FINITE SUBGROUPS OF THE CREMONA GROUP OF THE PLANE

JÉRÉMY BLANC

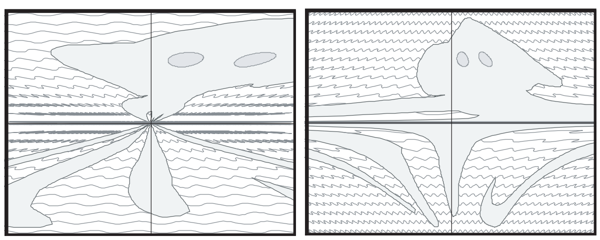
1. Intro

These are notes for the 35th Autumn School in Algebraic Geometry, entitled *Subgroups of Cremona groups* in Lukecin, Poland, September 23 – September 29, 2012.

Our aim is to study the finite subgroups of the group $\mathrm{Bir}(\mathbb{P}^2)$ of birational transformations of the plane $\mathbb{P}^2$. It is a group, with a law given by composition of maps. Our base field will be an algebraic closed field $\mathbf{k}$ of characteristic zero. (In fact, most of the results hold also in characteristic p, if p does not divide the order of the group).

1.1. The Cremona group - informal introduction.

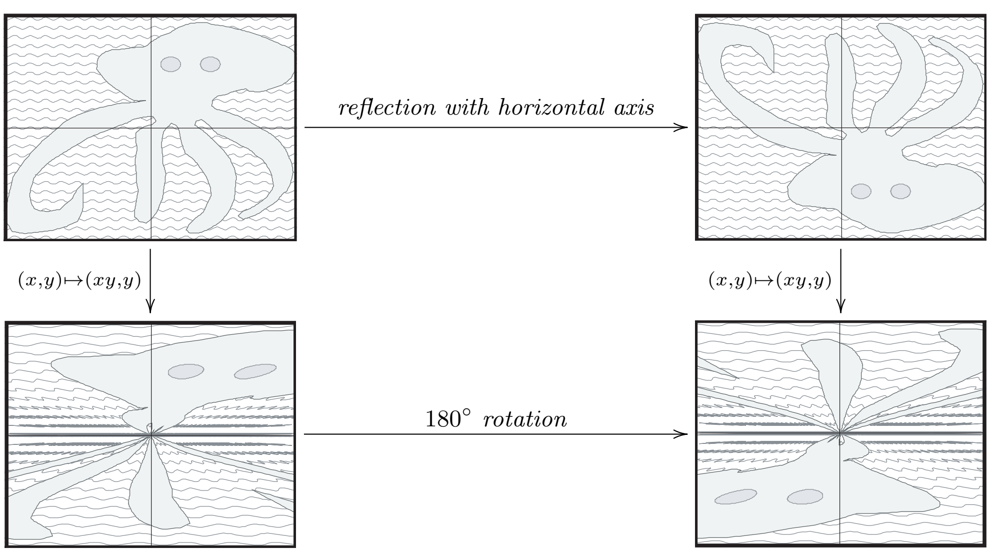
A *birational map* of the plane is a transformation defined by quotients of polynomials. Such a map always sends most algebraic curves to algebraic curves, except for a finite number of curves, which might be mapped to points. Moreover, some points might have no image. For this reason, we denote the map by the symbol "$\dashrightarrow$". For example, the map $(x, y) \dashrightarrow (xy, y)$ has inverse $(x, y) \dashrightarrow (x/y, y)$:



On the left, the image of the octopus by $(x, y) \dashrightarrow (xy, y)$, which sends the line $y = 0$ to a point.
On the right, the image of the octopus by $(x, y) \dashrightarrow (x/y, y)$; the point $(0, 0)$ has no image.

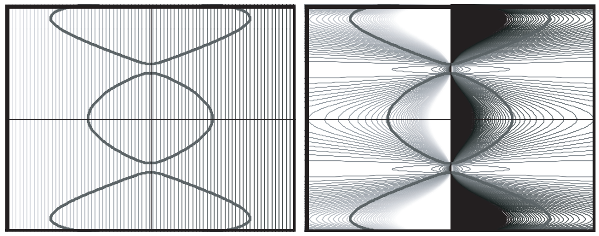
The set of all birational maps of the plane is called *the Cremona group*.

Note that the reflection and the 180° rotation, which are not conjugate in the automorphism group, are conjugate in the Cremona group (in fact, this result is true in any dimension, as proved in [Bla06], Theorem 1). We can show this explicitly:



Thus, the two conjugacy classes of involutive isometries belong to the same conjugacy class of the Cremona group.

However, there are infinitely many conjugacy classes of involutions in the Cremona group; each class belongs to one of three families, called de Jonquières, Geiser and Bertini involutions.



The image of some vertical lines by a de Jonquières involution. The set of fixed points is in grey.

Our aim is to obtain such a classification for involutions, but also for some other finite groups of birational maps. In fact we shall work with the complex plane rather than the real plane. This simplifies many of the results (but not the figures).

So, let the story begin....

1.2. **A long history of results.** As we explained above, our subject is the Cremona group of the plane, which is the group of birational maps of the surface $\mathbb{P}^2(\mathbf{k})$ (or $\mathbf{k}^2$), or equivalently, the group of $\mathbf{k}$-automorphisms of the field $\mathbf{k}(X,Y)$.

This very large group has been a subject of research for many years. We refer to [AlC02] for a modern survey about its elements. Some presentations of the group by generators and relations are available (see [Giz82], [Isk83] and [Bla12]), but these results does not provide substantial insight into the algebraic properties of the group. For example, given an abstract group, it is not possible to say whether it is isomorphic to a subgroup of the Cremona group. Moreover, the results of [Giz82], [Isk83] and [Bla12] do not allow one to decide whether the Cremona group is isomorphic to a linear group or wether it is a simple group.

In fact, the group $\mathrm{Bir}(\mathbb{P}^2)$ is not simple [CL10] and not linear (not conjugate to a subgroup of $\mathrm{GL}(n,\mathbb{K})$) for any n and any $\mathbb{K}$).

The study of the finite subgroups of the Cremona group is one way to understand the group. It begun over one hundred years ago and it appears unlikely that it will be totally completed in the near future. Let us give a historical review of the main results:

- The first results are attributed to Bertini, for his work on involutions in 1877 (see [Ber77]). He identified three types of conjugacy classes, which are now known as de Jonquières, Geiser and Bertini involutions. However, his proof of the classification of conjugacy classes in each type is generally considered incomplete (see [Ba-Be00]).
- In 1895, S. Kantor [Kan95] and A. Wiman [Wim96] gave a description of finite subgroups. The list is exhaustive, but not precise in two respects:
  - Given some finite group, it is not possible using their list to say whether this group is isomorphic to a subgroup of the Cremona group.
  - The possible conjugation between the groups of the list is not considered.
- A great deal of work was done by the Russian school and in particular by M.K. Gizatullin, V.A. Iskovskikh and Yu. Manin.
  - They obtained many results on G-surfaces (rational surfaces with a biregular action of some group G, see below). Our main interest is in the classification of minimal G-surfaces into two types (see Proposition 8.5). The description of decompositions of birational maps into elementary links (see [Isk96]) is also a very useful tool.
  - We refer to the articles [Giz81], [Isk67], [Isk70], [Isk79] and [Isk96] for more information.
- The modern approach started with the work of L. Bayle and A. Beauville on involutions (see [Ba-Be00]). They used the classification of minimal G-surfaces to classify the subgroups of order 2 of the Cremona group. This is the first example of a precise description of conjugacy classes:
  - The number of conjugacy classes and their descriptions are precise and clear, parametrised by isomorphism classes of curves.
  - One can decide directly whether two involutions are conjugate or not.
- The techniques of [Ba-Be00] were generalised by T. de Fernex (see [dFe04]) to cyclic groups of prime order. The list is as precise as one can wish, except for two classes of groups of order 5, for which the question of their conjugacy is not answered, but done in [Be-Bl04] by A. Beauville and the author.
- A. Beauville has further classified the p-elementary maximal groups up to conjugation (see [Bea07]). He obtains for example the following results:
  - No group $(\mathbb{Z}/p\mathbb{Z})^3$ belongs to the Cremona group if p is a prime $\neq 2,3$.
  - There exist infinitely many conjugacy classes of groups isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$.

Note that the conjugacy classes of subgroups $(\mathbb{Z}/2\mathbb{Z})^4$ of the de Jonquières group are well described ([Bea07], Proposition 2.6). However it is not clear whether two groups non-conjugate in the de Jonquières group are conjugate in the Cremona group.

- More recently, I.V. Dolgachev and V.A. Iskovskikh had updated the list of S. Kantor and A. Wiman, using the modern theory of G-surfaces, the theory of elementary links of V.A. Iskovskikh, and the conjugacy classes of Weyl groups (see [Do-Iz09]). This text contains many new results and is currently the most precise classification of conjugacy classes of finite subgroups.
  However, the following questions remain open:
  - Given two subgroups of automorphisms of the same rational surface, not conjugate by an automorphism of the surface, are they birationally conjugate?
  - [Do-Iz09] gives a list of elements of the Cremona group which are not conjugate to linear automorphisms. All these elements have order ≤ 30. The complete list of possible orders is not given.
  - Given some finite group, it is still not possible using [Do-Iz09] to say whether the group is isomorphic to a subgroup of the Cremona group.

  Moreover, the conjugacy classes of automorphisms of conic bundles are only partially described.
- In [Bla09a], the classification of maximal algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ is described.
- In [Bla09b], it is explain when a finite abelian subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ is linearisable.
- In [Bla11], the complete classification of finite cyclic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ is given.
- In [Tsy11], more precise descriptions (and equation) of automorphisms of conic bundles are provided.
- The classification of finite nonsolvable subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ appears in [Tsy12].

We omit to speak about the case where $\mathbf{k}$ *has positive characteristic, or when* $\mathbf{k}$ *is non-algebraically closed, which were also studied (only partially) in other texts.*

1.3. **The techniques.** The main technique is the one which was used in most of the articles above:

- We consider finite subgroups of the Cremona group as biregular automorphisms of some complete rational smooth surfaces (or equivalently as G-surfaces).
- We use the classification of minimal G-surfaces (Proposition 8.5), which comprises two cases:
  - groups of automorphisms of conic bundles (see Section 9),
  - groups of automorphisms of surfaces where the canonical class is, up to a multiple, the only invariant class of divisors (see Section 10).
- We use some conjugacy invariants and tools with G-elementary links to decide when two subgroups are conjugate. These tools are described in Section 11.

2. Blow-ups

The notion of blow-up is the most fundamental one in the subject of birational geometry.

2.1. Blowing-up the origin in $\mathbb{A}^2$.

Definition 2.1. *We let* $\mathrm{Bl}_0(\mathbb{A}^2) \subset \mathbb{A}^2 \times \mathbb{P}^1$ *be the following projective variety*

$$\mathrm{Bl}_0(\mathbb{A}^2) = \{\big((x,y),(u:v)\big) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid xv = yu\},$$

and say that the map $\pi\colon \mathrm{Bl}_0(\mathbb{A}^2) \to \mathbb{A}^2$ *is* the blow-up of $p = (0,0) \in \mathbb{A}^2$.

It follows directly from the definition that the following hold:

(1) The preimage $E = \pi^{-1}(p) = \{p\} \times \mathbb{P}^1$ is isomorphic to $\mathbb{P}^1$
(2) The map π restricts to an isomorphism $\mathrm{Bl}_0(\mathbb{A}^2)\backslash E \to \mathbb{P}^n\backslash\{p\}$ whose inverse is $(x,y) \mapsto \big((x,y),(x:y)\big)$.
(3) In particular, π is a birational morphism whose inverse is a birational map not being a morphism (it is not defined exactly at p).

Remark 2.2. *The name "blow-up" of p comes from the fact that the point is replaced with a line E, so we want to say that the point is "blown-up". We often say that π "blows down E onto p" or that "π blows up p".*

In fact, π should be considered as a blow-down but it is more π^{-1} which should be called blow-up. Because this latter map is not a morphism, we often prefer to deal with π; that is why we say that π is also a blow-up.

2.2. **The surface** $\mathrm{Bl}_0(\mathbb{A}^2)$ **is smooth and is covered by two affine planes.** The variety $\mathbb{A}^2 \times \mathbb{P}^1$ is covered by the two open sets isomorphic to $\mathbb{A}^3$ given by the image of $(x,y,z) \mapsto ((x,y),(z:1))$ and $(x,y,z) \mapsto ((x,y),(1:z))$. The trace of $\mathrm{Bl}_0(\mathbb{A}^2)$ on each of the two subsets corresponds to the surface of equation $x = yz$ and $y = xz$. Both are isomorphic to $\mathbb{A}^2$, with coordinates (y,z) and (x,z) respectively, so are smooth. This shows in particular that $\mathrm{Bl}_0(\mathbb{A}^2)$ is smooth.

2.3. **Local description of the blow-up.** Let us describe the blow-up $\pi\colon \mathrm{Bl}_0(\mathbb{A}^2) \to \mathbb{A}^2$ in affine charts. We choose two open subsets $U, V \subset \mathrm{Bl}_0(\mathbb{A}^2)$ where respectively $v \neq 0$ and $u \neq 0$. As we observed above, these two open subsets are isomorphic to $\mathbb{A}^2$ via the maps

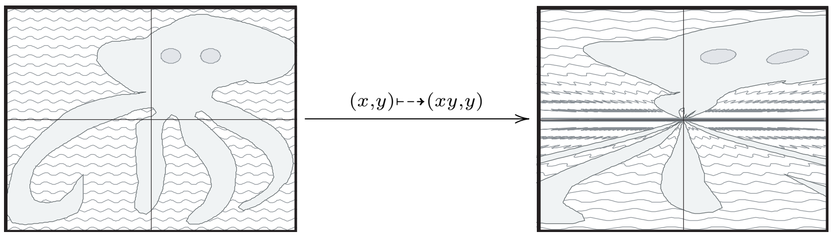
$$\begin{array}{rcl} \mathbb{A}^2 & \to & U \\ (y,u) & \mapsto & ((yu,y),(u:1)) \end{array} \qquad\qquad \begin{array}{rcl} \mathbb{A}^2 & \to & V \\ (x,v) & \mapsto & ((x,xv),(1:v)) \end{array}$$

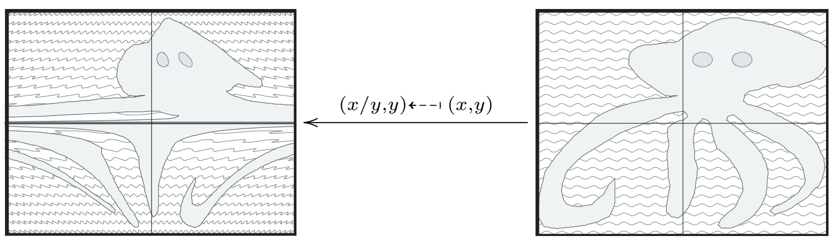
In local coordinates, we can thus describe the blow-up by

$$\begin{array}{rcl} \mathbb{A}^2 & \to & \mathbb{A}^2 \\ (y,u) & \mapsto & (yu,y) \end{array} \qquad\qquad \begin{array}{rcl} \mathbb{A}^2 & \to & \mathbb{A}^2 \\ (x,v) & \mapsto & (x,xv) \end{array}$$

These two maps have the same behaviour (they only differ by an exchange of coordinates).

The blow-up and its inverse are thus described in local coordinates by the following pictures:





2.4. **Blowing-up the point** $(1:0:\dots:0)$ **in** $\mathbb{P}^n$**.**

Definition 2.3. *We let* $Y \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ *be the following projective variety*

$$Y = \{\big((x_0:x_1:\dots:x_n),(y_1:\dots:y_n)\big) \in \mathbb{P}^n \times \mathbb{P}^{n-1} \mid x_i y_j = x_j y_i \textit{ for } 1 \leq i,j \leq n\},$$

and say that the map $\pi\colon Y \to \mathbb{P}^n$ *is* the blow-up of $p = (1:0:\dots:0) \in \mathbb{P}^n$.

It follows directly from the definition that the following hold:

(1) The preimage $E = \pi^{-1}(p) = \{p\} \times \mathbb{P}^{n-1}$ is isomorphic to $\mathbb{P}^{n-1}$
(2) The map π restricts to an isomorphism $Y \backslash E \to \mathbb{P}^n \backslash \{p\}$ whose inverse is $(x_0:\dots:x_n) \mapsto \big((x_0:\dots:x_n),(x_1:\dots:x_n)\big)$.
(3) In particular, π is a birational morphism whose inverse is a birational map not being a morphism (it is not defined exactly at p).

If $X \subset \mathbb{P}^n$ is a subset, we define by $\tilde{X} \subset Y$ the *strict transform* of X, which is equal to $\overline{\pi^{-1}(X \backslash \{p\})} \cap \pi^{-1}(X)$.

Example 2.4. *Let* $L \subset Y$ *be a line passing through* p*. It corresponds to the image of a morphism* $\mathbb{P}^1 \to \mathbb{P}^n$ *given by*

$$(u:v) \to (u:a_1 v:\dots:a_n v)$$

where $(a_1:\dots:a_n) \in \mathbb{P}^{n-1}$*. The set* $\pi^{-1}(L)$ *is the union of* E *with the image of the morphism* $\mathbb{P}^1 \to Y$ *given by*

$$(u:v) \to ((u:a_1 v:\dots:a_n v),(a_1:\dots:a_n)).$$

This latter is equal to $\tilde{L}$*, and is the preimage of* $(a_1:\dots:a_n) \in \mathbb{P}^{n-1}$ *under the projection* $Y \to \mathbb{P}^{n-1}$ *on the second factor.*

The strict transform of the lines passing through p *are thus disjoint in* Y *and are the fibres of the projection* $Y \to \mathbb{P}^{n-1}$.

