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# Algebraic contact manifolds, their generalizations and applications

*PhD dissertation*

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Author's declaration:
I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Supervisors' declaration:
The dissertation is ready to be reviewed.

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## Abstract

This dissertation revolves around smooth complex algebraic varieties equipped with a holomorphic contact structure (usually called *contact manifolds*). We sketch the historical motivation to study them coming from the Riemannian geometry along with the conjecture of LeBrun and Salamon and present selected results concerning their classification. In particular, we discuss classical works concerning the Mori theory of projective contact manifolds and very recent results exploiting action of the torus. These theorems require assuming that fundamental linear systems of some Fano varieties are big enough. Consequently, we present state-of-the-art answers for the questions concerning dimensions of such systems for mildly singular, high-index Fano varieties.

Then, we move to considerations concerning some natural generalizations of the notion of a contact manifold. Our focus is on the case where we allow the variety to have some mild singularities. This situation is parallel to the theory of symplectic singularities, so we define our notion of *a singular contact variety* and study it having in mind the correspondence between contact and symplectic manifolds.

Finally, we explain how contact structures arise in the theory of partial differential equations. We focus on the case of a symplectic Monge-Ampère equation in dimension 3 over the field of complex numbers and show equivalence between the Hitchin moment map and Kushner-Lychagin-Rubtsov invariant and how they determine the cocharacteristic variety of the equation.



**Keywords:** Algebraic contact manifold, Fano variety, fundamental divisor, nonvanishing, holomorphic contact structure, symplectic singularity, nilpotent orbit, Lagrangian Grassmannian, Monge-Ampère equation

**AMS MSC 2020 classification:** 14M99, 14C20, 14C40, 14E15, 14E30, 14J30, 14J42, 14J45, 14M17, 35A30, 58J70, 58A20

## Streszczenie

Niniejsza rozprawa porusza problematykę gładkich zespolonych rozmaitości algebraicznych wyposażonych w holomorficzną strukturę kontaktową. Przedstawiamy historyczną motywację do badania ich pochodzącą z geometrii riemannowskiej i hipotezę LeBruna-Salamona oraz wybrane wyniki dotyczące klasyfikacji tych obiektów. W szczególności, omawiamy teorię Mori rzutowych rozmaitości kontaktowych wraz z niedawno rozwiniętą metodą wykorzystującą działanie na rozmaitości algebraicznego torusa. Zastosowanie tej metody wymaga założeń dotyczących fundamentalnych systemów liniowych pewnych rozmaitości Fano. Wobec tego, omawiamy aktualny stan wiedzy w dziedzinie badania takich systemów dla osobliwych rozmaitości Fano o dużym indeksie.

W dalszej części rozważamy pewne uogólnienia pojęcia rozmaitości kontaktowej. Koncentrujemy się na sytuacji w której rozważana rozmaitość dopuszcza wymierne osobliwości (*osobliwe rozmaitości kontaktowe*). Ta idea ma odpowiednik w klasycznej teorii osobliwości symplektycznych, wobec tego badamy podstawowe własności tych obiektów, mając na uwadze odpowiedniość między rozmaitościami kontaktowymi i symplektycznymi.

Wreszcie omawiamy jak struktury kontaktowe pojawiają się w naturalny sposób w teorii równań różniczkowych cząstkowych. Przedstawiamy bardziej szczegółowo przypadek równania Monge'a-Ampère'a w trzech wymiarach nad ciałem liczb zespolonych i pokazujemy równoważność między odwzorowaniem momentu Hitchina i niezmiennikiem Kushnera-Lychagina-Rubtsova i jak wyznaczają one rozmaitość kocharakterystyczną równania.

**Słowa kluczowe:** algebraiczna rozmaitość kontaktowa, rozmaitość Fano, dywizor fundamentalny, nieznikanie, holomorficzna struktura kontaktowa, osobliwość symplektyczna, orbita nilpotentna, Lagranżowski grassmannian, równanie Monge-Ampère'a

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# Contents

CHAPTER 1

# Introduction

The study of smooth complex projective varieties equipped with a holomorphic contact structure (from now on called simply *projective contact manifolds*) is a rich subject, utilizing different tools, especially those developed for varieties of negative Kodaira dimension. In particular, the seminal result in the theory [**KPSW00**, Thm 1.1] comes from an application of the Mori theory, and a promising modern approach [**BW22**] relies on the methods analysing actions of tori. Moreover, there exists a link between the contact and symplectic geometry provided by the construction of the symplectization. Nevertheless, the problem of the full classification of projective contact manifolds remains open in the Fano case, which is the most interesting one. The only known – and conjecturally the only existing – examples are homogeneous coadjoint varieties and this scarcity makes the study of contact manifolds challenging.

