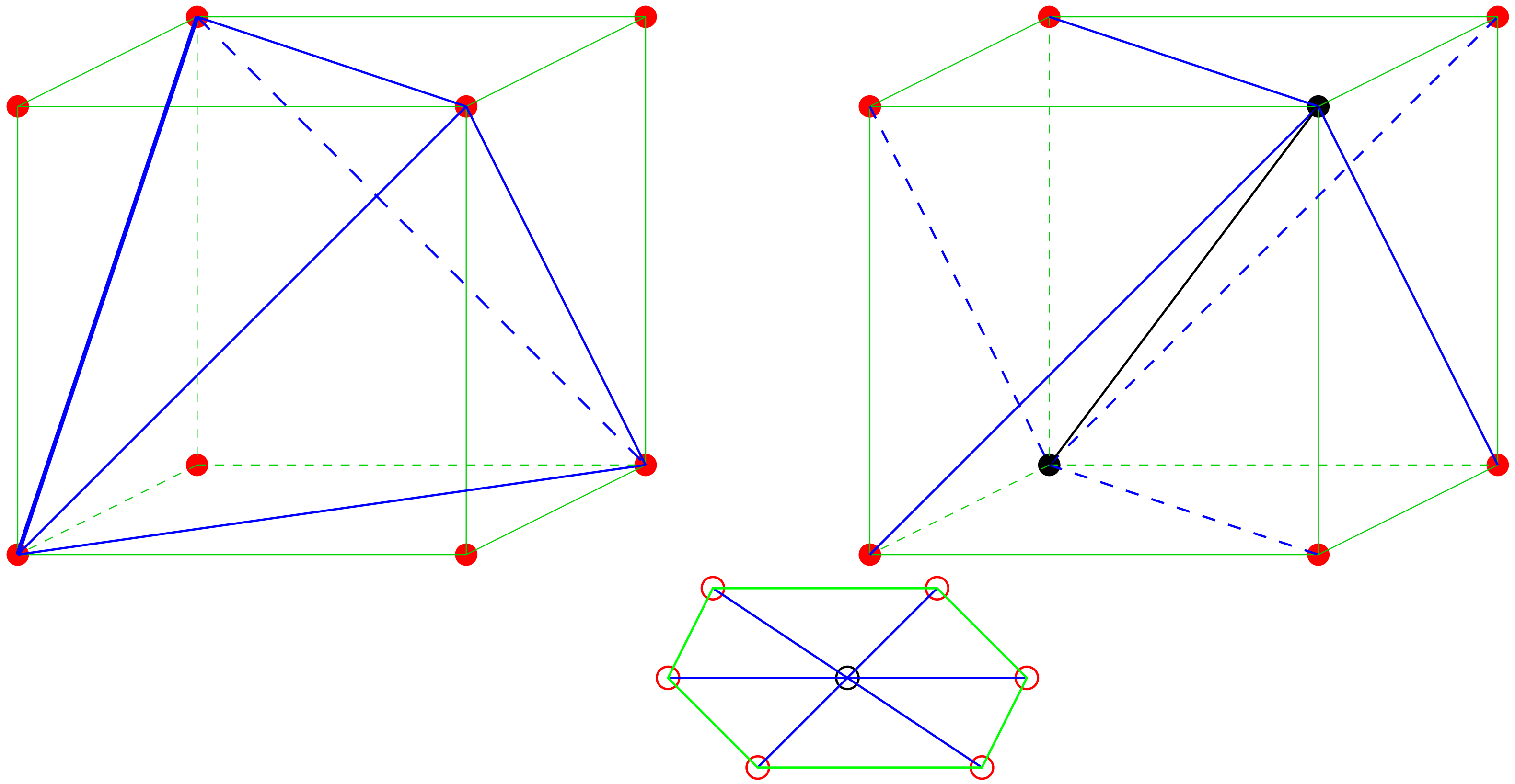
blow-up of $\mathbb{P}^3$ in 2 points



blow-up of $\mathbb{P}^3$ in 2 points



blow-up of $\mathbb{P}^3$ in 4 points



$A_1 \times A_n$, A_{n+2} and D_{n+3}

We get a diagram of essential elementary contractions of Fano spaces associated to X_r^d

$$\begin{array}{ccccccc}
\mathcal{X}^d_{d+1} & \leftarrow & \mathcal{X}^d_{d+2} & \leftarrow & \mathcal{X}^d_{d+3} & & \\
 & \nwarrow & & \nwarrow & & \nwarrow & \\
\cdots & \leftarrow & \mathcal{X}^{d+1}_{d+2} & \leftarrow & \mathcal{X}^{d+1}_{d+3} & \leftarrow & \mathcal{X}^{d+1}_{d+4}
\end{array}$$

and a diagram of inclusion of facets

$$\begin{array}{ccccccc}
\mathrm{Mov}(\mathcal{X}^d_{d+1}) & \rightarrow & \mathrm{Mov}(\mathcal{X}^d_{d+2}) & \rightarrow & \mathrm{Mov}(\mathcal{X}^d_{d+3}) & & \\
 & \searrow & & \searrow & & \searrow & \\
\cdots & \rightarrow & \mathrm{Mov}(\mathcal{X}^{d+1}_{d+2}) & \rightarrow & \mathrm{Mov}(\mathcal{X}^{d+1}_{d+3}) & \rightarrow & \mathrm{Mov}(\mathcal{X}^{d+1}_{d+4})
\end{array}$$

marked polytopes again

Repeat the same arguments as for the del Pezzo case for $r = d+1,\ d+2,\ d+3$:

- facets of Δ_r^d are Δ_{r-1}^d or Δ_{r-1}^{d-1}
- vertices are associated to $\mathbb{P}^1, \ldots, \mathbb{P}^d$
- Δ_r^d are simple (cosimplicial) at vertices $\mathbb{P}^2, \ldots, \mathbb{P}^d$ and of special type at $\mathbb{P}^1$
- marked polynomial satisfy some differential equations
- duals of Δ_r^d are known polytopes

D_{n+3}: Gosset's demicubes (1900)