Definition 2.5. *Let X be a quasi-projective variety and $p \in X$ be a point. We take an isomorphism of X with a closed subset of $\mathbb{P}^n$, which sends p onto $(1:0:\cdots:0)$ and let $\pi\colon Y \to \mathbb{P}^n$ be the blow-up of this point as in Definition 2.3. The* blow-up *of X at p is the morphism $\tilde{X} \to X$ given by the restriction of π to the strict transform of X under the map p.*

Note that this definition depends *a priori* of the embedding of X in $\mathbb{P}^n$. In fact, we can see that if $\pi\colon \tilde{X} \to X$ and $\pi'\colon \tilde{X}' \to X$ are two blow-ups of the same point, there exists an isomorphism $\varphi\colon \tilde{X} \to \tilde{X}'$ that makes the following diagram commutative:

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \tilde{X}' \arrow[dl, "\pi'"] \\ & X. & \end{tikzcd}$$

This can be done by hand or using a more intrinsic way of blowing-up like in [Har77]. It follows from the definition that the blow-up $\tau\colon \tilde{X} \to X$ of p restricts to an isomorphism $\tilde{X}\backslash\tau^{-1}\{p\} \to X\backslash\{p\}$ and hence is birational. If X is smooth, irreducible and projective, then $\tilde{X}$ has the same properties.

Taking an open subset $U \subset X$ containing p, one can study the blow-up of p in X by looking at the blow-up of p in U and then glueing this one with $X\backslash\{p\}$. When U is for example isomorphic to $\mathbb{A}^2$, it suffices to study the map $\mathrm{Bl}_0(\mathbb{A}^2) \to \mathbb{A}^2$ given in §2.3. Since we will blow-up only rational surfaces, this will be what we will have in mind when we blow-up points.

2.5. **Birational morphisms are sequences of blow-ups.** The following shows that any birational morphism between smooth projective surfaces is a sequence of blow-ups. Note that the result is false in dimension ≥ 3.

Proposition 2.6. [Har77, Chapter V, Corollary 5.4, page 411], [Bea96, Proposition II.11, page 16] *Let $\eta\colon X \to Y$ be a birational morphism between smooth projective surfaces. There exists a sequence of birational morphisms $\eta_1, \ldots, \eta_k$ between smooth projective surfaces such that $\eta_i\colon X_i \to X_{i-1}$ is the blow-up of a point $p_i \in X_{i-1}$ for $i = 1, \ldots, k$ and such that $X_k = Y$, $X_0 = X$ and $\eta = \eta_1 \circ \cdots \circ \eta_k$.*

Remark 2.7. *In the above description, the point $p_i \in X_{i-1}$ corresponds to a point of X_j with $j < i-1$ if $\eta_{j+1} \circ \cdots \circ \eta_i$ is an isomorphism in a neighbourhood of p_i. The blow-up being local, we can thus exchange the order in this case and replace the sequence $p_1, \ldots, p_n$ with the sequence $p_1, \ldots, p_i, p_j, p_{i+1}, \ldots, p_{j-1}, p_{j+1} \ldots, p_k$.*

2.6. **Bubble spaces.** For any smooth projective surface X, we denote by $\mathcal{B}(X)$ the *bubble space* of X, which is the set of points that belong to X as proper or infinitely near points. More precisely, $\mathcal{B}(X)$ can be viewed as the set of equivalence classes of triplets (y, Y, π), where y is a point of a smooth projective surface Y and $\pi\colon Y \to X$ is a birational morphism (which is a sequence of blow-ups) and where we say that (y, Y, π) is equivalent to (y', Y', π') if $(\pi')^{-1} \circ \pi\colon Y \dashrightarrow Y'$ is a birational map which restricts to an isomorphism on a open neighbourhood of y, sending y onto y'.

Points equivalent to (x, X, id) where id is the identity map, are *proper points* (or points of X), and other are *infinitely near points*.

A point $y \in \mathcal{B}(X)$ is *infinitely near to* $y' \in \mathcal{B}(X)$ if these points correspond respectively to triplets (y, Y, π) and (y', Y', π') such that $(\pi')^{-1} \circ \pi\colon Y \to Y'$ is a birational morphism which contracts a curve containing y onto the point $y' \in Y'$; if y belongs to the strict transform of the curve obtained by blow-up y', we say that y if *proximate* to y' (and write $y \succ y'$), and if moreover $(\pi')^{-1} \circ \pi$ is locally the blow-up of y', we say that the point y is in *the first neighbourhood of y'*.

A point y is in the *n-th neighbourhood of y'* if there exists a sequence of points $y_1, \ldots, y_n$, where $y_n = y$, where y_i is in the first neighbourhood of y_{i-1} for $i \geq 2$ and y_1 is in the first neighbourhood of y'.

Exercise 1. *Let X be a smooth projective surface. Show that any point $y \in \mathcal{B}(X)$ is either a proper point or is in the n-th neighbourhood of a unique point $x \in X$, for some $n \geq 1$. If $n > 1$, y is in the first neighbourhood of a unique point $y' \in \mathcal{B}(X)$ which is in the $(n-1)$-th neighbourhood of x.*

2.7. **Birational morphisms and subsets of bubble spaces.** Let X be a smooth projective surface and let $\pi\colon Y \to X$ be a birational morphism. We can associate to it a subset of $\mathcal{B}(X)$, which will be denote by $\mathcal{B}(\pi^{-1})$, which is the set of *base-points of* π^{-1}. Decomposing π as $\pi_1 \circ \cdots \circ \pi_k$, as in Proposition 2.6, where $\pi_i\colon X_i \to X_{i-1}$ is the blow-up of a point $p_i \in X_{i-1}$ for $i = 1, \ldots, k$, we say that $\mathcal{B}(\pi^{-1})$ is the union of the p_i (or more precisely of the triplets $(p_i, X_{i-1}, \pi_1 \circ \cdots \circ \pi_{i-1})$). We can see that this set does not depend on the factorisation. Moreover, if $\pi\colon Y \to X$ and $\pi'\colon Y' \to X$ are two birational morphisms such that $\mathcal{B}(\pi^{-1}) = \mathcal{B}(\pi'^{-1})$, there exists an isomorphism $\tau\colon Y \to Y'$ such that $\pi' \circ \tau = \pi$.

3. Intersection form on projective surfaces and relations with blow-ups

In this section, we assume the ground field $\mathbf{k}$ to be algebraically closed.

3.1. Intersection of two curves on a surface.

Proposition 3.1. [Har77, Chapter V, Theorem 1.1, page 357] *Let X be a smooth projective surface. There exists an unique bilinear symmetric form (called* intersection form*)*

$$\begin{array}{ccc} \mathrm{Div}(X)\times\mathrm{Div}(X) & \to & \mathbb{Z} \\ (C,D) & \mapsto & C\cdot D \end{array}$$

having the following properties

1. *If C and D are smooth curves meeting transversally, then $C\cdot D$ is equal to $\#(C\cap D)$, which is the number of points of $C\cap D$.*
2. *If C, C' are linearly equivalent, then $C\cdot D = C'\cdot D$.*

In particular, this yields an intersection form

$$\begin{array}{ccc} \mathrm{Pic}(X)\times\mathrm{Pic}(X) & \to & \mathbb{Z} \\ (C,D) & \mapsto & C\cdot D. \end{array}$$

Example 3.2. *Take $X=\mathbb{P}^2$, the intersection form is given by the following: if C, D are two curves of degree m and n, $C\cdot D = m\cdot n$. Indeed, recall that $\mathrm{Pic}(X)=\mathbb{Z}L$, where L is the divisor of a line. The fact that $L\cdot L=1$ follows from the fact that two distinct lines intersect into one point and are linearly equivalent. This shows that $C\cdot D=(mL)\cdot(nL)=mn$.*

Let us recall what is the *multiplicity of a curve at a point.* If $C\subset X$ is a curve in a smooth projective surface and $p\in X$ is a point, we can define the multiplicity $m_p(C)$ of C at p. Taking a local equation f of C, it can be defined as the integer k such that $f\in \mathfrak{m}^k\backslash\mathfrak{m}^{k+1}$, where $\mathfrak{m}$ is the maximal ideal of the ring of functions $\mathcal{O}_{p,X}$. If we can find an open neighbourhood U of p in X with $U\subset\mathbb{A}^2$, the point p can be choosed to $(0,0)$ in this affine neighbourhood, and the equation of C is a polynomial

$$\sum_{i=0}^{r} P_i(x,y)=0$$

where all P_i are homogeneous polynomials in two variables. The multiplicity $m_p(C)$ is equal to the lowest i such that P_i is not equal to 0. We always have the following:

1. $m_p(C)\geq 0$;
2. $m_p(C)=0 \Leftrightarrow p\notin C$;
3. $m_p(C)=1 \Leftrightarrow p$ is a smooth point of C;
4. $m_p(C)\geq 2 \Leftrightarrow p$ is a singular point of C.

If $\pi\colon Y\to X$ is a blow-up of a point $p\in X$, we have a map $\pi^*\colon \mathrm{Pic}(X)\to\mathrm{Pic}(Y)$, which sends a curve or a divisor on X onto its pull-back. Moreover, if $C\subset X$ is an irreducible curve, the strict transform of C on Y is defined to be the closure of $\pi^{-1}(C\backslash\{p\})$, and written often $\tilde{C}$. Observe that $\tilde{C}\subset Y$ is again an irreducible curve.

Lemma 3.3. *Let X be a smooth surface, let $p\in X$ be a point, let $\pi\colon Y\to X$ be the blow-up of p, and let $C\subset X$ be an irreducible curve. In $\mathrm{Pic}(Y)$, we have*

$$\pi^*(C)=\tilde{C}+m_p(C)E,$$

where $\tilde{C}$ is the strict transform of C and where $E=\pi^{-1}(p)$. In particular, $\tilde{C}^2=C^2-m_p(C)^2$.

Proof. Take local coordinates x, y at p and write $k=m_p(C)$. The curve C is given by

$$p_k(x,y)+p_{k+1}(x,y)+\cdots+p_r(x,y),$$

where p_i are homogeneous polynomials of degree i. The blow-up can be viewed as $(u,v)\mapsto(uv,v)$. The pull-back of C becomes

$$v^k(p_k(u,1)+vp_{k+1}(u,1)+\cdots+v^{r-k}p_r(x,y)),$$

so it decomposes into k times the exceptional divisor E (here $v=0$), and the strict transform. □

Proposition 3.4. [Har77, Chapter V, Proposition 3.2, page 386], [Bea96, Proposition II.3, page 12]
Let X be a smooth surface, let $x \in X$ be a point, and let $\pi\colon Y \to X$ be the blow-up of x. We denote by $E \subset Y$ the curve $\pi^{-1}(p)$, which is isomorphic to $\mathbb{P}^1$.

$$\mathrm{Pic}(Y) = \pi^*(\mathrm{Pic}(X)) \oplus \mathbb{Z}E$$

The intersection form on Y is induced by the intersection form on X via the following formulas:

$$\begin{array}{rcl} \pi^*(C)\cdot\pi^*(D) & = & C\cdot D \textit{ for any } C,D \in \mathrm{Pic}(X),\\ \pi^*(C)\cdot E & = & 0 \textit{ for any } C \in \mathrm{Pic}(X)\\ E\cdot E & = & -1 \end{array}$$

In particular, the curve obtained by blowing-up a point in a smooth surface is isomorphic to $\mathbb{P}^1$ and has self-intersection -1. We will say that it is a (-1)-*curve*. In fact, we have the following converse statement, due to G. Castelnuovo:

Proposition 3.5. [Har77, V, Theorem 5.7, page 414] *Let $E \subset X$ be a curve in a smooth projective surface. The following are equivalent:*
(i) There exists a morphism $\pi\colon X \to Y$, where Y is a smooth projective surface, which contracts E onto a point p and which is an isomorphism outside of E (π is the blow-up of $p \in Y$).
(ii) $E \cong \mathbb{P}^1$ and $E^2 = -1$ (i.e. E is a (-1)-curve).

We will prove this Proposition in the case where X is rational (Corollary 7.8).

If S is a smooth projective surface, $C \subset S$ is an irreducible curve and $p \in \mathcal{B}(S)$, we can define the multiplicity of C at p which is a non-negative integer $m_p(C)$. The element p corresponds to a triplet (p, X, π), where $\pi\colon X \to S$ is a birational morphism. Then $m_p(C)$ is the multiplicity of the strict transform $\tilde{C} \subset X$ of C.

Proposition 3.6. *Let S be a smooth projective surface and let $C, D \subset S$ be two distinct irreducible curves. We have*

$$C \cdot D = \sum_{p \in \mathcal{B}(S)} m_p(C) \cdot m_p(D).$$

In particular, only finitely many points $p \in \mathcal{B}(S)$ satisfy $m_p(C) \cdot m_p(D) > 0$.

Proof. Note that $m_p(C)$ is positive only for points that belong to C, as proper or infinitely near points, and the same for C.

If C and D are disjoint, then $C \cdot D = 0$ and the result is obvious.

Otherwise, we let $p \in S$ be a point of the intersection, and let $\pi\colon S' \to S$ be the blow-up of p. By Lemma 3.3, the strict transform $\tilde{C}, \tilde{D} \subset S'$ of C, D are linearly equivalent to $\pi^*(C) - m_p(C)E$ and $\pi^*(D) - m_p(D)E$. In particular, $\tilde{C} \cdot \tilde{D} = C \cdot D - m_p(C) \cdot m_p(D)$. The result follows thus by induction on the non-negative integer $C \cdot D$. □

4. Del Pezzo surfaces

Recall that the canonical divisor on an algebraic variety X is given by the divisor of a differential form. Choosing two distinct differential forms give linearly equivalent divisors. In particular, we have an unique element $K_X \in \mathrm{Pic}(X)$ that we call the *canonical divisor*.

Proposition 4.1 (Ramification formula)**.** *Let X be a smooth surface, let $p \in X$ be a point, let $\pi\colon Y \to X$ be the blow-up of p, and let $E = \pi^{-1}(p)$. Then*

$$K_Y = \pi^*(K_X) + E.$$

Proof. Take local coordinates u, v at p so that this point corresponds to $u = v = 0$. Let ω be a differential form on X, which locally corresponds to $du \wedge dv$. It has no pole or zero at p (but certainly outside of the local neighbourhood). The divisor of ω, equal to K_X, corresponds thus locally to the trivial divisor.

The blow-up can be viewed locally as $(u, v) \mapsto (uv, v)$, and the differential form $\pi^*(\omega)$ becomes $d(uv) \wedge dv = v \cdot du \wedge dv$. In these coordinates, v is the equation of the divisor E and $\eta^*(K_X)$ is the trivial divisor.

The canonical divisor of Y is the divisor of the differential form $\pi^*(\omega)$, which is equal thus equal to $\pi^*(K_X)+E$: this equation was computed locally above p, and is clear outside because π restricts to an isomorphism. □

Proposition 4.2 (Adjunction formula)**.** [Har77, Chapter V, Proposition 1.5, page 361],[Bea96, I.15, page 8]
Let $C \subset S$ be an irreducible curve on a surface S. We have $C \cdot (C + K_S) = -2 + 2 \cdot g(C)$, where $g(C) = H^1(C, \mathcal{O}_C)$ is the arithmetical genus of C.

Remark 4.3. *In fact, if $X \subset Y$ is an hypersurface in a smooth projective variety, we have $K_X = (K_Y + X)_{|X}$. In the case where X is a curve, the canonical divisor of X has degree $-2 + 2 \cdot g(C)$ where $g(C)$ is the arithmetical genus.*

Let us recall that a divisor D on an algebraic projective variety X is *ample* if nD is *very ample* for some $n \geq 1$, which means that the linear system $|nD|$ induces a closed embedding of X into a projective space. We also have the following more algebraic criterion:

Proposition 4.4 (Nakai-Moishezon Criterion)**.** [Har77, Chapter V, Theorem 1.10, page 365] *A divisor D on a smooth projective surface X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for any (irreducible) curve $C \subset X$.*

Definition 4.5. *A* del Pezzo surface *is a smooth projective surface S such that its anti-canonical divisor $-K_S$ is ample.*

Proposition 4.6 (Descriptions of del Pezzo surfaces)**.** *Let $\pi \colon S \to \mathbb{P}^2$ be a birational morphism, where S is a smooth projective surface. The following conditions are equivalent:*

1. *$-K_S$ is ample (i.e. S is a del Pezzo surface);*
2. *the morphism π is the blow-up of $0 \leq r \leq 8$ points of $\mathbb{P}^2$ (no infinitely near point) such that no 3 are collinear, no 6 are on the same conic, no 8 lie on a cubic having a double point at one of them;*
3. *$K_S^2 \geq 1$ and any irreducible curve of S has self-intersection ≥ -1;*
4. *$C \cdot (-K_S) > 0$ for any effective divisor C.*

Remark 4.7. *In general a divisor is ample if and only if it intersects positively the adherence of the cone of effective divisors.*

In our case (when the surface is rational and the divisor is the anti-canonical divisor), the equivalence of assertions 1 and 4 shows that the criterion is true, even if we omit the adherence in the statement, or when we omit the fact that $C^2 > 0$.

Proof. The implication $(1 \Rightarrow 4)$ is the easy sense of Kleiman's ampleness criterion: as $-K_S$ is ample, $-mK_S$ is very ample, for some integer $m > 0$, in which case $m \cdot (-K_S) \cdot C$ is the degree of C in the corresponding embedding, which must be positive.

If one point p blown-up by π is infinitely near to a point q (which is thus blown-up), the strict transform of the curve obtained by the blow-up of q is a smooth curve $C \subset S$, isomorphic to $\mathbb{P}^1$ which has self-intersection $C^2 \leq -2$ on S. In particular Adjunction formula (Proposition 4.2) yields $C \cdot K_S \geq 0$. Hence, all assertions are false in this case. We can thus assume that π is the blow-up of r points $p_1, \ldots, p_r \in \mathbb{P}^2$. And $\mathrm{Pic}(S)$ is generated by L, the pull-back of a line of $\mathbb{P}^2$, and $E_1, \ldots, E_r$, the divisors contracted on $p_1, \ldots, p_r$, which have self-intersection -1. Moreover $L \cdot E_i = E_i \cdot E_j = 0$ for $i \neq j$.