The aforementioned methods exploiting torus actions use lower bounds on the dimension of the fundamental linear system for prime and smooth Fano manifolds (in this particular case, this is the linear system of the ample generator of the Picard group). Computing such bounds is an interesting problem on its own, and positive answers are of utmost importance for the general theory of Fano varieties. Moreover, proving the nonemptiness of this system can be seen as a particular case of the effective nonvanishing conjecture of Kawamata and Ambro. As of today, we only have partial results obtained for the large values of index, with the most recent being an original work of the author (Theorem 3.3.8). It is enough to conclude that the nonvanishing conjecture holds for all Fano varieties of dimension 5 or less, but in higher dimensional cases the problem remains open.

Taking a different point of view, adjoint varieties can be seen as a part of a bigger geometric picture. They arise as projectivizations of minimal nilpotent orbits of the adjoint action for a simple Lie group. Nilpotent orbits of semisimple Lie groups are a classical object of study, in part because they constitute model examples of (singular) symplectic varieties. On the other hand, their projectivizations and the contact structures on them have not attracted a comparable interest of the community and there exist only a handful of isolated results. Consequently, there is room to explore possible generalizations of the notion of a contact manifold to some singular setting, using projectivized orbits as examples and both the theory of symplectic varieties and contact manifolds as guides. The author defines one such generalization that fits in the described picture (Definition 5.3.1) and shows its basic properties, along with pointing out some obstacles that arise in the singular setting. However, this constitutes only the first step on the way and there are numerous questions that can be investigated.

## 1.1. Overview of this dissertation and some original results

Although complex contact structures constitute a major theme of the dissertation, we begin our discussion in Section 1.3 by providing the historical context that comes from the Riemannian geometry. Namely, we present the conjecture of LeBrun and Salamon [**LS94**] and its link to Fano contact manifolds. As the author is by no means an expert in this field, the role of this section – besides providing a background – is to motivate some additional assumptions that appear in the theory.

Chapter 2 serves as an extension of the introduction. Namely, it begins with a brief discussion of basic classes of singularities and their properties that are used through the dissertation. The main purpose of this chapter is to present the seminal results of the Mori theory of projective contact manifolds and it does not contain any original results. The main theorem of the chapter is the amalgam of the works of Druel [**Dru99**], Kebekus-Peternell-Sommese-Wiśniewski [**KPSW00**] and Demailly [**Dem02**] that provides a full classification of possible Mori contractions of projective contact manifolds along with a list of all projective toric varieties that admit a contact structure (Theorem 2.3.2).

In Chapter 3 we make a brief detour from the contact structures and study linear systems associated to some ample divisors on general Fano varieties, allowing mild singularities. Besides the anticanonical one, we are mainly interested in the linear system associated to the fundamental divisor, i.e. an ample divisor $L$ that is not divisible in the Picard group and whose multiplicity is the anticanonical divisor, see Definition 3.1.6 for a more precise discussion. We present methods and computations used to estimate the dimension of such linear system (equivalently: the dimension of the space of sections of the associated line bundle) and restrict possible singularities of a general element. Both of those problems are open in full generality, but there are partial results in low dimensions or for high value of index. First presented cases are classical or were obtained earlier by other researchers, but we also show the following result that was proved by Höring and the author [**HŚ20**]:

Theorem (3.3.7). *Let $Y$ be a Gorenstein-Fano variety of dimension 5 with canonical singularities. Then we have $h^0(Y, -K_Y) \geq 4$. If a general element $D \in |-K_Y|$ is reduced, then it has canonical singularities.*

This theorem allows in particular to claim a corollary, whose usefulness is twofold: it is a special case of the effective nonvanishing conjecture of Kawamata and Ambro, and it can be used in the study of contact manifolds, as we explain in Chapter 4:

Corollary (3.3.6). *Let $Y$ be a Gorenstein-Fano variety of dimension at most 5 with canonical singularities and $L$ be its fundamental divisor. Then $h^0(Y, L) \geq 2$.*

We also prove the following original result, that can be seen as a partial generalization of the theorem obtained in collaboration with Höring:

Theorem (3.3.8). *Let $Y$ be a smooth Fano variety of dimension $n$ and index $i_Y = n-4$ with $\mathrm{Pic}(Y) = \mathbb{Z} \cdot L$ for $L$ - ample. Then we have:*

$$h^0(Y, L) \geq n - 1.$$

Consequently, presented results constitute the state of the art for the problem of existence of sections for the fundamental linear system and the intention of the author was to give a presentation that would be explanatory for a non-expert. We also use discussed results to give a slightly different proof showing the lower bound for the dimension of an automorphism group of contact Fano manifolds in dimensions up to 9 (Corollary 3.5.1) that was first proved by Buczyński, Weber and Wiśniewski [**BW22**, Thm 6.1]). We conclude the chapter by proposing a variant of the effective non-vanishing conjecture for smooth and prime Fano manifolds and discussing additional examples supporting it with arbitrary index (Conjecture 4).

In Chapter 4 we discuss the role played by the estimates on the dimension of the fundamental linear system in theory of contact manifolds. We concentrate on the methods based on torus action, developed very recently by Buczyński, Occhetta, Romano, Solá Conde, Weber and Wiśniewski in [**BW22**], [**RW22**] and [**ORCW21**], as they were the original motivation for the author's research reported in Chapter 3. Although the torus action methods are interesting on their own, a full survey is beyond the scope of this dissertation. Instead, our goal is to show how the machinery developed in the works mentioned above utilizes results discussed in Chapter 3 and how improving them would strengthen the evidence for the LeBrun-Salamon conjecture. The fruit of our considerations is the following theorem, whose formulation, but not the proof, is an original contribution of the author.

THEOREM (4.1.3). *Let $(X, L)$ be a Fano contact manifold of dimension $2n+1 \geq 3$ with a reductive group of automorphisms $G$. Assume that:*

- *$h^0(X, L) \neq 0$, i.e. the effective nonvanishing holds for $X$,*
- *for every smooth Fano variety $Y$ of dimension at most $n$ with $b_2(Y) = 1$ we have $h^0(Y, \mathcal{L}) \geq 2$, where $\mathcal{L}$ is the fundamental divisor on $Y$.*

*Then $G$ is a simple group and either:*

(1) *$X$ is the adjoint variety associated with $G$;*
(2) *$G = \mathrm{SL}(2)$, $h^0(X, L) = 3$, the maximal torus $\mathbb{C}^* \subset G$ acts on $X$ in such a way that the source and the sink of the action are isolated points and* $\dim(X) \geq 11$.

On the other hand, Chapter 5 introduces singular contact varieties, a natural generalization of the notion of the contact manifold that was proposed by the author. Consequently, Chapter 5 presents original and not yet published results, with the exception of introductory Section 5.2. It surveys some results from the theory of symplectic singularities that influenced and guided author's research. Additionally, we discuss the exploration of similar ideas in works of others, notably Campana-Flenner [**CF02**] and Fu [**Fu06**]. The first original result in this chapter is the following correspondence:

THEOREM (5.3.5). *If $X$ is a (singular) contact variety then there exists a principal $\mathbb{C}^*$-bundle over $X$ having the structure of a symplectic variety with a homogeneous symplectic form of weight 1. Going the other way around, if we have a principal $\mathbb{C}^*$-bundle over some base $Z$ such that the total space has symplectic singularities and the symplectic form is homogeneous of weight 1, then $Z$ is a singular contact variety.*

This correspondence can be used to prove an analogue of Kaledin's stratification for symplectic varieties:

THEOREM (5.3.8). *Let $X$ be a contact variety. Then there exists a finite stratification $X = X_0 \supset X_1 \supset \ldots \supset X_k$ such that:*

(1) *$X_{i+1}$ is the singular part of $X_i$.*
(2) *The normalization of every irreducible component of each stratum is a contact variety.*

The author also studies finite quotients of contact varieties and uses a recent result of Cao and Höring [**CH22**] to show that projective contact varieties are uniruled. Concerning birational morphisms, the author discusses the relation between being crepant and preserving the contact structure and in particular shows:

THEOREM (5.3.10, 5.3.13). *A birational morphism from a projective contact manifold is necessarily crepant. A resolution of singularities of projective singular contact variety is a contact manifold if it is crepant.*

These general results are then applied to provide a classification of projective singular contact threefolds:

THEOREM (5.4.3, 5.4.4). *Let $(X, L)$ be a projective singular contact variety in dimension 3. Then $X$ admits a crepant resolution $f\colon \mathbb{P}(TS) \to X$, where $S$ is a ruled surface over some smooth projective curve $B$. Going the other way around, if $S$ is any ruled surface over some smooth projective curve $B$, then we can associate to it a projective singular* *contact threefold $X$ with $\rho(X) = 2$ that is a locally trivial bundle over $B$ whose fibers are cones over rational curves. $X$ is constructed from $\mathbb{P}(TS)$ via the crepant contraction of the image of a section $\sigma\colon S \to \mathbb{P}(TS)$ onto a curve isomorphic to $B$.*

*Moreover, there exists a unique $X$ with $\rho(X) = 1$. Such $X$ is Fano and it is the projectivized nilpotent cone in the algebra $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$, obtained from $\mathbb{P}(T(\mathbb{P}^1 \times \mathbb{P}^1))$ by the crepant morphism contracting images of two sections onto two disjoint rational curves.*

The author was motivated to study such generalizations, as the class of contact manifolds is not closed with respect to taking quotients or birational modifications, so in particular one cannot easily construct examples. However, these operations are conditionally allowed in the framework proposed by the author.

Chapter 6 contains a presentation of elements of the geometric theory of partial differential equations and explains how contact structures (real analytic or complex holomorphic) arise naturally in this context. It is an active field of research and our demonstration focuses on some results that were obtained by Gutt, Manno, Moreno and the author **[GMMŚ21]** that concern symplectic Monge-Ampère equations in 3 dimensions. In particular, we discuss how in our case the (co)characteristic variety is determined in terms of geometric constructions. We can sum up original results presented in this chapter as follows:

THEOREM (6.3.10, 6.3.12). *For a symplectic Monge-Ampère equation $\mathcal{E}_\eta$ in dimension 3, which is a hyperplane section of the Lagrangian Grassmannian $\mathrm{LGr}(3, F)$ for the symplectic vector space $F$ of dimension 6, and $\eta \in \mathbb{P}(\Lambda^3_0(F^*))$, the projectivization of the Kushner-Lychagin-Rubtsov invariant coincides with the Hitchin moment map. Moreover, by the projective duality this invariant also determines the characteristic variety of $\mathcal{E}_\eta$.*

Both the KLR invariant and the Hitchin moment map can be considered as maps from the space of (effective) 3-forms on 6 dimensional, symplectic vector space $F$ to the space of quadratic forms on $F$ and were first defined and studied in the real case, whereas **[GMMŚ21]** focuses on the complex case.

## 1.2. Contact manifolds

We begin by giving a precise definition of objects around which the dissertation revolves:

DEFINITION 1.2.1. Let $X$ be a holomorphic manifold of dimension $2n + 1$ for $n \geq 0$. We say that it is a contact manifold if there exists a subbundle $F$ of the tangent bundle of rank $2n$ (i.e. *the contact distribution*) and a line bundle $L$ fitting into the exact sequence:

$$0 \to F \to TX \xrightarrow{\vartheta} L \to 0, \tag{1}$$

such that the induced morphism $d\vartheta\colon \Lambda^2 F \to L$ is nowhere degenerate. Equivalently, one can demand that $\vartheta \wedge (d\vartheta)^{\wedge n}$ as an element of $H^0(X, \Omega_X^{2n+1} \otimes L^{n+1})$, where $\Omega_X^{2n+1}$ is a sheaf of $2n + 1$ differential forms, has no zeroes. We will usually call $L$ from the definition *the contact line bundle* and $\vartheta$ *the (twisted) contact form.*

REMARK 1.2.2. We make a deliberate decision to include the one dimensional case in our considerations. Observe that on any smooth curve one may simply put $L = TX$, $\vartheta = \mathrm{id}$ and $F = 0$, so the existence of the contact structure does not give any information on the underlying variety (besides its smoothness). For this reason, in many works it is assumed (sometimes silently) that $2n + 1 \geq 3$ and we will do so explicitly when necessary. Our alternative convention is useful for a uniform formulation of statements concerning contact subvarieties in bigger contact varieties (manifolds), in particular Theorem 5.3.8.