$(4 \Rightarrow 2, 3)$ We first prove that assertion 4 implies that any irreducible curve of S has self-intersection ≥ -1. Suppose that some irreducible curve C of S has self-intersection ≤ -2. The adjunction formula (Proposition 4.2) gives $C \cdot (C + K_S) = -2 + 2 \cdot g(C) \geq -2$, whence $C \cdot (-K_S) \leq 2 + C^2 \leq 0$, which contradicts assertion 4.

If three are collinear, say p_1, p_2, p_3, the strict transform of the line passing through the points is equivalent to $L - \sum_i E_i$ where the sum has at least 3 terms, so has self-intersection ≤ -2. The same holds if six lie on a conic: $(2L - \sum_{i=1}^{6} E_i)^2 = -2$ or if 8 lies a cubic which is singular at one of the points: $(3L - 2E_1 - \sum_{i=2}^{8} E_i)^2 = -2$.

Furthermore, if the number of blown-up points is at least 9, there exists a cubic passing through 9 of the blown-up points, which is irreducible as the points are in general position. The strict transform of this curve intersects the anti-canonical divisor of S non positively. The number of blown-up points is then at most 8, and so $K_S^2 \geq 1$. We get assertions 2 and 3.

$(3 \Rightarrow 2)$ The fact that $(K_S)^2 \leq 1$ implies that the number of points is at most 8, and the fact that $C^2 \geq -1$ for any irreducible curve C implies that the points have the good configuration (no 3 collinear, no 6 on the same conic,...)

$(2 \Rightarrow 1)$ Applying Nakai-Moishezon Criterion (Proposition 4.4), we only need to show that $-K_X \cdot C > 0$ for any irreducible curve $C \subset X$ (the fact that $(-K_X)^2 > 0$ is given by the fact that at most 8 points are blown-up).

Let $C \subset X$ be an irreducible curve. If $C = E_i$ for some i, then $C \cdot (-K_X) = 1$. Otherwise, C is linearly equivalent to $dL - \sum_{i=1}^{r} a_i E_i$ with $d > 0$. Moreover, $a_i = C \cdot E_i \geq 0$ for $i = 1, \ldots, r$.

If $d = 1$, the curve is the strict transform of a line, which can only pass through 2 points by (2); this implies that $\sum_{i=1}^{r} a_i \leq 2$, whence $C \cdot (-K_X) \geq 1$. The cases $d = 2$ and $d = 3$ are similar, we have $\sum_{i=1}^{r} a_i \leq 5$ (respectively ≤ 8), so $C \cdot (-K_X) \geq 1$.

It remains to study the cases where $d \geq 4$. Since $r \leq 9$, $-K_X$ is effective (it is linearly equivalent to the strict transform of a cubic passing through the r points), and because $d \geq 4$ and C is irreducible, C has no common component with $-K_X$ so $C \cdot (-K_X) \geq 0$. It remains to show that $C \cdot K_X = 0$ is not possible. Otherwise, by adjunction formula we would have $C^2 = -2 + 2g$ for some integer $g \geq 0$ (which is the arithmetic genus of C). We get thus

$$\sum_{i=1}^{r} (a_i)^2 = d^2 + 2 - 2g, \qquad \sum_{i=1}^{r} a_i = 3d.$$

Since $(\sum_{i=1}^{r} a_i)^2 \leq r \sum_{i=1}^{r} (a_i)^2 \leq 8 \sum_{i=1}^{r} (a_i)^2$, we get $9d^2 \leq 8(d^2 + 2 - 2g)$ which implies that $d^2 \leq 8(2 - 2g)$. Since $d \geq 4$, the only possibility should be $d = 4$, $g = 0$. But the equality implies that $r = 8$ and all a_i are equal, which contradicts the equality $\sum_{i=1}^{r} a_i = 3d$. □

5. Resolution of indeterminacies

Proposition 5.1. [Bea96, Theorem II.7, page 14] *Let X be a projective smooth surface and let $\varphi \colon X \dashrightarrow Y$ be a rational map, where Y is a projective variety. There exists a birational morphism $\eta \colon Z \to X$, which is a finite sequence of blow-ups (so Z is a smooth projective surface), and a morphism $\pi \colon Z \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} & Z \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ X \arrow[rr, dashed, "\varphi"'] & & Y. \end{tikzcd}$$

Corollary 5.2. *Let $\varphi \colon X \dashrightarrow Y$ be a birational map between projective smooth surfaces. There exists a projective smooth surface Z and birational morphisms $\eta_X \colon Z \to X$, $\eta_Y \colon Z \to Y$, which are finite sequences of blow-ups such that the following diagram commutes:*

$$\begin{tikzcd} & Z \arrow[dl, "\eta_X"'] \arrow[dr, "\eta_Y"] & \\ X \arrow[rr, dashed, "\varphi"'] & & Y. \end{tikzcd}$$

Proof. Follows from Propositions 2.6 and 5.1. □

6. Hirzebruch surfaces

Definition 6.1. *We define the n-th Hirzebruch surface $\mathbb{F}_n$ to be*

$$\mathbb{F}_n = \{((x:y:z),(u:v)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid yv^n = zu^n\},$$

and let $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$ be the map given by the projection on the second factor.

Remark 6.2. *Note that $\mathbb{F}_0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, via $((x:y:z),(u:v)) \mapsto ((x:y),(u:v))$.*

Lemma 6.3. *The following maps*

$$\begin{array}{rccc} \rho_{1,n} \colon & \mathbb{P}^1 \times \mathbb{A}^1 & \to & \mathbb{F}_n \\ & ((\alpha:\beta),t) & \to & ((\alpha:\beta:\beta t^n),(1:t)) \end{array}$$

and

$$\begin{array}{rccc} \rho_{2,n} \colon & \mathbb{P}^1 \times \mathbb{A}^1 & \to & \mathbb{F}_n \\ & ((\alpha:\beta),t) & \to & ((\alpha:\beta t^n:\beta),(t:1)) \end{array}$$

are open embeddings. Writing $U_{1,n}, U_{2,n} \subset \mathbb{F}_n$ the images of the two morphisms, we have $U_{1,n} \cup U_{2,n} = \mathbb{F}_n$, so this yields an open covering of $\mathbb{F}_n$.

In particular, $\mathbb{F}_n$ is a smooth projective surface, and $\pi \colon \mathbb{F}_n \to \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle.

Proof. We let the proof in exercise. □

Remark 6.4. *With the notation above, we can compute that $\rho_{1,n} \circ (\rho_{2,n})^{-1}$ is the birational involution of $\mathbb{P}^1 \times \mathbb{A}^1$ given by $((\alpha:\beta),t) \dashrightarrow ((\alpha:\beta t^n),\frac{1}{t})$. We can thus view $\mathbb{F}_n$ as the union of two copies of $\mathbb{P}^1 \times \mathbb{A}^1$ glued along $\mathbb{P}^1 \times \mathbf{k}^*$ via this map.*

In fact, we can see that $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ (see [Bea96], *Proposition IV.1 for more details on these surfaces).*

Lemma 6.5. *For any integer $n \geq 0$, the map $\rho_{1,n+1} \circ (\rho_{1,n})^{-1}$ yields a birational map $\varphi_n \colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$ whose resolution of indeterminacies is*

$$\begin{tikzcd} & Y \arrow[dl, "\eta_n"'] \arrow[dr, "\eta_{n+1}"] & \\ \mathbb{F}_n \arrow[rr, dashed, "\varphi_n"] & & \mathbb{F}_{n+1} \end{tikzcd}$$

where η_n, η_{n+1} are two birational morphisms having the following properties:

1. *η_n is the blow-up of $p = \rho_{2,n}((1:0),0) \in \mathbb{F}_n$;*
2. *η_{n+1} is the blow-up of $q = \rho_{2,n+1}((0:1),0) \in \mathbb{F}_{n+1}$;*
3. *the curve contracted by η_{n+1} on q is the strict transform by η_n of the fibre of π_n passing through p;*
4. *the curve contracted by η_n on p is the strict transform by η_{n+1} of the fibre of π_{n+1} passing through q.*

Proof. For $k = n, n+1$, Lemma 6.3 shows that $\rho_{1,k}$ induces an isomorphism between $\mathbb{P}^1 \times \mathbb{A}^1$ and $U_{1,k}$, which is an open subset of $\mathbb{F}_k$.

In particular, $\varphi_n = \rho_{1,n+1} \circ (\rho_{1,n})^{-1}$ restricts to an isomorphism between $U_{1,n}$ and $U_{1,n+1}$, and is therefore a birational map from $\mathbb{F}_n$ and $\mathbb{F}_{n+1}$.

Recalling that the map $U_{1,k} \dashrightarrow U_{2,k}$ is given by $((\alpha:\beta),t) \dashrightarrow ((\alpha:\beta t^k),\frac{1}{t})$ (with coordinates given by $\mathbb{P}^1 \times \mathbb{A}^1$), the map φ_n restricts to a birational map from $U_{1,n}$ to $U_{1,n+1}$ which is the identity and thus to a birational map $U_{2,n} \dashrightarrow U_{2,n+1}$ given by

$$((\alpha:\beta),t) \dashrightarrow ((\alpha t:\beta),t).$$

We define $Y' = \{((a:b),(c:d),t) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{A}^1 \mid tad = bc\}$, and let $\eta_n' \colon Y' \to U_{2,n}$ be given by $((a:b),(c:d),t) \mapsto ((a:b),t)$ and $\eta_{n+1}' \colon Y' \to U_{2,n+1}$ be given by $((a:b),(c:d),t) \mapsto ((c:d),t)$. We observe that η_n' is the blow-up of $p = ((1:0),0) \in U_{2,n}$, that η_{n+1}' is the blow-up of $q = ((0:1),0) \in U_{2,n+1}$ and that the restriction of φ_n is given by $(\eta_0) \circ (\pi_0)^{-1}$. Moreover, the curve contracted by η_n' is sent by η_{n+1}' onto the curve $t = 0$, which corresponds to the fibre of π_{n+1} passing through q. Similarly, the curve contracted by η_{n+1}' is sent by η_n' onto the curve $t = 0$, which corresponds to the fibre of π_n passing through p. We extend the morphism η_n' and η_{n+1}' (by taking the identity outside of $U_{2,n}$ and $U_{2,n+1}$) and obtain a morphism $\eta_n \colon Y \to \mathbb{F}_n$ which is the blow-up of p and a morphism $\eta_{n+1} \colon Y \to \mathbb{F}_{n+1}$, which is the blow-up of q. □

Definition 6.6. *We denote by $E_n \subset \mathbb{F}_n$ the curve given by $E_n = \{((1:0:0),(u:v)) \mid (u:v) \in \mathbb{P}^1\}$.*

The curve here is a section of the line bundle $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$, which is a special one for $n \geq 1$, as we will below (Lemma 6.9).

Lemma 6.7. *For any $n \geq 0$, the Picard group $\mathrm{Pic}(\mathbb{F}_n)$ is given by:*

1. *$\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}f \oplus \mathbb{Z}E_n$, where f is the divisor of a fibre of π_n;*
2. *the intersection form on $\mathrm{Pic}(\mathbb{F}_n)$ is given by*

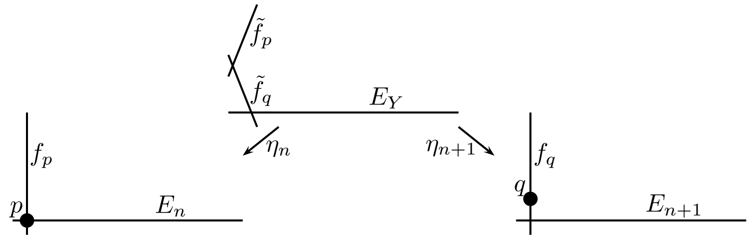
$$f^2 = 0, (E_n)^2 = -n, E_n \cdot f = 1;$$

3. *moreover, the canonical divisor $K_{\mathbb{F}_n} \in \mathrm{Pic}(\mathbb{F}_n)$ is equal to $-2E_n - (2+n)f$.*

Proof. Note that $E_0 \cdot f = 1$ is given by the fact that E_n is a section, and $f^2 = 0$ is given by intersecting two distinct fibres, both equivalent to f. Assuming the first two assertions, let us see that the canonical divisor $K_{\mathbb{F}_n} \in \mathrm{Pic}(\mathbb{F}_n)$ is equal to $-2E_n - (2+n)f$. Indeed, $K_{\mathbb{F}_n} = aE_n + bf$, and adjunction formula (Proposition 4.2) for E_n and f implies that $a = E_n \cdot K_{\mathbb{F}_n} = -2 - n$ and $b = f \cdot K_{\mathbb{F}_n} = 0$.

We prove now the result (Assertion 1 and the fact that $(E_n)^2 = -n$) by induction on n. For $n = 0$, $\mathbb{F}_n$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and the fibres of the two projections are E_0 and f. This implies that $\mathrm{Pic}(\mathbb{F}_n)$ is generated by E_0 and f and that $(E_0)^2 = 0$.

We then use the map $\varphi_n \colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$ given in Lemma 6.5, which is given by the blow-up $\eta_n \colon Y \to \mathbb{F}_n$ of $p \in E_n$, followed by the contraction $\eta_{n+1} \colon Y \to \mathbb{F}_{n+1}$ of the strict transform $\tilde{f}_p \subset Y$ of the fibre f_p of π_p passing through p onto the point $q \in \mathbb{F}_{n+1}$. Note that E_n is sent by φ_n onto E_{n+1}; we denote by $E_Y \subset Y$ the strict transform of these curves. We also denote by $f_q \subset \mathbb{F}_{n+1}$ the fibre of π_{n+1} passing through q and by $\tilde{f}_q \subset Y$ its strict transform, which is contracted by η_n onto p. The situation is described by the following diagram:



Because p belongs to E_n, the strict transform E_Y of E_n on Y has self-intersection $-n-1$ and its image in $\mathbb{F}_{n+1}$ has the same self-intersection because E_Y does not intersect the curve $\tilde{f}_p$, contracted by η_{n+1}. This yields $(E_{n+1})^2 = -(n+1)$. To prove the result, we only need to show that $\mathrm{Pic}(\mathbb{F}_{n+1})$ is generated by E_{n+1} and f_q (the fact that we have a direct sum is given by the intersections).

Because η_{n+1} is the contraction of $\tilde{f}_p$, we have $\mathrm{Pic}(Y) = (\eta_{n+1})^*(\mathrm{Pic}(\mathbb{F}_{n+1})) \oplus \mathbb{Z}\tilde{f}_p$ (Proposition 3.4), so we need to see that $(\eta_{n+1})^*(f_q)$, $(\eta_{n+1})^*(E_{n+1})$ and $\tilde{f}_p$ generate $\mathrm{Pic}(Y)$. These are respectively equal to $\tilde{f}_p + \tilde{f}_q$, E_Y and $\tilde{f}_p$. We use the fact that η_n is the contraction of $\tilde{f}_q$ to obtain $\mathrm{Pic}(Y) = (\eta_n)^*(\mathrm{Pic}(\mathbb{F}_n)) \oplus \mathbb{Z}\tilde{f}_q$, and get (by induction hypothesis) that $\mathrm{Pic}(Y)$ is generated by $(\eta_{n+1})^*(f_p)$, $(\eta_{n+1})^*(E_n)$ and $\tilde{f}_q$, being respectively equal to $\tilde{f}_p + \tilde{f}_q$, E_Y and $\tilde{f}_q$. This achieves to give the result. □

Remark 6.8. *As we already observed, $\mathbb{F}_0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, via $((x:y:z),(u:v)) \mapsto ((x:y),(u:v))$. It has thus two line bundle structures given by the two projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.*

Writing $\mathbb{F}_0 = \{((x:y:z),(u:v)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid y = z\}$, as in Definition 6.1, the two maps correspond to π_0, introduced before, given by $((x:y:z),(u:v)) \mapsto (u:v)$ and π_0' given by $((x:y:z),(u:v)) \to (x:y)$.

A fibre of π_0 (respectively π_0') is a section of π_0' (respectively π_0), so the morphisms have distinct fibres. The following result (Assertion 3) implies that this is the only case where such phenomenon occurs.

Lemma 6.9. *The Hirzebruch surfaces $\mathbb{F}_n$ have the following properties:*

(1) *Any curve of $\mathbb{F}_n$ is linearly equivalent to $aE_n + bf$ for some $a, b \geq 0$. (Here $f \in \mathrm{Pic}(\mathbb{F}_n)$ denotes as before a fibre of $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$)*
(2) *Any section s of $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$ which is different from E_n is linearly equivalent to $E_n + (n+k)f$ for some integer $k \geq 0$. In particular, $s^2 = n + k$ and $s \cdot E_n = k$.*
(3) *If $\pi \colon \mathbb{F}_n \to \mathbb{P}^1$ is a morphism having a general fibre isomorphic to $\mathbb{P}^1$, there exists $\alpha \in \mathrm{Aut}(\mathbb{P}^1)$ such that either $\pi = \alpha \circ \pi_n$ or $n = 0$ and $\pi = \alpha \circ \pi_0'$, where $\pi_0' \colon \mathbb{F}_0 \to \mathbb{P}^1$ is the morphism given in Remark 6.8.*
(4) *If $\pi \colon \mathbb{F}_n \to \mathbb{P}^1$ is a morphism having a general fibre isomorphic to $\mathbb{P}^1$, the minimum of the self-intersections of the sections of π is equal to $-n$.*
(5) *The surfaces $\mathbb{F}_m$ and $\mathbb{F}_n$ are isomorphic if and only if $m = n$.*

Proof. Let $C \subset \mathbb{F}_n$ be an irreducible curve, that is linearly equivalent to $aE_n + bf$ for some $a, b \in \mathbb{Z}$ (Lemma 6.7). If C is a fibre of π_n or $C = E_n$ we have $\{a, b\} = \{0, 1\}$, hence $a, b \geq 0$. If $C \neq E_n$, we have $0 \leq C \cdot E_n = -na + b$ and $0 \leq C \cdot f = a$, hence $b \geq na \geq a \geq 0$. This proves (1). If C is moreover a section (distinct from E_n), we have $1 = C \cdot f = a$ and $k = b - n = C \cdot E_n \geq 0$. The section is thus linearly equivalent to $s = E_n + (k+n)f$ and $s^2 = n + 2k$, $s \cdot E_n = k$. This achieves to prove (2).

(3) Let $\pi \colon \mathbb{F}_n \to \mathbb{P}^1$ be a morphism having a general fibre f' isomorphic to $\mathbb{P}^1$. If π and π_n have the same fibres, we have $\pi = \alpha \circ \pi_n$ for some $\alpha \in \mathrm{Aut}(\mathbb{P}^1)$. We can thus assume that f' is not equal to a fibre of π_n. In particular, we have $f \cdot f' > 0$, so $f' \sim aE_n + bf$ with $a > 0, b \geq 0$, which yields $0 = (f')^2 = -na^2 + 2ab = a(2b - na)$, hence $2b = na$. Intersecting E_n with f' we get $b - an = -b$, so $b = 0$ and $f' = E_n$. Since $(f')^2 = 0$ this yields $n = 0$, and implies that the fibres of π are equal to the fibres of π_0'. Hence, $\pi = \alpha \circ \pi_0'$ for some $\alpha \in \mathrm{Aut}(\mathbb{P}^1)$.

$(4) - (5)$ By (2), the minimum of the self-intersections of the sections of π_n is equal to $-n$. Because π_0' is equal to $\pi_0\tau$, where τ is the automorphism of $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ that exchanges the two factors, the minimum of the self-intersections of the sections of π_0' is also equal to 0. Using (3), this achieves to prove (4), which implies (5). □

6.1. **Automorphisms of Hirzebruch surfaces.** Note that $\mathrm{GL}(2, \mathbf{k})$ acts linearly on $\mathbf{k}^2$, hence on the set of polynomials on two variables, which is a $\mathbf{k}$-vector space. It preserves the subvector space of homogeneous polynomials of degree n. The corresponding action yields an algebraic group $\mathbf{k}^{n+1} \rtimes \mathrm{GL}(2, \mathbf{k})$, whose elements are

$$\left(a_0 Y^n + a_1 X Y^{n-1} + \ldots + a_n X^n, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in \mathbf{k}^{n+1} \rtimes \mathrm{GL}(2, \mathbf{k}).$$

We observe that this group acts on the Hirzebruch surface $\mathbb{F}_n = \{((x:y:z),(u:v)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid yv^n = zu^n\}$. The $\left(a_0Y^n + a_1XY^{n-1} + ... + a_nX^n, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ sends $((x:y:z),(u:v)) \in \mathbb{F}_n$ onto

$$\begin{cases} ((xu^n + y(a_0v^n + a_1uv^{n-1} + \cdots + a_nu^n) : y(au+bv)^n : y(cu+dv)^n), (au+bv : cu+dv)) & \text{if } u \neq 0, \\ ((xv^n + z(a_0v^n + a_1uv^{n-1} + \cdots + a_nu^n) : z(au+bv)^n : z(cu+dv)^n), (au+bv : cu+dv)) & \text{if } v \neq 0. \end{cases}$$

The kernel of this action is easy to compute. It is the finite cyclic subgroup of $\mathrm{GL}(2,\mathbf{k})$ consisting of diagonal matrices of the form $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$, where $\mu^n = 1$. In particular, $\mathbf{k}^n$ is an algebraic subgroup of $\mathrm{Aut}(\mathbb{F}_n)$; $(a_0, \ldots, a_n)$ sends $((x:y:z),(u:v)) \in \mathbb{F}_n$ onto

$$\begin{cases} ((xu^n + y(a_0v^n + a_1uv^{n-1} + \cdots + a_nu^n) : yu^n : yv^n), (u:v)) & \text{if } u \neq 0, \\ ((xv^n + z(a_0v^n + a_1uv^{n-1} + \cdots + a_nu^n) : zu^n : zv^n), (u:v)) & \text{if } v \neq 0. \end{cases}$$

Denote by f_0 the fibre of $(1:0)$, which is the set of points where $v = 0$. The open set $\mathbb{F}_n \backslash (E_n \cup f_0)$ is naturally invariant by the action of $\mathbf{k}^n$, and is isomorphic to $\mathbb{A}^2$, via the map $(x,y) \mapsto ((x:y^n:1),(y:1))$. The action of $\mathbf{k}^n$ on this open subset corresponds thus to

$$(x,y) \mapsto (x + a_0 + a_1y + a_2y^2 + \cdots + a_ny^n, y).$$

One has the following result:

Lemma 6.10. *For $n = 0$, the group $\mathrm{Aut}(\mathbb{F}_n)$ acts transitively on $\mathbb{F}_n$.*
For $n \geq 1$, the action of $\mathrm{Aut}(\mathbb{F}_n)$ on $\mathbb{F}_n$ has two orbits: E_n and $\mathbb{F}_n \backslash E_n$.

Proof. The case $n = 0$ follows from the fact that $\mathrm{PGL}(2,\mathbf{k}) \times \mathrm{PGL}(2,\mathbf{k})$ acts transitively on $\mathbb{P}^1 \times \mathbb{P}^1$.

If $n \geq 1$, then E_n is the unique section of self-intersection $-n$ (Lemma 6.9), hence is invariant by $\mathrm{Aut}(\mathbb{F}_n)$. Its complement $\mathbb{F}_n \backslash E_n$ is thus also invariant. The group $\mathrm{GL}(2,\mathbf{k})$ acts transitively on the set of fibres and $\mathbf{k}^{n+1}$ acts transitively on $f \backslash E_n$, where f is any fibre. This implies that$\mathbf{k}^{n+1} \rtimes \mathrm{GL}(2,\mathbf{k})$ acts transitively on the two sets E_n and $\mathbb{F}_n \backslash E_n$. □

Exercise 2. *For $n \geq 1$, prove that $\mathrm{Aut}(\mathbb{F}_n)$ is the quotient of $\mathbf{k}^{n+1} \rtimes \mathrm{GL}(2,\mathbf{k})$ by the subgroup of $\mathrm{GL}(2,\mathbf{k})$ consisting of diagonal matrices of the form $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$, where $\mu^n = 1$. (One way to do is to use induction on n: the case of $\mathbb{F}_1$ comes from $\mathbb{P}^2$ and the automorphisms of $\mathbb{F}_{n+1}$ that fix the base-point of $(\varphi_n)^{-1}$ correspond to automorphisms of $\mathbb{F}_n$ fixing the base-point of φ_n).*

7. Elementary links and the decomposition theorem

In Section 6, we defined the Hirzebruch surfaces $\mathbb{F}_n$, for $n \geq 0$, as

$$\mathbb{F}_n = \{((x:y:z),(u:v)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid yv^n = zu^n\},$$

and defined $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$ to be the map given by the projection on the second factor (Definition 6.1).

By construction, there is a morphism $\mathbb{F}_n \to \mathbb{P}^2$, given by the first projection. It sends the curve $E_n = \{((1:0:0),(u:v)) \mid (u:v) \in \mathbb{P}^1\}$ onto the point $(1:0:0)$. For $n = 1$, this morphism is the blow-up of $(1:0:0) \in \mathbb{P}^2$, which is the contraction of the curve $E_1 \subset \mathbb{F}_1$ of self-intersection -1. Composing the morphism $\tau \colon \mathbb{F}_1 \to \mathbb{P}^2$, with an automorphism of $\mathbb{P}^2$, we obtain the blow-up of any point of $\mathbb{P}^2$. In particular, if $X \to \mathbb{P}^2$ is the blow-up of one point, then X is isomorphic to $\mathbb{F}_1$.

Definition 7.1. *If $\tau \colon \mathbb{F}_1 \to \mathbb{P}^2$ is the blow-up of a point, the birational map $\tau^{-1} \colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$ will be called a* link of type I.

After automorphisms, the map τ^{-1} above is the simplest case of a birational map starting from $\mathbb{P}^2$ and will be the first block in a decomposition of a birational map. By Corollary 5.2, any birational map from $\mathbb{P}^2$ to a smooth projective surface decomposes into a sequence of blow-ups followed by a sequence of blow-downs. However, this decomposition factors through surfaces that are complicate to understand. As we will see, it is possible to decompose the maps through only simple maps between "simple surfaces", i.e. $\mathbb{P}^2$ and the Hirzebruch surfaces.

Starting from $\mathbb{P}^2$, we can "go" to $\mathbb{F}_1$ via a link of type I. A next possible step is to go to $\mathbb{F}_0$ or $\mathbb{F}_2$ via a simple map as in Lemma 6.5. More generally, the links of type II that we will consider are the following:

Definition 7.2. *A birational map $\varphi\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_m$ is* a link of type II *if there exists a commutative diagram*

$$\begin{tikzcd} & Y \arrow[dl, "\eta_n"'] \arrow[dr, "\eta_m"] & \\ \mathbb{F}_n \arrow[rr, dashed, "\varphi"] \arrow[d, "\pi_n"'] & & \mathbb{F}_m \arrow[d, "\pi_n"] \\ \mathbb{P}^1 \arrow[rr, "\alpha"] & & \mathbb{P}^1, \end{tikzcd}$$

where $\alpha \in \mathrm{Aut}(\mathbb{P}^1)$ and η_n, η_m are blow-ups of one point of $\mathbb{F}_n, \mathbb{F}_m$ respectively.

Lemma 7.3. *For any point $p \in \mathbb{F}_n$, there exists an elementary link $\varphi\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_m$, unique up to automorphism of $\mathbb{F}_m$, whose unique base-point is p. Moreover, $m = n-1$ if $n \geq 1$ and p does not belong to E_n, and $m = n+1$ otherwise.*

Proof. The unicity follows from the unicity of blow-ups up to automorphisms; it remains to show the existence. Lemma 6.5 yields the existence of an elementary link $\varphi_n\colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$ such that φ_n has one base-point on E_n and $(\varphi_n)^{-1}$ has a base-point in $\mathbb{F}_{n+1}\backslash E_{n+1}$. Choosing $\alpha_n \in \mathrm{Aut}(\mathbb{F}_n)$ and $\alpha_{n+1} \in \mathrm{Aut}(\mathbb{F}_{n+1})$, the map $\psi_n = \alpha_{n+1}\varphi_n\alpha_n$ is an elementary link $\mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$. Moreover, the transitivity lemma above (Lemma 6.10) shows that we can obtain any point of E_n as base-point of ψ_n and any point of $\mathbb{F}_{n+1}\backslash E_{n+1}$ as base-point of $(\psi_n)^{-1}$. This yields the result. □

After blowing-up a point in $\mathbb{P}^2$ and having performed elementary links of type II, we can go back to $\mathbb{P}^2$ from $\mathbb{F}_1$:

Definition 7.4. *A birational map $\mathbb{F}_1 \to \mathbb{P}^2$ is a* link of type III *if it is the blow-up of a point of $\mathbb{P}^2$.*

The last link is the following:

Definition 7.5. *A birational map $\mathbb{F}_0 \to \mathbb{F}_0$ is a* link of type IV *if it is an automorphism which exchanges the fibres of the two projections.*

One of the main results of birational geometry of projective surfaces is that any birational map of $\mathbb{P}^2$ decomposes into the elementary links. To prove this result, we will show the following lemma, that is in fact more general because it also yields the description of minimal surfaces and Castelnuovo's contraction criterium for projective smooth rational surfaces:

Lemma 7.6. *Let X be equal to $\mathbb{P}^2$ or to an Hirzebruch surface $\mathbb{F}_n$ for some $n \geq 0$. Let $\eta\colon Y \to X$ be a birational morphism and let $E \subset Y$ be a smooth curve isomorphic to $\mathbb{P}^1$ and of self-intersection -1. Then, the following hold:*

1. *There exists a birational morphism $\tau\colon Y \to Z$, where Z is a smooth projective surface which contracts E onto a point q, and which is the blow-up of this point (in particular, τ restricts to an isomorphism $Y\backslash E \to Z\backslash\{q\}$).*
2. *There exists a sequence of birational maps $\varphi_1, \ldots, \varphi_k$, where $\varphi_i\colon X_{i-1} \dashrightarrow X_i$ is elementary link of type* I, II, III *or* IV *for $i = 1, \ldots, k$, where $X = X_0$, and such that $\psi = \varphi_k\varphi_{k-1}\ldots\varphi_1\eta\tau^{-1}$ is a birational morphism. (Note that $k = 0$ is allowed, which implies that $\psi = \eta\tau^{-1}$ is a birational morphism).*

In particular, the following diagram is commutative:

$$\begin{tikzcd} Y \arrow[rrrr, "\tau"] \arrow[dd, "\eta"'] & & & & Z \arrow[dd, "\psi"] \\ & & & & \\ X \arrow[r, dashed, "\varphi_1"] & X_1 \arrow[r, dashed, "\varphi_2"] & \cdots \arrow[rr, dashed, "\varphi_k"] & & X_k \end{tikzcd}$$

Proof. We decompose η into blow-ups: we set $Y_0 = X$, and for $i = 1, \ldots, m$, we let $p_i \in Y_i$ be a point and $\eta_i\colon Y_i \to Y_{i-1}$ be the blow-up of p_i, such that $\eta = \eta_m \circ \eta_{m-1} \circ \cdots \circ \eta_1$ (in particular, $Y = Y_m$). We let $\mathcal{E}_i \subset Y_i$ be the unique irreducible curve contracted by η_i, and let $E_i \in \mathrm{Pic}(Y)$ be equal to $(\eta_m \ldots \eta_{i+1})^*(\mathcal{E}_i)$.

Let $E \subset Y = Y_m$ be a smooth curve isomorphic to $\mathbb{P}^1$ and of self-intersection -1.

If E is contracted by η then because E has self-intersection -1 it is equal to E_i for some i, where p_i has no point p_j which is infinitely near to it. In particular, we can reorder the points and assume that $i = m$. We choose then $\tau = \eta_m$ and $k = 0$ to conclude.

We can now assume that E is not contracted by η. It is thus the strict transform of the curve $\eta(E) = E_X \subset X$, and is linearly equivalent to $\eta^*(E_X) - \sum a_i E_i$, where a_i is the multiplicity of E_X at p_i. We can assume that $X = \mathbb{F}_n$ (if $X = \mathbb{P}^2$, we blow-up a point and obtain $\mathbb{F}_1$) and write $d = E \cdot f_Y$, where $f_Y \in \mathrm{Pic}(Y)$ is the pull-back of a fibre of $\pi_n \colon \mathbb{F}_n \to \mathbb{P}^1$. Note that $d = E_X \cdot f_X$, where $f_X \in \mathrm{Pic}(X)$ is a fibre of π_n. Moreover, $-1 = E^2 = (E_X)^2 - \sum a_i^2$.

Suppose that $d = 0$, which implies that E_X is a fibre of π_n, so $(E_X)^2 = 0$. Because $E^2 = -1$, exactly one of the a_i is equal to 1 and the other are equal to 0. The point p_i corresponding to a_i is thus a proper point of $X = \mathbb{F}_n$ so we can assume that $i = 1$. We denote by $\tau' \colon X_1 \to \mathbb{F}_{n\pm 1}$ the contraction of the strict transform of E_X, which exists and yields an elementary link $\varphi \colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n'}$ (with $n' = n \pm 1$) equal to $\tau' \circ (\eta_1)^{-1}$. The morphism $\eta' \colon Y \to \mathbb{F}_{n'}$ given by $\eta_n \circ \cdots \circ \eta_2 \circ \tau'$ is a sequence of blow-ups; it contracts E onto a point $q \in \mathbb{F}_{n'}$, whose blow-up corresponds to τ'. Because no infinitely near point is blown-up, we can decompose η' in another sequence of blow-ups where q is the last point blown-up instead of begin the first one. The last blow-up will then be $\tau \colon Y \to Z$ and the remaining part will be a birational morphism $\psi \colon Z \to \mathbb{F}_{n'}$. We have thus $k = 1$, $X_k = \mathbb{F}_{n'}$ and $\varphi = \varphi_1$.

We can now assume that $d > 0$ and prove the result by induction on d.

Suppose that one of the a_i satisfies $2a_i > d$. We can order the points and assume that $a_1 > d$. We denote by $\tau' \colon X_1 \to \mathbb{F}_{n'}$ the contraction of the strict transform of the fibre $f_1 \subset X$ of π_n passing through p_1, which exists and yields an elementary link $\varphi \colon \mathbb{F}_n \dashrightarrow \mathbb{F}_{n'}$ (with $n' = n \pm 1$) equal to $\tau' \circ (\eta_1)^{-1}$. Denote by $q \in \mathbb{F}_{n'}$ the point blown-up by τ' and $F_q \in X_1$ the curve contracted on it. We replace $\eta \colon Y \to \mathbb{F}_n$ with the morphism $\eta' \colon Y \to \mathbb{F}_{n'}$ given by $\eta_n \circ \cdots \circ \eta_2 \circ \tau'$. This does not change d (because φ sends a general fibre of π_n onto a general fibre of $\pi_{n'}$) and does not change the multiplicities $a_2, \ldots, a_n$. However, a_1 is replaced with the multiplicity a_1' of $\eta'(E)$ at q, which is equal to $(\eta_n \circ \cdots \circ \eta_2)(E) \cdot F_q$. Because a_1 is equal to $(\eta_n \circ \cdots \circ \eta_2)(E) \cdot \mathcal{E}_1$ and $\mathcal{E}_1 + F_q$ corresponds to a fibre of the map to $\mathbb{P}^1$, we get $a_1 + a_1' = d$, hence $a_1' < a_1$ because $2a_1 > d$.

Doing this will all other possible points, we can assume that each of the a_i satisfies $2a_i \leq d$.