REMARK 1.2.3. One can equally well define contact manifolds in real differentiable or algebraic setting by replacing *holomorphic manifold* by *smooth manifold* or *smooth* *algebraic variety.* The only reason for our choice was to be consistent with the seminal works of LeBrun and Salamon that are recalled in the next section. In fact, in the further course of the dissertation we usually consider contact manifolds in the complex algebraic category. These two are consistent in the projective case, on which we are focused.

## 1.3. Historical context and the LeBrun-Salamon conjecture

The researchers' interest in holomorphic contact manifolds originated from the Riemannian geometry, more precisely the study of holonomy groups. For a given connected Riemannian manifold $M$ and a point $m \in M$, the holonomy group $\mathcal{H}_m$ is a subgroup of orthogonal transformations of $T_m M$ generated by parallel transport of tangent vectors along smooth loops. One can consider $\mathcal{H}_m$ as a subgroup of $O(n)$ for $n = \dim(M)$ and then its conjugacy class does not depend on the choice of basis in $T_m M$ nor on the base-point $m$. Moreover, by using only homotopically trivial loops in the definition one obtains a restricted holonomy group $\mathcal{H}_0$ that is a subgroup of $SO(n)$. It was shown by Berger [**Ber55**] that, excluding locally reducible and locally symmetric manifolds, $\mathcal{H}_0$ has to be one of the following groups:

| dimension | group | geometry |
|---|---|---|
| n | SO(n) | general |
| $2n \geq 4$ | U(n) | Kähler |
| $2n \geq 4$ | SU(n) | Ricci-flat Kähler |
| $4n \geq 8$ | Sp(n) | hyperkähler |
| $4n \geq 8$ | Sp(n) · Sp(1) | quaternion-Kähler |
| 7 | $G_2$ | exceptional |
| 8 | Spin(7) | exceptional |

By $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ we mean the Lie group abstractly isomorphic to $[\mathrm{Sp}(n)\times\mathrm{Sp}(1)]/\mathbb{Z}_2$, which is obtained by combining left action of some quaternionic matrix $\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$ and right action of unit-length quaternions Sp(1). Note that Berger's list contains some natural inclusions: every listed group is a subgroup of $\mathrm{SO}(n)$ and hyperkähler manifolds can be considered as a subset of quaternion-Kähler manifolds. They are however usually excluded from discussion by demanding the nonvanishing of the scalar curvature – quaternion-Kähler manifold is hyperkähler if and only if its scalar curvature is zero.

The result of Berger brought natural questions about the existence of examples of all geometries, the most interesting being non-symmetric manifolds. Such examples were constructed for all possible $\mathcal{H}_0$, but in the case of (strictly) quaternion-Kähler manifolds only for the negative scalar curvature. On the other hand, the positivity of the scalar curvature and completeness force the manifold to be compact, and the only known examples are symmetric spaces listed by Wolf [**Wol65**], corresponding to compact simple Lie groups. Conjecturally, this list exhausts all possibilities:

CONJECTURE 1 (LeBrun-Salamon conjecture, differential version, [**LS94**]). The only complete Riemannian manifolds with quaternion-Kähler holonomy and positive scalar curvature are the symmetric Wolf spaces.

The link between a smooth complex variety $X$ equipped with a holomorphic contact structure and a quaternion-Kähler manifold $M$ is provided by the following construction:

CONSTRUCTION 1.3.1 (The twistor space). For a given quaternion-Kähler manifold $M$ of (real) dimension $4n$ consider the bundle $\mathrm{End}(TM)$, whose fiber over $m \in M$ is $\mathrm{End}(TM_m)$, i.e. the endomorphism bundle. Let us denote by $H$ the space $\mathbb{C}^2$ equipped with a standard representation of Sp(1) and an invariant and skew symmetric form $\omega_H$, so that we can identify $\mathfrak{sp}(1)$ with $S^2(H)$. Locally, although not necessarily globally, we can define an associated vector bundle $S^2\mathbf{H}$ that can be considered as a coefficient bundle of imaginary quaternions acting on the tangent space to $M$ at each point. In particular, over some open subset it embeds into $\mathrm{End}(TM)$. Moreover, $\omega_H$ defines a norm on $S^2\mathbf{H}$, and by restricting ourselves to elements with norm $\sqrt{2}$ we obtain a real 2-sphere bundle that is globally defined, and we denote its total space by $X$. Any element $j$ of $X$ satisfies $j^2 = -1$ as an endomorphism of the tangent bundle, so $j$ defines an almost complex structure on the tangent space that is compatible with $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$-structure of $M$. Salamon [**Sal82**, Thm 4.1] showed that $X$ is equipped with a natural complex structure, so it is a complex manifold of dimension $2n+1$, called *the twistor space* of $M$. Moreover, the fibers of $X \to M$ are in fact complex rational curves and $X$ has a holomorphic contact structure.

The positivity of the scalar curvature of $M$ implies that $X$ is equipped with the Kähler-Einstein metric of positive scalar curvature [**Bes07**, Thm 14.80]. It follows that $c_1(X) > 0$, so $X$ is a Fano variety, i.e. the anticanonical class $\mathcal{O}(-K_X)$ is ample and in particular $X$ is projective. Going the other way around, LeBrun [**LeB95**, Thm A] showed that if $X$ is a Fano contact manifold, then it is a twistor space of some quaternion-Kähler manifold of positive scalar curvature if and only if it admits a Kähler-Einstein metric.

The twistor space of a Wolf space is also homogeneous, with the action of a corresponding complex Lie group [**Wol65**, Thm 6.1]. More precisely, it is a projective coadjoint variety, associated with an action of a simple Lie group (see Section 4.2.2 for a brief discussion). Therefore, we can formulate the following conjecture, which is stronger than Conjecture 1 as we do not assume the existence of the Kähler-Einstein metric:

Conjecture 2 (LeBrun-Salamon conjecture, algebraic version, [**LS94**]). The only complex contact Fano manifolds are the coadjoint varieties.

From now on, we will be interested in this version of the conjecture. As we do not know any contact Fano manifolds not equipped with a Kähler-Einstein metric, one may also conjecture that they always exist on such manifolds. Nevertheless, some of the known results, in particular those in Chapter 4 assume the reductivity of the group of the automorphisms of the variety, which can be deduced from the existence of the Kähler-Einstein metric, as in [**Mat57**, Théorème 1]. We can wrap up different flavours of LeBrun-Salamon conjecture and the relations between them using the following flowchart:

| Conjectures in complex algebraic geometry | | Conjecture in Riemannian geometry |
|---|---|---|
| Every contact Fano manifold of dimension $2n+1$ ($n \geq 2$) with a Kähler-Einstein metric is coadjoint | $\Longleftrightarrow$ | Every quaternion-Kähler manifold of dimension $4n$ ($n \geq 2$) with a positive scalar curvature is a Wolf space |
| $\Uparrow$ | | |
| Every contact Fano manifold of dimension $2n+1$ ($n \geq 1$) with reductive group of automorphisms is coadjoint | | |
| $\Uparrow$ | | |
| Every contact Fano manifold of dimension $2n+1$ ($n \geq 1$) is coadjoint | | |

## 1.4. Notation and conventions

From Chapter 2 to Chapter 5 we work over the field of complex numbers $\mathbb{C}$ and our focus is on algebraic varieties, often projective. In this text, an algebraic variety is in particular irreducible, reduced and this convention encompasses varieties of dimension 1, i.e. curves. Moreover, a *rational curve* is a curve birational to $\mathbb{P}^1$. Sometimes we note that a particular object can be equally well defined in the complex holomorphic setting, or use/reference a result that was originally stated in the analytical category but this is indicated by the use of phrases like *holomorphic manifold, Kähler space* etc. Consequently, in these chapters, the word *manifold* without any other adjectives means *smooth algebraic variety.* We abandon this convention in Chapter 6, as the results presented therein are motivated by the theory of partial differential equations.

We do not distinguish between vector bundles and associated locally free sheaves, unless it is needed, and the same holds for line bundles and Cartier divisors. However, to make a distinction between Weil and Cartier divisors, we use additive notation for the former and multiplicative for the latter (but we often move freely between both descriptions if it is allowed, for instance in case of smooth varieties). In particular, $K_Y$ is a Weil canonical divisor, and if it has a corresponding Cartier divisor, we will write $\mathcal{O}(K_Y)$.