We write $E_X = dE_n + bf$ for some integer $b \in \mathbb{N}$ (Lemma 6.9). If $E_X \cdot E_n < 0$, we have $E_X = E_n$ and $n = -1$ because $(E_X)^2 \geq E^2 = -1$, so it suffices to choose the contraction of E_n, which is an elementary link $\mathbb{F}_1 \to \mathbb{P}^2$, to get the result. We can thus assume that $E_X \cdot E_n = b - dn > 0$. In particular, $\epsilon = 2b - dn$ satisfies $\epsilon > b > dn$. Since $E = d\eta^*(E_n) + b\eta^*(f) - \sum a_i E_i$ and $K_Y = \eta^*(K_X) + \sum E_i$ with $K_X \cdot E_n = -2 - n$, $K_X \cdot f = -2$, we obtain

$$\begin{array}{rcccccc} -1 & = & E^2 & = & -d^2 n + 2bd - \sum a_i^2 & = & \epsilon d - \sum a_i^2 \\ -1 & = & K_Y E & = & (-2-n)d - 2b + \sum a_i & = & -\epsilon - 2d + \sum a_i, \end{array}$$

hence

$$\sum a_i = \epsilon + 2d - 1, \quad \sum a_i^2 = \epsilon d + 1.$$

The fact that $0 \leq 2a_i < d$ for each i yields

$$0 < \sum a_i(d - 2a_i) = d \sum a_i - 2 \sum a_i^2 = 2d^2 - d - \epsilon d - 2,$$

which implies that $2d^2 - \epsilon d > d + 2 > 0$, hence $2d > \epsilon = 2b - nd > b > nd$. In particular, $n < 2$, so $n = 0$ or $n = 1$.

If $n = 0$, the inequality $2d > \epsilon$ yields $d > b$. Performing an elementary link of type IV, we exchange E_0 and f, and thus exchange d with b, which decreases d as we wanted.

If $n = 1$, the inequality $b > nd$ yields $b > d$. We claim that at least one of the a_i satisfies $a_i > b - d$. Indeed, otherwise we would have

$$0 \geq \sum a_i^2 - \sum a_i(b - d) = 2bd - d^2 + 1 - (2b + d - 1)(b - d) = b(3d - 2b) + (b - d) - 1$$

which implies that $3d - 2b \leq 0$ and contradicts $2d > \epsilon = 2b - d$.

Note that $b - d = E_X \cdot E_1$, so the point p_i corresponding to the multiplicity a_i with $a_i > b - d$ does not belong to E_1. We can also assume that it is a proper point of $\mathbb{F}_1$. Contracting the curve E_1 (via $\tau \colon \mathbb{P}^2 \to \mathbb{F}_1$) then blowing up the point (going to $\mathbb{F}_1$), the intersection with the fibre decreases. Indeed, it is equal to the degree of $\tau(E_X)$ minus the multiplicity at the point $\tau(E_1)$ or the point p_i, which is respectively $b - d$ and a_i. □

Lemma 7.6 gives the following corollaries

Corollary 7.7. *Any smooth projective rational surface S admits a birational morphism $\kappa \colon S \to X$ where surface is equal to $\mathbb{P}^2$ or to an Hirzebruch surface $\mathbb{F}_n$ for some $n \neq 1$.*

Proof. The fact that $n = 1$ can be avoided is clear, because of a birational morphism $\mathbb{F}_1 \to \mathbb{P}^2$. Because S is rational, there exists a birational map $\varphi \colon X \dashrightarrow S$, where $X = \mathbb{P}^2$. We decompose it with two birational morphisms $\pi \colon Y \to S$, $\eta \colon Y \to X$ such that $\varphi = \pi\eta^{-1}$ (Corollary 5.2). If π is an isomorphism, it suffices to choose $\kappa = \eta\pi^{-1}$. Otherwise, π is a sequence of contractions. Denoting by $E \subset Y$ the first curve contracted, which is a (-1)-curve, we use Lemma 7.6, and obtain a birational morphism $\tau \colon Y \to Z$, which contracts E, and a birational morphism $\psi \colon Z \to X'$ where X' is an Hirzebruch surface or $\mathbb{P}^2$. We replace X with X', η with ψ, Y with Z and

π with $\pi\tau^{-1}$, which contracts one curve less that π. The result follows thus by induction on the number of curves contracted by π, which is the number of blow-ups we need in decomposing it. □

Corollary 7.8 (Castelnuovo's contraction criterium for rational surfaces)**.** *Let Y be a smooth projective rational surface and $E \subset Y$ be a (-1)-curve. There exists a birational morphism $\tau\colon Y \to Z$, where Z is a smooth projective surface which contracts E onto a point q, and which is the blow-up of this point (in particular, τ restricts to an isomorphism $Y \backslash E \to Z \backslash \{q\}$).*

Proof. By Corollary 7.7, there exists a birational morphism $\eta\colon Y \to X$, where X is $\mathbb{P}^2$ or an Hirzebruch surface. The existence of τ is thus given by Lemma 7.6. □

Corollary 7.9. *Let S be a smooth projective rational surface and $p \in S$ be a point. There exists an open subset $U \subset S$ isomorphic to $\mathbb{A}^2$ which contains p.*

Proof. If S is equal to $\mathbb{P}^2$, the result is obvious. If S is equal to an Hirzebruch surface $\mathbb{F}_n$, the result is given by Lemma 6.3. We assume the result true for a projective rational surface S and prove it for the blow-up $\pi\colon S' \to S$ of one point q. By Corollary 7.7, this will imply the results for all projective rational surfaces. Let $U \subset S$ be an open subset isomorphic to $\mathbb{A}^2$. If q does not belong to U, then $\pi^{-1}(U)$ is isomorphic to U, and hence to $\mathbb{A}^2$. If U contains q; we can assume that q corresponds to the origin of $\mathbb{A}^2$ (by use of a translation). By Lemma **??**, the set $\pi^{-1}(U)$ is isomorphic to $\mathrm{Bl}_0(\mathbb{A}^2)$, which is covered by open subsets isomorphic to $\mathbb{A}^2$ (see §2.3). The covering of S by open subsets isomorphic to $\mathbb{A}^2$ gives thus the covering of S' by open subsets isomorphic to $\mathbb{A}^2$. □

Corollary 7.10. *Let X_1, X_2 be two surfaces, such that X_i is equal to $\mathbb{P}^2$ or to an Hirzebruch surfaces $\mathbb{F}_n$ for $i = 1, 2$. Any birational map $\varphi\colon X_1 \dashrightarrow X_2$ decomposes into automorphisms and elementary links of type* I, II, III, IV

Proof. We decompose φ with two birational morphisms $\pi\colon Y \to X_1$, $\eta\colon Y \to X_2$ such that $\varphi = \pi\eta^{-1}$ (Corollary 5.2). If both are isomorphisms, the result is trivial. If only η is an isomorphism, the only possibility is that π is the blow-up of one point, and φ is a link of type I. Otherwise, π is a sequence of contractions and we proceed as in Corollary 7.7: we denote by $E \subset Y$ the first curve contracted, which is a (-1)-curve, we use Lemma 7.6, and obtain a birational morphism $\tau\colon Y \to Z$, which contracts E, and a birational morphism $\psi\colon Z \to X'$ where X' is an Hirzebruch surface or $\mathbb{P}^2$, and where $\psi\tau\eta^{-1}$ is a sequence of elementary links.

We replace X with X', η with ψ, Y with Z and π with $\pi\tau^{-1}$, which contracts one curve less that π. The result follows thus by induction on the number of curves contracted by π, which is the number of blow-ups we need in decomposing it. □

In fact, Corollary 7.10 implies that any birational map of $\mathbb{P}^2$ is generated by $\mathrm{Aut}(\mathbb{P}^2)$ and by $(x : y : z) \dashrightarrow (yz : xz : xy)$, which is the famous Noether-Castelnuovo theorem.

Exercise 3. *Using the decomposition of birational maps from $\mathbb{P}^2$ to $\mathbb{P}^2$ into elementary links, shows that* $\mathrm{Bir}(\mathbb{P}^2)$ *is generated by* $\mathrm{Aut}(\mathbb{P}^2)$ *and by the birational maps preserving a given pencil of lines through one point. Deduce the Noether-Castelnuovo theorem.*

We can now prove a more general version of Proposition 4.6:

Proposition 7.11. *Let X be a smooth projective rational surface. The following are equivalent:*

(1) *X is a del Pezzo surface;*
(2) *X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, to $\mathbb{P}^2$ or to the blow-up of $1 \le r \le 8$ points of $\mathbb{P}^2$ such that no 3 are collinear, no 6 are on the same conic, no 8 lie on a cubic having a double point at one of them.*
(3) *$K_X^2 \ge 1$ and any irreducible curve of X has self-intersection ≥ -1;*
(4) *$C \cdot (-K_X) > 0$ for any effective divisor C.*

Proof. The canonical divisor of $\mathbb{P}^1 \times \mathbb{P}^1$ is equivalent to $-2f_1 - 2f_2$, where f_1, f_2 are the fibres of the two projections (Lemma 6.7). Since $C = -\frac{1}{2}K_{\mathbb{P}^1\times\mathbb{P}^1} = f_1 + f_2$ satisfies $C^2 > 0$ and $C \cdot f_i > 0$ for each i, it intersects any curve of $\mathbb{P}^1 \times \mathbb{P}^1$ positively. It is thus ample by Nakai-Moishezon Criterion. (It is in fact very ample because it gives the embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ given by $((x : y), (u : v)) \to (xu : xv : yu : yv))$. This shows that $\mathbb{P}^1 \times \mathbb{P}^1$ is a del Pezzo surface, and moreover that all assertions are true for $X = \mathbb{P}^1 \times \mathbb{P}^1$.

If there exists a birational morphism $\pi\colon X \to \mathbb{P}^2$, the equivalence of all assertions is provided by Proposition 4.6.

Otherwise, Corollary 7.7 yields a birational morphism $\pi\colon X \to \mathbb{F}_n$ for some $n \ne 1$. If $n = 0$, either π is an isomorphism (in which case $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, case already done) or π factors through the blow-up $\tau\colon Z \to \mathbb{F}_0$ of one point p. An elementary link centred at p gives rise to a birational morphism $Z \to \mathbb{F}_1$, and thus to a birational morphism $X \to \mathbb{P}^2$. We can thus assume that $n \ge 2$. In this case, the strict transform on X of the exceptional section of $\mathbb{F}_n$ gives a smooth curve $C \subset X$ isomorphic to $\mathbb{P}^1$ and of self-intersection ≤ -2. Adjunction formula yields $K_X \cdot C + C^2 = -2$, hence $K_X \cdot C \leq 0$. This shows that assertions $1), 3), 4)$ are not satisfied. The fact that 2) is not satisfied follows from Proposition 4.6. □

Definition 7.12. *A smooth projective surface X is said to be* minimal *if any birational morphism $X \to Y$, where Y is a smooth projective surface, is an isomorphism.*

Proposition 7.13. *Let S be a smooth projective rational surface. The following are equivalent:*

(1) *S is minimal;*
(2) *S does not contain any (-1)-curve;*
(3) *S is isomorphic to $\mathbb{P}^2$ or to $\mathbb{F}_n$ for $n \neq 1$.*

Proof. If S does not contain any (-1)-curve, any birational morphism starting from S to a smooth projective surface is an isomorphism: there is no curve to contract. This yields $2) \Rightarrow 1)$.

Suppose that S is minimal. Corollary 7.7 yields the existence of a birational morphism $S \to X$, where X is $\mathbb{P}^2$ or $\mathbb{F}_n$ for $n \neq 1$. If S is minimal, then this morphism has to be an isomorphism, which shows that $S = \mathbb{P}^2$ or $S = \mathbb{F}_n$ for $n \neq 1$. We thus have $1) \Rightarrow 3)$.

It remains to show $3) \Rightarrow 2)$. It is clear that $\mathbb{P}^2$ does not contain any (-1)-curve: every curve has self-intersection d^2, where d is its degree. By Lemma 6.9, a curve C on $\mathbb{F}_n$, is equivalent to $aE_n + bf$ for some $a, b \geq 0$. Its square is equal to $-a^2n + 2ab = a(2b - an)$. If $C^2 = -1$, we thus have $a = 1$, which means that C is a section of self-intersection -1. This is only possible when $n = 1$ by Lemma 6.9. □

8. Regularisation of finite groups

8.1. **Base-points and action on Bubble spaces.** Let X, Y is a smooth projective surface and $\varphi\colon X \dashrightarrow Y$ be a birational map. By Corollary 5.2, there exist a projective smooth surface Z and birational morphisms $\eta_X\colon Z \to X$, $\eta_Y\colon Z \to Y$, which are finite sequences of blow-ups such that $\varphi = \eta_Y \circ (\eta_X)^{-1}$. This gives a resolution of indeterminacies

$$\begin{tikzcd} & Z \arrow[dl, "\eta_X"'] \arrow[dr, "\eta_Y"] & \\ X \arrow[rr, dashed, "\varphi"'] & & Y. \end{tikzcd}$$

We can moreover assume that the resolution is minimal, which means that no (-1)-curve of Z is contracted by both η_X and η_Y. In this case, we say that the base-points of φ (respectively of φ^{-1}) are the points blown-up by η_X (respectively by η_Y). These points form two subsets of the bubble spaces $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ respectively, that we denote by $\mathcal{B}(\varphi) \subset \mathcal{B}(X)$ and $\mathcal{B}(\varphi^{-1}) \subset \mathcal{B}(Y)$.

Lemma 8.1. 1) *If $\pi\colon S \to S'$ is the blow-up of a point $p \in S'$, then π induces a bijection $\pi_\bullet\colon \mathcal{B}(S) \to \mathcal{B}(S')\backslash\{p\}$.*

2) *Let X, Y is a smooth projective surface and $\varphi\colon X \dashrightarrow Y$ be a birational map. Then φ induces a bijection $\varphi_\bullet\colon \mathcal{B}(X)\backslash\mathcal{B}(\varphi) \to \mathcal{B}(Y)\backslash\mathcal{B}(\varphi^{-1})$.*

3) *If $\varphi\colon X \dashrightarrow Y$ and $\psi\colon Y \dashrightarrow Z$ are birational maps between smooth projective surfaces, we have $\psi_\bullet(\varphi_\bullet(p)) = (\psi\varphi)_\bullet(p)$ for any point $p \in \mathcal{B}(X)\backslash\mathcal{B}(\varphi)$ such that $\varphi_\bullet(p) \in \mathcal{B}(Y)\backslash\mathcal{B}(\psi)$.*

Remark 8.2. *It follows from the construction that all bijections above preserve the partial order given by $p > q$ if p is infinitely near to q.*

Proof. 1) Any point of $\mathcal{B}(S)$ corresponds to a triplet (x, X, η), where $x \in X$ and $\eta\colon X \to S$ is a birational morphism. It naturally corresponds to the triplet $(x, X, \pi \circ \eta)$ which gives an element of $\mathcal{B}(S')$. Moreover, any point except p is obtained by this process.

2) We take a minimal resolution of indeterminacies of φ, given by two birational morphisms $\eta_X\colon Z \to X$, $\eta_Y\colon Z \to Y$ such that $\varphi = \eta_Y \circ (\eta_X)^{-1}$, where Z is a smooth projective surface. Applying (1) finitely many times we see that η_X induces a bijection $\mathcal{B}(Z) \to \mathcal{B}(X)\backslash\mathcal{B}(\varphi)$ and η_Y induces a bijection $\mathcal{B}(Z) \to \mathcal{B}(Y)\backslash\mathcal{B}(\varphi^{-1})$. This yields the bijection $\varphi_\bullet\colon \mathcal{B}(X)\backslash\mathcal{B}(\varphi) \to \mathcal{B}(Y)\backslash\mathcal{B}(\varphi^{-1})$.

Assertion 3) follows from the construction of the bijections. □

8.2. **Regularisation of finite subgroups.** Recall that $\mathrm{Bir}(\mathbb{P}^2)$ is the group of birational maps of $\mathbb{P}^2$, also called *Cremona group.*

Proposition 8.3. *Let X be a smooth projective surface. Let $G \subset \mathrm{Bir}(X)$ be a finite subgroup. Let $\mathcal{B}(G) = \bigcup_{g \in G} \mathcal{B}(g)$.*

There exists a birational morphism $\pi\colon S \to X$, which is the blow-up of the points of $\mathcal{B}(G)$, and such that $\pi^{-1}G\pi \subset \mathrm{Aut}(S)$.

Proof. Because $\mathcal{B}(g)$ is finite for any $g \in G$, the set $\mathcal{B}(G)$ is finite. Moreover, if $p \in \mathcal{B}(g), q \in \mathcal{B}(X)$ and p is infinitely near to q, then $q \in \mathcal{B}(g)$. This implies that the same conditions holds for $\mathcal{B}(G)$ and thus that there exists a birational $\pi\colon S \to X$ which is the blow-up of the points of $\mathcal{B}(G)$ (we let π be the identity if $\mathcal{B}(G) = \emptyset$).

It remains to prove that $\hat{g} = \pi^{-1} g \pi \in \mathrm{Aut}(S)$ for any $g \in G$. As before, we take a minimal resolution of indeterminacies of g, given by two birational morphisms $\eta_1\colon Z \to X$, $\eta_2\colon Z \to X$ such that $g = \eta_2 \circ (\eta_1)^{-1}$. Because $\pi\colon S \to X$ is the blow-up of $\mathcal{B}(G)$ and η_1 blows-up $\mathcal{B}(g)$, the map $\tau_1 = (\eta_1)^{-1}\pi\colon S \to Z$ is a birational morphism, which blows-up $((\eta_1)^{-1})_\bullet(\mathcal{B}(G)\backslash\mathcal{B}(g))$. Similarly, η_2 blows-up $\mathcal{B}(g^{-1})$, so $\tau_2 = (\eta_2)^{-1}\pi\colon S \to Z$ is a birational morphism, which blows-up $((\eta_2)^{-1})_\bullet(\mathcal{B}(G)\backslash\mathcal{B}(g^{-1}))$:

$$\begin{tikzcd} S \arrow[dd, "\pi"'] \arrow[dr, "\tau_1"] & & S \arrow[dd, "\pi"] \arrow[dl, "\tau_2"'] \\ & Z \arrow[dl, "\eta_1"'] \arrow[dr, "\eta_2"] & \\ X \arrow[rr, dashed, "g"'] & & X. \end{tikzcd}$$

It remains to see that $(\tau_2)^{-1} \circ \tau_1$ is an automorphism of S, which amounts to see that τ_1, τ_2 blow-up the same set, i.e. that

$$((\eta_1)^{-1})_\bullet(\mathcal{B}(G)\backslash\mathcal{B}(g)) = ((\eta_2)^{-1})_\bullet(\mathcal{B}(G)\backslash\mathcal{B}(g^{-1})).$$

Recall that η_1 and η_2 blow-up respectively the sets $\mathcal{B}(g)$ and $\mathcal{B}(g^{-1})$, and induce bijections $(\eta_1)_\bullet\colon \mathcal{B}(Z) \to \mathcal{B}(X)\backslash\mathcal{B}(g)$, $(\eta_2)_\bullet\colon \mathcal{B}(Z) \to \mathcal{B}(X)\backslash\mathcal{B}(g^{-1})$, which induce a bijection $g_\bullet\colon \mathcal{B}(X)\backslash\mathcal{B}(g) \to \mathcal{B}(X)\backslash\mathcal{B}(g^{-1})$ given by $g_\bullet \circ (\eta_1)_\bullet = (\eta_2)_\bullet$ (see Lemma 8.1). Applying $(\eta_2)_\bullet$ to both sides of the equality, we need to see that

$$g_\bullet(\mathcal{B}(G)\backslash\mathcal{B}(g)) = (\mathcal{B}(G)\backslash\mathcal{B}(g^{-1})).$$

If $p \in \mathcal{B}(G)\backslash\mathcal{B}(g)$, there exists $h \in G$ such that $p \in \mathcal{B}(h)$. Because p is not a base-point of g, the element $g_\bullet(p)$ is well-defined element of $\mathcal{B}(X)$, which is in fact a base-point of hg^{-1} and is not a base-point of g^{-1}. In particular, $g_\bullet(p)$ belongs to $\mathcal{B}(G)\backslash\mathcal{B}(g^{-1})$. Doing the same with g^{-1} shows that $g_\bullet$ induces a bijection from $\mathcal{B}(G)\backslash\mathcal{B}(g)$ to $\mathcal{B}(G)\backslash\mathcal{B}(g^{-1})$. □

Using Proposition 8.3, the study of finite sugbroups of $\mathrm{Bir}(\mathbb{P}^2)$ can be done by looking at finite groups acting biregularly on smooth projective rational surfaces. T

Here is a simple but important observation: a birational map $\varphi : S \dashrightarrow \mathbb{P}^2$ yields an isomorphism between $\mathrm{Bir}(S)$ and $\mathrm{Bir}(\mathbb{P}^2)$. Choosing any smooth projective rational surface S, a finite group $G \subset \mathrm{Aut}(S)$ corresponds to a subgroup of $\mathrm{Bir}(\mathbb{P}^2)$: we choose a birational $\varphi : S \dashrightarrow \mathbb{P}^2$ and get $\varphi G \varphi^{-1} \subset \mathrm{Bir}(\mathbb{P}^2)$. The choice of φ only replace the group obtained with another one in its conjugacy class.

We say that two groups $G \subset \mathrm{Bir}(S)$ and $G' \subset \mathrm{Bir}(S')$ are *birationally conjugate* if there exists a birational map $\varphi : S \dashrightarrow S'$ such that $\varphi G \varphi^{-1} = G'$. This means that the two groups represent the same conjugacy class in the Cremona group.

Definition 8.4.
- *We denote by (G, S) a pair in which S is a smooth projective rational surface and $G \subset \mathrm{Aut}(S)$ is a group acting biregularly on S. A pair (G, S) is also classically called a G-*surface.
- *Let (G, S) be a G-surface. We say that a birational map $\varphi : S \dashrightarrow S'$ is* G-equivariant *if the G-action on S' induced by φ is biregular. The birational map φ is called* a birational map of G-surfaces.
- *We say that a pair (G, S) is* minimal *(or equivalently that G* acts minimally on *S) if any G-equivariant birational morphism $\varphi : S \to S'$ is an isomorphism.*

When G is the identity, a pair (G, S) is minimal if the surface S is minimal, so S is $\mathbb{P}^2$ or a Hirzebruch surface.

Given some finite subgroup $G \subset \mathrm{Bir}(\mathbb{P}^2)$, the modern approach (used in [Ba-Be00], [dFe04], [Bea07] and [Do-Iz09]) is to see this subgroup as a group of automorphisms of a surface S (using Proposition 8.3) and to suppose that the pair (G, S) is minimal by contracting orbits of disjoint (-1)-curves.

Then, we apply the following very powerful result, due to Yu. Manin in the abelian case (see [Man67]) and V.A. Iskovskikh in the general case (see [Isk79]), which classifies the minimal G-surfaces.

Proposition 8.5. *Let S be a smooth projective rational surface and $G \subset \mathrm{Aut}(S)$ be a finite subgroup of automorphisms of S. If the pair (G, S) is minimal then one* and only one *of the following holds:*

1. *The surface S has a conic bundle structure invariant by G, and* $\mathrm{rk}\ \mathrm{Pic}(S)^G = 2$*, i.e. the fixed part of the Picard group is generated by the canonical divisor and the divisor class of a fibre.*

2. rk $\mathrm{Pic}(S)^G = 1$, *i.e. the fixed part of the Picard group is generated by the canonical divisor.*

Let us comment on this proposition:

- In the first case, there exists a morphism $\pi : S \to \mathbb{P}^1$ with general fibres rational and irreducible and such that every singular fibre is the union of two rational curves F, F' with $F^2 = F'^2 = -1$ and $FF' = 1$. The group G embeds in the group of automorphisms of the generic fibre $\mathbb{P}^1_{\mathbf{k}(x)}$ of π. This group is abstractly $\mathrm{PGL}_2(\mathbf{k}(x)) \rtimes \mathrm{PGL}(2, \mathbf{k})$ and can be viewed as the group of birational maps of $\mathbb{P}^1 \times \mathbb{P}^1$ that preserve the first projection (see Section 9.2 and in particular Proposition **??**).

  The *de Jonquières involutions* are examples of this case: they are given in this context as birational maps of $\mathbb{P}^1 \times \mathbb{P}^1$ of the form
  $$(x_1 : x_2) \times (y_1 : y_2) \dashrightarrow (x_1 : x_2) \times (y_2 \prod_{i=1}^n (x_1 - a_i x_2) : y_1 \prod_{i=1}^n (x_1 - b_i x_2)),$$
  for some $a_1, ..., a_n, b_1, ..., b_n \in \mathbf{k}$ all distinct.
- In the second case, S is a *Del Pezzo surface* (see Lemma **??**). We recall that a Del Pezzo surface is a rational surface whose anti-canonical divisor $-K_S$ is ample. In fact, it is either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of r points $A_1, A_2, ..., A_r \in \mathbb{P}^2$, $1 \le r \le 8$, in general position. (See [Dem76] and [Bea96] for more details.) We describe in Chapter **??** the finite abelian subgroups $G \subset \mathrm{Aut}(S)$ when the group of invariant divisors is of rank 1 (case 2 of Proposition 8.5).

  Here is an example: the 3-torsion of the diagonal torus of $\mathrm{PGL}(4, \mathbf{k})$, isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$, acting on the cubic surface in $\mathbb{P}^3$ with equation $w^3 + x^3 + y^3 + z^3 = 0$. This surface is a Del Pezzo surface of degree 3 (see Section **??**). Other famous examples are Geiser and Bertini involutions, acting minimally respectively on Del Pezzo surfaces of degree 2 and 1 (see Sections **??** and **??**).

Remark 8.6.
- *In the second case, no conic bundle structure can be invariant (for this would imply that both the canonical divisor and the fibre are invariant, so* rk $\mathrm{Pic}(S)^G > 1$*). However, in the first case, the underlying surface may be a Del Pezzo surface (see Section* **??***).*
- *Although the two cases are distinct, they are not birationally distinct (see Section* **??***).*

9. Automorphisms of conic bundles

In this chapter we give some description of the case of conic bundles (second case of Proposition 8.5).

9.1. **Description of conic bundles.** We first describe conic bundles without goups. We begin with the definition of a conic bundle:

Definition 9.1. *Let S be a rational surface and $\pi : S \to \mathbb{P}^1$ a morphism. We say that the pair (S, π) is a* conic bundle *if*
- *A general fibre of π isomorphic to $\mathbb{P}^1$.*
- *There is a finite number of exceptions: these singular fibres are the union of rational curves F and F' such that $F^2 = F'^2 = -1$ and $FF' = 1$.*

Note that the condition that S be rational is in fact induced by the others (using for example a Noether-Enriques theorem whose proof can be found in [Bea96], Theorem III.4). Moreover a conic bundle is not in general a bundle (it is not locally trivial) but is a conic section of a $\mathbb{P}^2$-bundle.

In this context, the natural notion of minimality is defined as follows:

Definition 9.2.
- *Let (S, π) and (S', π') be two conic bundles. We say that $\varphi : S \dashrightarrow S'$ is a* birational map of conic bundles *if φ is a birational map which sends a general fibre of π on a general fibre of π'.*
- *We say that a conic bundle (S, π) is* minimal *if any birational morphism of conic bundles $(S, \pi) \to (S', \pi')$ is an isomorphism.*

The following lemma is classical:

Lemma 9.3. *Let (S, π) be a conic bundle. The following conditions are equivalent:*

(1) *(S, π) is minimal.*
(2) *The fibration π is smooth, i.e. no fibre of π is singular.*
(3) *S is a Hirzebruch surface.*

Proof. Let $\varphi : (S, \pi) \to (S', \pi')$ be some birational morphism of conic bundles. The morphism φ is not an isomorphism if and only if it contracts at least one (-1)-curve F. If this is the case, as the conic bundle structure is preserved, F does not intersect a regular fibre and so $\pi(F)$ is a point. This implies that F is contained in some singular fibre. This yields 2) $\Rightarrow$ 1).

3) $\Rightarrow$ 2) is easy (see Lemma 6.3).

1) $\Rightarrow$ 3): If (S, π) is minimal, there is no (-1)-curve contained in a fibre (the contraction of it would give a birational morphism which is not an isomorphism and would give another conic bundle). In particular, the rank of the Picard group of S is 2. Corollary 7.7 implies that S is an Hirzebruch surface. □

Here is a useful result:

Lemma 9.4. *Let (S, π) be a conic bundle on a surface $S \not\cong \mathbb{P}^1 \times \mathbb{P}^1$.*

(1) *There exists a section s of π of self-intersection $s^2 < 0$.*
(2) *For any section s of self-intersection $s^2 = -n < 0$, there exists a birational morphism of conic bundles $p\colon S \to \mathbb{F}_n$ which sends s onto the exceptional section E_n.*
(3) *Suppose that s, t are two sections of self-intersection $-n < 0$; denote by r the number of singular fibres of π and by k the number of these fibres where s, t intersect the same component. We have*
$$r = 2s \cdot t + 2n + k$$
and there exist birational morphisms of conic bundles $p_0 : S \to \mathbb{F}_0$ and $p_1 : S \to \mathbb{F}_1$.

Proof. There exists a birational morphism of conic bundles $p\colon (S, \pi) \to (\mathbb{F}_n, \pi_n)$ by Lemma 9.3. If $n > 0$, the strict transform of the exceptional section yields a section of negative self-intersection on S. If $n = 0$, the fact that $S \not\cong \mathbb{P}^1 \times \mathbb{P}^1$ implies that p blows-up at least one point of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, so the strict transform of the section of self-intersection 0 passing through the point has negative intersection on S. This yields assertion (1).

Let s be a section of π of self-intersection $-n < 0$ and r be the number of singular fibres of π. If $r = 0$, the lemma is trivial: we take in assertion (2) the identity map and (3) is trivially true because only one section of π has self-intersection < 0 (Lemma 6.9). We suppose now that $r \geq 1$.

We denote by $F_1, ..., F_r$ the irreducible components of the singular fibres which do not intersect s. Blowing these down, we get a birational morphism of conic bundles $p : S \to \mathbb{F}_m$, for some integer $m \geq 0$. The image of s has again self-intersection $-n < 0$, so $m = n$ and s is sent on the exceptional section E_n. This yields (2).

Let t be another section on S of self-intersection $-n$. The Picard group of S is generated by $s = p^*(E_n)$, the divisor f of a fibre of π and $F_1, ..., F_r$. Write some section $t = s + bf - \sum_{i=1}^r a_i F_i$, for some integers $b, a_1, ..., a_r$, where a_i is equal to $t \cdot F_i$, so belongs to $\{0, 1\}$. Moreover, $a_i = 0$ if and only if s, t intersect the other component of the fibre where F_i is. In particular, $k = r - \sum_{i=1}^r a_i$. We compute $t^2 = s^2 - \sum_{i=1}^r a_i^2 + 2b = s^2 - (r - k) + 2b$, which is equal to s^2 by hypothesis, so $2b = r - k$. We then compute $s \cdot t = b - n$, and find thus
$$r = 2b + k = 2s \cdot t + 2n + k,$$
which yields (3). We have in particular $r \geq 2n$. We can thus contract $f - F_1, f - F_2, ..., f - F_n, F_{n+1}, F_{n+2}, ..., F_r$, we obtain a birational morphism p_0 of conic bundles which sends s on a section of self-intersection 0 and thus whose image is $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Similarly, the morphism $p_1 : S \to \mathbb{F}_1$ is given by the contraction of $f - F_1, f - F_2, ..., f - F_{n-1}, F_n, F_{n+1}, ..., F_r$. □

The above lemma shows that $r \geq 2n$. The extremal case is when $r = 2n$, a case called *exceptional* in [Do-Iz09]:

Definition 9.5. *A conic bundle (S, π) is said to be* exceptional *if it has $2n$ singular fibres and two sections of self-intersection $-n$.*

Note that such conic bundle can be viewed as obtained by blowing-up $2n$ points on a section of self-intersection n of $\mathbb{F}_n$.

9.2. **Finite groups of automorphisms: two representations.**

Example 9.6. *Let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection on the first factor.*

The group of birational maps $\mathbb{P}^1 \times \mathbb{P}^1$ which send a general fibre of π_1 onto a general fibre of π_1 is the group of birational maps of the form:
$$(x, y) \dashrightarrow \left(\frac{ax + b}{cx + d}, \frac{\alpha(x)y + \beta(x)}{\gamma(x)y + \delta(x)}\right),$$
where $a, b, c, d \in \mathbf{k}$, $\alpha, \beta, \gamma, \delta \in \mathbf{k}(x)$, and $(ad - bc)(\alpha\delta - \beta\gamma) \neq 0$.

This group, called the de Jonquières group*, is isomorphic to* $\mathrm{PGL}(2, \mathbf{k}(x)) \rtimes \mathrm{PGL}(2, \mathbf{k})$.

Definition 9.7. *Let (S,π) be a conic bundle. The group of automorphisms of S that leave the conic bundle structure invariant (i.e. that send every fibre to another fibre) is denoted by* $\mathrm{Aut}(S,\pi)$.

The study of finite groups of automorphisms of conic bundles, is in fact equivalent to the study of finite subgroup of the de Jonquières group given before. If $G \subset \mathrm{Aut}(S,\pi)$, we have a birational morphism of conic bundles $(S,\pi) \to (S',\pi')$, where (S',π') is minimal and hence isomorphic to $(\mathbb{F}_n,\pi_n)$ for some n. Performing elementary links, we can go to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and see G as a subgroup of the de Jonquières group. The converse also holds.

9.3. **The exact sequence.** Let $G \subset \mathrm{Aut}(S,\pi)$ be some subgroup acting (biregularly) on a conic bundle (S,π). We have a natural homomorphism $\overline{\pi} : G \to \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2,\mathbf{k})$ that satisfies $\overline{\pi}(g)\pi = \pi g$, for every $g \in G$. We observe that the group $G' = \mathrm{Ker}(\overline{\pi})$ of automorphims that leaves every fibre invariant embeds in the subgroup $\mathrm{PGL}(2,\mathbf{k}(x))$ of automorphisms of the generic fibre $\mathbb{P}^1(\mathbf{k}(x))$.

We use the exact sequence

$$1 \to G' \to G \xrightarrow{\overline{\pi}} \overline{\pi}(G) \to 1 \tag{9.1}$$

to restrict the structure of G.

When G is finite, so are G' and $\overline{\pi}(G)$. These are moreover finite subgroups of $\mathrm{PGL}(2,\mathbf{k})$ and $\mathrm{PGL}(2,\mathbf{k}(x))$ so either cyclic, diedral, Alt_4, Alt_5, Sym_4.

9.4. **Exceptional conic bundles and $\mathbb{Z}/2\mathbb{Z}$-conic bundles.**

Proposition 9.8. *Let (S,π) be a conic bundle with at least one singular fibre, and assume the existence of a finite group $G \subset \mathrm{Aut}(S,\pi)$ such that (G,S) is minimal. Then, one of the following occurs:*

(1) *the conic bundle (S,π) is exceptional;*
(2) *the subgroup $G' \subset G$ of elements that preserve any fibre (i.e. that act trivially on the basis of the fibration) is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$.*

Proof. Let $s \in S$ be a section of π of negative self-intersection (which exists by Lemma 9.4), and let us take one singular fibre, whose components are F_1, F_2; we choose the order so that s intersects F_1. Because the pair (G,S) is minimal, there exists $g \in G$ such that $g(F_1) = F_2$ and $g(F_2) = F_1$ (otherwise we can contract the orbit of F_1). In particular, the section $t = g(s)$ is distinct from s and has the same self-intersection.

1) Assume first the existence of $h \in G$ which is not trivial but acts trivially on $\mathrm{Pic}(S)$ and on the basis of the fibration. It preserves thus s and t (because they have negative self-intersection) and fixes them pointwise. The set of points of S fixed by h being smooth, we have $s \cdot t = 0$. Moreover, the sections s and t intersect distinct components of any singular fibre (otherwise we would have three fixed points on a component, which is impossible). Lemma 9.4 implies that the number of singular fibres is equal to $2n$, where $s^2 = -n$. Hence, (S,π) is exceptional.

2) We can now assume that any non-trivial element of the subgroup $G' \subset G$ of elements acting trivially on the basis induces a non-trivial action on the Picard group. This implies that each element of G' has order 1 or 2: the square of each element preserves the two components of each singular fibre so acts trivially on the Picard group (contract fibres to go to $\mathbb{F}_n$ with $n > 0$, the element corresponds to an automorphism of $\mathbb{F}_n$ so acts trivially on the Picard group). Because $G' \subset \mathrm{PGL}(2,\mathbf{k}(x))$, it can be isomorphic to $\{1\}$, $\mathbb{Z}/2$ or $(\mathbb{Z}/2\mathbb{Z})^2$. It remains to exclude the case where G' is trivial.

We assume that G' is trivial and look for a contradiction. The element $g \in G$ that exchanges F_1 and F_2 has even order. Replacing g with an odd power, we can assume that g has order 2^m.

a) Suppose that $m = 1$. The point $F_1 \cap F_2$ is fixed by g and cannot be an isolated fixed point because g does not act trivially on the tangent directions. In particular, g fixes some curve passing through the point, so acts trivially on the basis, which contradicts the fact that G' is trivial.

b) Suppose that $m > 1$. The element $g' = g^{2^{m-1}}$ has order 2 and preserves the two components F_1, F_2. Observe that the action of g' on F_1 and F_2 is the same because g, g' commute. The fixed locus of g' being smooth, it does not act trivially on F_1 and F_2, so fixes exactly three points on the fibre: $x = F_1 \cap F_2, y_1 \in F_1 \backslash F_2$ and $y_2 = g(y_1) \in F_2 \backslash F_1$. Contracting F_1 we conjugate g' to an automorphism of order 2 fixing the point q being the image of F_1. Because the action on F_1 is not trivial, the point q cannot be an isolated fixed point. There exists thus a curve of S of fixed points of g' passing through F_1, so $g' \in G'$, which again contradicts the fact that G' is trivial. □

The automorphisms of exceptional bundles (S,π) are nice algebraic groups of dimension 1, easy to describe. See [Do-Iz09, Proposition 5.3] and [Bla09a, Lemma 4.3.3].

The second case of Proposition 9.8 is more complicated to study. One can have some descriptions on the number of fibres where the involutions in G' act non-trivially (i.e. exchange the two components) and relate this to the genus of the curves fixed. Apart from this, it is quite hard to describe really all possibilities for the groups G, up to conjugation. This is partly done in [Do-Iz09], but not achieved. It is only completely achieved in the case where G is cylic [Bla11]. Some more detailed work on conic bundles was also done recently by V.I. Tsygankov: [Tsy11], [Tsy12].

10. Actions on del Pezzo surfaces with fixed part of the Picard group of rank one

As we observed above, del Pezzo surfaces are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ or blow-ups of $1 \le r \le 8$ points of the plane in general position. In fact, the automorphism groups of these varieties are nice algebraic groups, that can be completely described.

The cases of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are very classical: $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}(3, \mathbf{k})$ and $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) = (\mathrm{PGL}(2, \mathbf{k}) \times \mathrm{PGL}(2, \mathbf{k})) \rtimes \mathbb{Z}/2\mathbb{Z}$. There is a huge list of finite subgroups, see [Do-Iz09] for more details.

If S is the blow-up of $1 \le r \le 8$ points in $\mathbb{P}^2$ in general position, the group $\mathrm{Aut}(S)$ acts on the set of (-1)-curves, which is finite. The kernel of this action correspond naturally to the group of automorphisms of $\mathbb{P}^2$ that fix the r points. This kernel is thus infinite when $r \le 3$ and finite when $r \ge 4$. The group $\mathrm{Aut}(S)$ is thus an algebraic group, which is finite if and only if $r \ge 4$.

Recall that the degree of a del Pezzo surface S is equal to $(K_S)^2 \in \{1, \dots, 9\}$, and corresponds classically to the degree of S viewed in $\mathbb{P}^d$ via the embedding $|-K_S|$, for $d \ge 3$. (When $d = 1, 2$, the divisor $-K_S$ is ample but not very ample, and S was not called del Pezzo in the classical sense, but nowadays the notion has been extended).

Proposition 10.1. *Let $\pi : S \to \mathbb{P}^2$ be the blow-up of $1 \le r \le 8$ points $p_1, ..., p_r \in \mathbb{P}^2$ in general position (no 3 collinear, no 6 on the same conic, no 8 on the same cubic singular at one of the points). The (-1)-curves of S are:*

- *The r exceptional curves corresponding to the total pull-backs $\pi^{-1}(p_1)$,...,$\pi^{-1}(p_r)$ of the blown-up points.*
- *The strict transforms of the curves of degree d passing through the p_i's with multiplicities given in the following table:*

r	*degree* d	*multiplicities at the points*	*number of such curves for* $r = 1$,	2,	3,	4,	5,	6,	7,	8
≥ 2	1	$(1,1)$		1	3	6	10	15	21	28
≥ 5	2	$(1,1,1,1,1)$					1	6	21	56
≥ 7	3	$(2,1,1,1,1,1,1)$							7	56
8	4	$(2,2,2,1,1,1,1,1)$								56
8	5	$(2,2,2,2,2,2,1,1)$								28
8	6	$(3,2,2,2,2,2,2,2)$								8

Remark 10.2. *The number of exceptional curves of a Del Pezzo surface is well known for a long time, see for example* [Dem76, Table 3, page 35].

Proof. We denote by $E_i = \pi^{-1}(p_i)$ the exceptional curve corresponding to the blow-up of A_i for $i = 1, ..., r$ and by L the pull-back of a general line of $\mathbb{P}^2$. Apart from these, the divisor class D of an exceptional curve of S is equal to $mL - \sum a_i E_i$ for some non-negative integers $m, a_1, ..., a_r$, with $m > 0$. The self-intersection $D^2 = -1$ and the adjunction formula $D(K_S + D) = -2$ (see Proposition 4.2) give the following relations:

$$\begin{array}{rcl} \sum_{i=1}^{r} a_i^2 & = & m^2 + 1, \\ \sum_{i=1}^{r} a_i & = & 3m - 1, \end{array} \tag{10.1}$$

which imply that $\sum_{i=1}^{r} a_i(a_i - 1) = (m-1)(m-2)$. Applying Cauchy-Schwarz to the vectors $(1, \dots, 1)$ and $(a_1, \dots, a_r)$ we have $\left(\sum_{i=1}^{r} a_i\right)^2 \le r \sum_{i=1}^{r} {a_i}^2$, hence $(3m-1)^2 \le r(m^2+1)$, which yields

$$P(m) \le 0, \text{ where } P(m) = (9-r)m^2 - 6m + (1-r).$$

Because $9 - r > 0$, $P(m)$ corresponds to a parabola, which minimum is at $m = \frac{3}{9-r} > 0$. Because $P(0) = 1 - r \le 0$, and $P(6) = 50(8-r) \ge 0$, the positive values of m for which $P(m) \le 0$ are between 1 and 6. Replacing m with $1, 2, 3, 4, 5, 6$ in the above equation gives the solutions above. It is then easy to check that all numeric solutions really give a curve by computing the dimension of the system of curves of degree d with multiplicity $a_1, \dots, a_r$ at the points, which is positive. Moreover, the curves of this systems are irreducible because of the fact that the points are in general position. □

Corollary 10.3. *Let $\pi : S \to \mathbb{P}^2$ be the blow-up of $1 \le r \le 2$ points. The pair* $(\mathrm{Aut}(S), S)$ *is not minimal.*

Proof. Proposition 10.1 gives the number of (-1)-curves. On the blow-up of one point (which is $\mathbb{F}_1$, there is only E_1, which is thus invariant by $\mathrm{Aut}(S)$. On the blow-up of two points we have E_1, E_2 and $L - E_1 - E_2$. Because E_1 and E_2 do not intersect each other but both intersect $L - E_1 - E_2$, the curve $L - E_1 - E_2$ is invariant. Contracting the (-1)-curve which is invariant, we see that $(\mathrm{Aut}(S), S)$ is not minimal. □

In case 2 of Proposition 8.5 (when rk $\mathrm{Pic}(S)^G = 1$), the following lemma (which was first used in [dFe04, Proposition 4.1.4]) restricts the possibilities for the surface S and the group G:

Lemma 10.4 (Size of the orbits). *Let S be a Del Pezzo surface, which is the blow-up of $1 \leq r \leq 8$ points of $\mathbb{P}^2$ in general position, and let $G \subset \mathrm{Aut}(S)$ be a finite subgroup of automorphisms with* rk $\mathrm{Pic}(S)^G = 1$. *Then:*

- *$G \neq \{1\}$;*
- *the size of any orbit of the action of G on the set of exceptional divisors is divisible by the degree of S, which is $9 - r$;*
- *in particular, the order of G is divisible by the degree of S.*

Proof. It is clear that $G \neq \{1\}$, since rk $\mathrm{Pic}(S) > 1$. Let $D_1, D_2, ..., D_k$ be k exceptional divisors of S, forming an orbit of G. The divisor $\sum_{i=1}^k D_i$ is fixed by G and thus is a multiple of K_S. We can write $\sum_{i=1}^k D_i = -aK_S$, for some rational number $a \in \mathbb{Q}$.

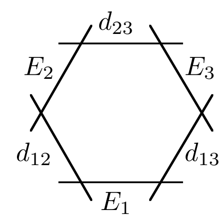
Since the D_i's are irreducible and rational, we deduce from the adjunction formula $D_i(K_S + D_i) = -2$ (Proposition 4.2) that $D_i \cdot K_S = -1$. Intersecting D_1 with $\sum_{i=1}^k D_i = -aK_S$, we find that a is an integer. Moreover, we get

$$K_S \cdot \textstyle\sum_{i=1}^k D_i = \sum_{i=1}^k K_S \cdot D_i = -k = K_S \cdot (-aK_S) = a(r-9).$$

Consequently, the degree $9 - r$ divides the size k of the orbit. □

10.1. **The del Pezzo surface of degree 6.** A del Pezzo surface S of degree 6 is the blow-up of three points of $\mathbb{P}^2$ which are not collinear. Up to isomorphism, these points can be chosen to be $p_1 = (1:0:0)$, $p_2 = (0:1:0)$ and $p_3 = (0:0:1)$, so the surface S is unique. We can view it in $\mathbb{P}^6$ as the image of the rational map $(x:y:z) \dashrightarrow (x^2y : x^2z : xy^2 : xyz : xz^2 : y^2z : yz^2)$, given by the linear system of cubics passing through p_1, p_2 and p_3. We may also view it in $\mathbb{P}^2 \times \mathbb{P}^2$, defined as $\{(x:y:z) \times (u:v:w) \mid ux = vy = wz\}$, where the blow-down is the projection on one copy of $\mathbb{P}^2$, explicitly $p : (x:y:z) \times (u:v:w) \mapsto (x:y:z)$. Denoting by $d_{ij} \subset S$ the strict transform of the line passing through p_i and p_j, the six (-1)-curves of S are given by

$$\begin{aligned}
E_1 &= \{(1:0:0) \times (0:a:b) \mid (a:b) \in \mathbb{P}^1\} = p^{-1}(p_1),\\
E_2 &= \{(0:1:0) \times (a:0:b) \mid (a:b) \in \mathbb{P}^1\} = p^{-1}(p_2),\\
E_3 &= \{(0:0:1) \times (a:b:0) \mid (a:b) \in \mathbb{P}^1\} = p^{-1}(p_3),\\
d_{23} &= \{(0:a:b) \times (1:0:0) \mid (a:b) \in \mathbb{P}^1\},\\
d_{13} &= \{(a:0:b) \times (0:1:0) \mid (a:b) \in \mathbb{P}^1\},\\
d_{12} &= \{(a:b:0) \times (0:0:1) \mid (a:b) \in \mathbb{P}^1\}.
\end{aligned}$$



Lemma 10.5.
- *The (-1)-curves of S form a hexagon: it is connected and each curve intersects two others.*
- *The action of $\mathrm{Aut}(S)$ on the hexagon gives rise to the exact sequence*

$$1 \to (\mathbf{k}^*)^2 \to \mathrm{Aut}(S) \xrightarrow{\rho} \mathrm{Sym}_3 \times \mathbb{Z}/2\mathbb{Z} \to 1.$$

- *The exact sequence splits and $\mathrm{Aut}(S) = (\mathbf{k}^*)^2 \rtimes (\mathrm{Sym}_3 \times \mathbb{Z}/2\mathbb{Z})$, where:*
  - *$(\mathbf{k}^*)^2$ is generated by automorphisms of the form $(x:y:z) \times (u:v:w) \mapsto (x:\alpha y:\beta z) \times (\alpha\beta u : \beta v : \alpha w)$, $\alpha, \beta \in \mathbf{k}^*$.*
  - *$(\mathbf{k}^*)^2 \rtimes \mathrm{Sym}_3$ is the lift on S of the group of automorphisms of $\mathbb{P}^2$ that leave invariant the set $\{A_1, A_2, A_3\}$.*
  - *$\mathbb{Z}/2\mathbb{Z}$ is generated by the automorphism*
  $$(x:y:z) \times (u:v:w) \mapsto (u:v:w) \times (x:y:z),$$
  which corresponds to the standard quadratic transformation $(x:y:z) \dashrightarrow (yz:xz:xy)$ of $\mathbb{P}^2$. It exchanges E_i and d_{jk}, for $\{i,j,k\} = \{1,2,3\}$.
  - *Sym_3 acts on the torus $(\mathbf{k}^*)^2$ by permuting the coordinates, and the action of $\mathbb{Z}/2\mathbb{Z}$ is the inversion.*

Proof. The first assertion follows directly from the description of exceptional divisors given above. By rotating the hexagon we find E_1, d_{12}, E_2, d_{23}, E_3, d_{13} and then E_1 again.

As $\mathrm{Aut}(S)$ preserves the exceptional divisors and the intersection form, it must preserve the hexagon. So the action of $\mathrm{Aut}(S)$ on the hexagon gives rise to a homomorphism

$$\rho : \mathrm{Aut}(S) \to \mathrm{Sym}_3 \times \mathbb{Z}/2\mathbb{Z}.$$

As any element of the kernel leaves invariant every exceptional divisor, it comes from an automorphism of $\mathbb{P}^2$ that fixes the three points p_1, p_2 and p_3. The kernel of ρ thus consists of automorphisms of the form $(x:y:z) \times (u:$

$v : w) \mapsto (x : \alpha y : \beta z) \times (\alpha\beta u : \beta v : \alpha w)$, with $\alpha, \beta \in \mathbf{k}^*$, and is the lift of the torus $\mathcal{T}$ of diagonal automorphisms of $\mathbb{P}^2$.

Note that the group Sym_3 of permutations of the variables x, y and z (and the corresponding variables u, v and w), generated by the two automorphisms

$$(x : y : z) \times (u : v : w) \mapsto (y : x : z) \times (v : u : w),$$
$$(x : y : z) \times (u : v : w) \mapsto (z : y : x) \times (w : v : u),$$

is sent by ρ on Sym_3. The group generated by the automorphism

$$(x : y : z) \times (u : v : w) \mapsto (u : v : w) \times (x : y : z)$$

is sent by ρ on $\mathbb{Z}/2\mathbb{Z}$. This gives the surjectivity of ρ and an obvious section. The other assertions are evident. $\square$

Proposition 10.6. *Let S be the del Pezzo surface of degree 6 and let $G \subset \mathrm{Aut}(S)$. The pair (G, S) is minimal if and only if the action on the hexagon of (-1)-curves is transitive.*

Proof. If the action is transitive, no orbit of (-1)-curves can be contracted so (G, S) is minimal. If the action is not transitive, there are two or three disjoint (-1)-curves of the hexagon which are invariant, so (G, S) is not minimal. $\square$

There are plenty of possibilities for the pair (G, S) above. One example is a group isomorphic to Sym_4 [Do-Iz09, §6.2].

Exercise 4. *On the del Pezzo surface S of degree 6, show that there is only one finite cyclic group $G \subset \mathrm{Aut}(S)$ such that (G, S) is minimal, up to conjugation in $\mathrm{Aut}(S)$. Show that this group is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ by a birational map $S \dashrightarrow \mathbb{P}^2$. (The answer to this question can be found in* [Bla09b, Lemma 9.7]*).*

10.2. **The del Pezzo surface of degree 5.** As for the del Pezzo surface of degree 6, there is a single isomorphism class of del Pezzo surfaces of degree 5. Consider the del Pezzo surface S_5 of degree 5 defined by the blow-up $p : S_5 \to \mathbb{P}^2$ of the points $p_1 = (1 : 0 : 0)$, $p_2 = (0 : 1 : 0)$, $p_3 = (0 : 0 : 1)$ and $p_4 = (1 : 1 : 1)$. There are 10 exceptional divisors on S_5, namely the divisor $E_i = p^{-1}(p_i)$, for $i = 1, ..., 4$, and the strict pull-back d_{ij} of the line of $\mathbb{P}^2$ passing through p_i and p_j, for $1 \le i < j \le 4$. There are 5 sets of 4 skew exceptional divisors on S_5, namely

$$F_1 = \{E_1, d_{23}, d_{24}, d_{34}\}, \quad F_2 = \{E_2, d_{13}, d_{14}, d_{34}\}, \quad F_3 = \{E_3, d_{12}, d_{14}, d_{24}\},$$
$$F_4 = \{E_4, d_{12}, d_{13}, d_{23}\}, \quad F_5 = \{E_1, E_2, E_3, E_4\}.$$

Proposition 10.7. *The action of $\mathrm{Aut}(S_5)$ on the five sets $F_1, ..., F_5$ of four skew exceptional divisors of S_5 gives rise to an isomomorphism*

$$\rho : \mathrm{Aut}(S_5) \to \mathrm{Sym}_5.$$

Furthermore, the actions of Sym_n, $\mathrm{Alt}_m \subset \mathrm{Aut}(S_5)$ on S_5 given by the canonical embedding of these groups into Sym_5 are fixed-point free if and only if $n = 3, 4, 5$, respectively $m = 4, 5$.

Proof. Since any automorphism in the kernel of ρ leaves E_1, E_2, E_3 and E_4 invariant and hence is the lift of an automorphism of $\mathbb{P}^2$ that fixes the 4 points, the homomorphism ρ is injective.

We now prove that ρ is also surjective. Firstly, the lift of the group of automorphisms of $\mathbb{P}^2$ that leave the set $\{p_1, p_2, p_3, p_4\}$ invariant is sent by ρ on $\mathrm{Sym}_4 = \mathrm{Sym}_{\{F_1,F_2,F_3,F_4\}}$. Secondly, the lift of the standard quadratic transformation $(x : y : z) \dashrightarrow (yz : xz : xy)$ is an automorphism of S_5, as its lift on S_6 is an automorphism, and as it fixes the point p_4; its image by ρ is $(F_4\ F_5)$. $\square$

Remark 10.8. *The structure of $\mathrm{Aut}(S_5)$ is classical and can be found for example in* [Wim96] *and* [Do-Iz09].

Exercise 5. *Show that the cyclic subgroups G of order 5 of $\mathrm{Aut}(S_5)$ (which are all conjugate) are such that (G, S) is minimal. Prove that however G is conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ by a birational map $S \dashrightarrow \mathbb{P}^2$ (as before, the answer can be found in* [Bla09b, Lemma 9.8]*)*

Proposition 10.9. *Let S be the del Pezzo surface of degree 5 and let $G \subset \mathrm{Aut}(S)$. The pair (G, S) is minimal if and only if the action of G on the set $\{F_1, \ldots, F_5\}$ is transitive.*

Proof. If the action is transitive, G contains an element of order 5 and (G, S) is minimal by Exercise 5. If the action is not transitive, one of the sets F_i is invariant and can thus be contracted, so (G, S) is not minimal. $\square$

10.3. **Del Pezzo surfaces of degree 4.** Let S be a smooth del Pezzo surface of degree 4, which is the blow-up of 5 points $p_1,\dots,p_5\in\mathbb{P}^2$ such that no 3 are collinear. Here the isomorphic classes are infinite (of dimension 1). The anti-canonical morphism induced by $|-K_S|$ gives rise to an embedding into $\mathbb{P}^4$ and the image is the intersection of two quadrics ([Hos96] and [Bea07]). The surface is in fact given by $\sum_{i=0}^5 x_i^2=\sum_{i=0}^5\lambda_i x_i^2=0$ for some λ_i general enough so that the surface is smooth [Do-Iz09, Lemma 6.1]. We then find that $\mathrm{Aut}(S)$ contains a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$ given by the diagonal 2-torsion of $\mathrm{PGL}(5,\mathbf{k})$. The group $\mathrm{Aut}(S)$ is in fact isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4\rtimes D$ where D is the subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ preserving the 5 points [Bla09b, Lemma 9.11].

There are many interesting examples of subgroups $G\subset\mathrm{Aut}(S)$ with (G,S) minimal (see [Do-Iz09, Theorem 6.9]).

10.4. **Del Pezzo surfaces of degree 3.** Let S be a smooth del Pezzo surface of degree 3, which is the blow-up of 6 points $p_1,\dots,p_6\in\mathbb{P}^2$ such that no 3 are collinear and no 6 lie on the same conic. The anti-canonical morphism induced by $|-K_S|$ gives rise to an embedding into $\mathbb{P}^3$ and the image is a smooth cubic surface and all smooth cubic surfaces are obtained by this way [Kol96, Theorem III.3.5]. In general, $\mathrm{Aut}(S)$ is trivial, but there are plenty of examples of cubic surfaces with non-trivial groups of automorphisms [Do-Iz09, Table 4].

Exercise 6. *Using that a smooth cubic surface S is the blow-up of 6 points of $\mathbb{P}^2$ in general position, show that it contains exactly 27 lines, which are the 27 (-1)-curves of S.*

10.5. **Del Pezzo surfaces of degree 2.** Let S be a smooth del Pezzo surface of degree 2, which is the blow-up of 7 points $p_1,\dots,p_7\in\mathbb{P}^2$ such that no 3 are collinear and no 6 lie on the same conic. The anti-canonical morphism $\rho\colon S\to\mathbb{P}^2$ is surjective but of course not an isomorphism. Because $(K_S)^2=2$, it is a double covering. The preimage of a general line corresponds to a smooth cubic through the 7 points and is thus of genus 1. By Riemann-Hurwitz, the restriction of the double covering to the curve has four ramification points. This shows that the ramification curve of ρ is a quartic, and is in fact smooth because S is smooth. One can thus see S as a surface given by

$$w^2=F_4(x,y,z)$$

in a weighted projective space $\mathbb{P}(2,1,1,1)$ (which is the set of equivalence classes on $\mathbf{k}^4\backslash\{0\}$, where $(w,x,y,z)\sim(\lambda^2w,\lambda x,\lambda y,\lambda z)$ for $\lambda\in\mathbf{k}^*$).

Moreover, all smooth quartic curves are obtained by this way (see [Kol96, Theorem III.3.5], and [KSC04, Corollary 3.54]). We can also see that the 56 (-1)-curves of S correspond to the 28 bitangents of the smooth quartic. The involution ι_G of S given by the double covering is called *Geiser involution*. In fact, we have an exact sequence

$$1\to<\iota_G>\to\mathrm{Aut}(S)\to\mathrm{Aut}(\Gamma)\to 1$$

where $\mathrm{Aut}(\Gamma)$ is the group of automorphisms of the curves, and also the group of automorphisms of $\mathbb{P}^2$ that preserve the quartic (by adjunction formula, the canonical divisor of Γ is the trace of an hyperplane so any automorphism of Γ extends to $\mathbb{P}^2$).

The study of automorphisms of quartic curves is a classical study. The description of automorphisms of del Pezzo surfaces can be deduced from this work (see [Do-Iz09, §6.6]).

10.6. **Del Pezzo surfaces of degree 1.** Let S be a smooth del Pezzo surface of degree 1, which is the blow-up of 8 points $p_1,\dots,p_8\in\mathbb{P}^2$ such that no 3 are collinear, no 6 lie on the same conic and not all lie on the same cubic with a singular point at one of them.

The anti-canonical morphism $|-K_S|$ gives a rational map $S\dashrightarrow\mathbb{P}^1$ whose fibres correspond to the cubics of $\mathbb{P}^2$ through the 8 points $p_1,\dots,p_8$. This pencil has one ninth base-point (because two cubics intersect into 9 points), which is a special point of S fixed by any automorphism, and is the unique base-point of the anti-canonical map $S\dashrightarrow\mathbb{P}^1$.

The linear system $|-2K_S|$ induces a morphism to $\mathbb{P}^3$, which lies in a quadric cone Q. The restriction yields a double covering of Q, ramified over the vertex v of Q and a smooth curve C of genus 4. Moreover C is the intersection of Q with a cubic surface. (See [Ba-Be00], [dFe04], [Do-Iz09].)

Note that a quadric cone is isomorphic to the weighted projective plane $\mathbb{P}(1,1,2)$ and the ramification curve C has equation of degree 6 there. Up to a change of coordinates, we may assume that the surface S has the equation

$$w^2=z^3+F_4(x,y)z+F_6(x,y)$$

in the weighted projective space $\mathbb{P}(3,1,1,2)$, where F_4 and F_6 are forms of respective degree 4 and 6 (see [Kol96, Theorem III.3.5], and [KSC04, Corollary 3.54]). Note that multiple roots of F_6 are not roots of F_4, since S is non-singular. The point $v=(1:0:0:1)=(-1:0:0:1)$ is the vertex of the quadric.

We denote by $\sigma(w:x:y:z)=(-w:x:y:z)$ the involution associated to the 2-covering. This is classically called the *Bertini* involution of the surface.

If $F_4 = 0$, the surface is a triple covering of $\mathbb{P}(3,1,1)$, ramified over v and the hyperelliptic curve of genus 2 of equation $z = 0$, $w^2 = F_6(x,y)$. In this case we denote the automorphism of order 3 corresponding to this covering by $\rho(w : x : y : z) = (w : x : y : \omega z)$ (where ω is a 3-rd root of unity).

The family of del Pezzo surfaces of degree 1 are those with the more complicated group of automorphisms. These are classified in [Do-Iz09, §6.7].

11. Conjugacy classes between examples

After studying possible minimal pairs (G,S), one has to determine if two pairs (G,S) and (G,S') are birationally conjugate, which means that there exists a birational map $\varphi\colon S \dashrightarrow S'$ which is G-equivariant.

There are two main tools for this:

11.1. **Curves of fixed points.** If $g \in \mathrm{Aut}(S)$ pointwise fixes an irreducible (smooth) curve Γ of genus > 0 (i.e. a non-rational curve) and $\varphi : S \dashrightarrow S'$ is a birational map such that $g' = \varphi g \varphi^{-1} \in \mathrm{Aut}(S')$, the image of Γ is a curve Γ' that is also pointwise fixed by g'. The fact that Γ has genus > 0 is important because it implies that Γ is not contracted by φ. Moreover, φ restricts to a birational map $\Gamma \dashrightarrow \Gamma'$, which is necessarily an isomorphism because both curves are smooth.

This is an important tool to decide when two elements are conjugate.

Example 11.1. *Let $\Gamma \subset \mathbb{P}^2$ be a smooth cubic curve given by $F(x,y,z) = 0$. Let $S_F \subset \mathbb{P}^3$ be the smooth cubic surface given by $w^3 = F(x,y,z)$ and let $G_F \subset \mathrm{Aut}(S_F)$ be the cyclic group of order 3 given by $\{(w : x : y : z) \mapsto (\omega w : x : y : z) \mid \omega^3 = 1\}$. The curve of S_F of equation $w = 0$, $F(x,y,z)$ is pointwise fixed by G_F.*

Taking all different cubic curves yields infinitely many conjugacy classes of subgroups of order 3 of $\mathrm{Bir}(\mathbb{P}^2)$. In fact, the conjugacy classes in this family are parametrised by the isomorphism classes of the curves.

Exercise 7. *Let $\alpha \in \mathbf{k}^*$ be a k-th root of unity for some integer $k \geq 2$. Let $g, h \in \mathrm{Aut}(\mathbb{P}^2)$ be given by $g\colon (x : y : z) \mapsto (x : \alpha y : \alpha^2 z)$ and $h\colon (x : y : z) \mapsto (x : \alpha y : z)$. Prove that g, h are conjugate in $\mathrm{Bir}(\mathbb{P}^2)$. Compare the fixed locus of g and h when $k \geq 3$.*

Proposition 11.2 ([Ba-Be00], [dFe04], [Be-Bl04])**.** *Let $G_1, G_2 \subset \mathrm{Bir}(\mathbb{P}^2)$ be two subgroups of prime order p. The groups G_1, G_2 are conjugated in $\mathrm{Bir}(\mathbb{P}^2)$ if and only if one of the following occur:*

(1) *Both G_1 and G_2 pointwise fix the the same irreducible curve of positive geometric genus;*
(2) *There is not irreducible curve of positive genus fixed by G_1 or G_2 (in which case both are conjugate to $\{(x : y : z) \mapsto (\alpha x : y : z) \mid \alpha^p = 1\}$)*

This result has being generalised in [Bla11, Theorem 1] to any cyclic group of finite order. The conjugacy classes are given by the curves of positive genus pointwise fixed by the non-trivial elements of the group *and* by the action of the group on these curves.

11.2. **G-elementary links.** Another important tool is provided by the study of G-elementary links, which are generalisations of the links we described before.

We deal with two sets of G-surfaces:

($\mathbb{D}$) A pair (G,S), where S is a smooth projective rational surface, $G \subset \mathrm{Aut}(S)$ is a finite group such that $\mathrm{Pic}(S)^G$ has rank 1.
($\mathbb{C}$) A triple (G,S,π), where S is a smooth projective rational surface, (S,π) is a conic bundle and $G \subset \mathrm{Aut}(S,\pi)$ is such that $\mathrm{Pic}(S)^G$ has rank 2.

Remark 11.3. 1) *If (G,S) is a minimal pair, then either (G,S) is in $(\mathbb{D})$ or (G,S,π) is in $(\mathbb{D})$ for some conic bundle structure (S,π) on S (Proposition 8.5).*

2) *The two sets $(\mathbb{D})$ and $(\mathbb{C})$ correspond to distinct pairs (G,S) because of the assertion on the rank of the invariant part of the Picard group.*

3) *If (G,S) is a pair in $(\mathbb{D})$, then S is a del Pezzo surface (take the orbit of a curve and intersect it with the canonical divisor, you find a positive number) and the pair (G,S) is minimal (we cannot contract anything G-equivariantly otherwise the invariant Picard group would have a rank which decrease: impossible).*

4) *If (G,S,π) is a triplet in $(\mathbb{C})$, then it is possible that (G,S) is not minimal: take for example $S = \mathbb{F}_1$, $\pi = \pi_1$ and any finite group $G \subset \mathrm{Aut}(\mathbb{F}_1)$. We have then enlarged the set of minimal pairs.*

5) *If (G,S,π) is a triplet in $(\mathbb{C})$, it is possible that S is a del Pezzo surface (see Example 11.6 below).*

We can define as before four types or elementary links. All such links are birational maps $\varphi\colon (G,S) \dashrightarrow (G,S')$ of pairs (i.e. a G-equivariant birational map $S \dashrightarrow S'$):

***G*-links of type** I: we have $(G,S) \in (\mathbb{D})$, $(G',S',\pi') \in (\mathbb{C})$ and the map φ is the blow-up of one orbit of points of S under the action of G.

G-**links of type** II**:** the map φ decomposes as $\varphi = (\eta') \circ (\eta)^{-1}$, where

$$\begin{tikzcd} & Z \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ S \arrow[rr, dashed, "\varphi"'] & & S' \end{tikzcd}$$

is a minimal resolution of φ and where η and η' are blow-ups of a G-orbit (of points of S and S' respectively). Moreover, either $(G, S) \in (\mathbb{D})$, $(G', S') \in (\mathbb{D})$ or $(G, S, \pi) \in (\mathbb{C})$, $(G', S', \pi) \in (\mathbb{C})$. In this latter case, we moreover ask that there exist an automorphism α of $\mathbb{P}^1$ such that $\pi\eta = \alpha\pi'\eta'$ (this means that φ sends a general fibre of π onto a general fibre of π').

G-**links of type** III**:** as in the case without group, these are inverse of links of type I: we have $(G, S, \pi) \in (\mathbb{C})$, $(G', S') \in (\mathbb{D})$ and the map φ is a birational morphism which is the blow-up of one orbit of points of S' under the action of G.

G-**links of type** IV**:** we have $(G, S, \pi) \in (\mathbb{C})$, $(G', S', \pi') \in (\mathbb{C})$ and the map φ is an isomorphism $S \to S'$ which is not compatible with π and π' (we take another conic bundle structure on the same surface).

Remark 11.4. 1) *If we take G to be the identity, all G-elementary links above are the classical elementary links defined before.*

2) *In the case where G is the identity, a link of type* II *cannot go from $\mathbb{D}$ to $\mathbb{D}$ because only $(G, \mathbb{P}^2)$ belongs to $\mathbb{D}$. But with a non-trivial G, such a link exists (see Example 11.5 below).*

Example 11.5. *Let G be the group with 2 elements, which acts on $\mathbb{P}^2$ via $(x : y : z) \mapsto (x : z : y)$ and acts on $\mathbb{P}^1 \times \mathbb{P}^1$ via $((a : b), (c : d)) \mapsto ((c : d), (a : b))$. Both pairs $(G, \mathbb{P}^2)$ and $(G, \mathbb{P}^1 \times \mathbb{P}^1)$ belong to $(\mathbb{D})$. The birational map $\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by $(x : y : z) \dashrightarrow ((x : y), (x : z))$ is a G-link of type* II*: it decomposes as $\varphi = (\eta') \circ (\eta)^{-1}$, where $\eta \colon Z \to \mathbb{P}^2$ is the blow-up of the two points $(0 : 0 : 1)$, $(0 : 1 : 0)$ and $\eta' \colon Z \to \mathbb{P}^1 \times \mathbb{P}^1$ is the blow-up of the fixed point $((0 : 1), (0 : 1))$.*

Example 11.6. *Let $G \subset \mathrm{Aut}(\mathbb{P}^2)$ be a finite group admitting an orbit consisting of 4 points such that no 3 are collinear. The blow-up $\eta \colon S \to \mathbb{P}^2$ of the four points yields a link of type* I *given by η^{-1}. The triplet (G, S, π) where the conic bundle $\pi \colon S \to \mathbb{P}^1$ a morphism with fibres being the strict transform of the conic through the four points is in $\mathcal{C}$. In this case, S is the del Pezzo surface of degree 5.*

The following result

Proposition 11.7 ([Cor95])**.** (*see also* [Isk96, Theorem 2.5]) *For $i = 1, 2$, let (G, S_i) be a pair such that either $(G, S_i) \in (\mathbb{D})$ or $(G, S_i, \pi_i) \in \mathbb{C}$ for some conic bundle π_i on S_i. Any birational map $(G, S_1) \dashrightarrow (G, S_2)$ decomposes into G-elementary links of type* I, II, III, IV.

A precise combinatorial description of the possible G-links is given in [Isk96, Theorem 2.6] (whose statement takes 7 pages): The values of $(K_S)^2$, $(K_{S'})^2$ and the size of the G-orbit involved in each links are given. The same kind of result can be found in [Do-Iz09, Propositions 7.13, 7.14,7.15].

In particular, we have the following:

Proposition 11.8. ([Do-Iz09, Corollary 7.11]) *Let $(G, S) \in (\mathbb{D})$ be such that every G-orbit on S consists of at least $(K_S)^2$ points. Then the pair (G, S) is supperrigid: every birational map $(G, S) \dashrightarrow (G', S')$, where (G', S') is another pair in $(\mathbb{D})$ or $(\mathbb{C})$, is an isomorphism.*

Let $(G, S, \pi) \in (\mathbb{C})$ be such that $(K_S)^2 < 0$. Then (G, S) is supperrigid.

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