All projectivized vector bundles $\mathbb{P}(\mathcal{E})$ are taken in the sense of Grothendieck. For consistency of the presentation we extend this convention also to projectivizations of vector spaces.

The letter $L$ always means a line bundle. In the context of Fano varieties, so in particular in Chapter 3, it is used to denote the fundamental line bundle.

In the case of varieties, we reserve the letter $X$ for those equipped with the contact structure, possibly in some generalized sense (mind however that $X$ and $x$ can also denote vectors or coordinates, in particular in Section 4.2 and Chapter 6). In the context of contact structures, $F$ always means the contact distribution, and $\vartheta$ is always the twisted form. Moreover, in this context, the letter $L$ denotes the contact line bundle. This convention overrides the one for Fano varieties, if the considered variety is both Fano and contact and a conflict arises (but in the smooth projective case this can happen only if $X$ the projective space, as shown in Theorem 2.3.2). Depending on which elements of contact structure we want to emphasize, we will write either $(X, F)$ or $(X, F, L)$ or $(X, F, L, \vartheta)$.

If $Y$ denotes a variety, then we use $\widetilde{Y}$ for its resolution of singularities.
For the $k$-th Veronese map we use the symbol $v_k$.
$G$ denotes an algebraic (Lie) group and $\mathfrak{g}$ its associated Lie algebra.
The symbol $\iota$ with a subscript, i.e. $\iota_v$ denotes a contraction of a differential form with a (multi)vector $v$.

Finally, results described as *folklore* or *well-known* are the ones that – usually because of their simplicity – were stated without reference nor explanation in the works consulted by the author. However, the dissertation requires more detailed arguments than a research paper. Therefore, the use of one of those descriptions means that the proof provided in the dissertation cannot be attributed to the author.

CHAPTER 2

# Mori theory of projective contact manifolds

## 2.1. Singularities and their basic properties

We will commence our discussion by recalling some terminology related to singular varieties along with their basic properties that will be used throughout the dissertation. For a more detailed survey see [**KM98**].

To begin with, we say that a commutative Noetherian local ring $R$ is Cohen–Macaulay if $\dim(R) = \operatorname{depth}(R)$. Then, a variety (or a scheme) $Y$ is Cohen–Macaulay (or CM) if and only if the local ring $\mathcal{O}_{Y,y}$ is Cohen–Macaulay for any $y \in Y$.

The notion of being CM is quite technical, however it is important for us as a minimal assumption under which Serre duality holds.

Theorem 2.1.1 (Serre duality, [**Har77**, Ch. III, Thm 7.6]). *Let $Y$ be a CM projective scheme of pure dimension $n$. Then there exists a coherent sheaf $\omega_Y^o$, called the dualizing sheaf, such that for any coherent sheaf $\mathcal{E}$ the natural maps:*

$$\operatorname{Ext}^i_Y(\mathcal{E}, \omega_Y^o) \to H^{n-i}(X, \mathcal{E})^*$$

*are isomorphisms. In the case where $\mathcal{E}$ is a vector bundle, we can further identify*

$$\operatorname{Ext}^i_Y(\mathcal{E}, \omega_Y^o) \simeq H^i(Y, \mathcal{E}^* \otimes \omega_Y^o).$$

Now we are ready to define the class of varieties that will frequently appear in the whole dissertation.

Definition 2.1.2. We say that a variety (or a scheme) $Y$ is *Gorenstein* if and only if it is CM and the dualizing sheaf $\omega_Y^o$ is locally free and equal to the canonical sheaf $\omega_Y = \bigwedge^{\dim Y} \Omega^1_Y$, where $\Omega^1_Y$ denotes the sheaf of Kähler differentials.

We will also need to define a class of singularities that are useful in conducting cohomological reasonings.

Definition 2.1.3. Let $Y$ be a variety. We say that it has *rational singularities* if for any resolution of singularities $f\colon \widetilde{Y} \to Y$ the following two conditions hold:

(1) $f_*\mathcal{O}_{\widetilde{Y}} = \mathcal{O}_Y$ (i.e. $Y$ is normal by Stein Factorization Theorem [**Har77**, Ch. III, Cor. 11.5]),
(2) $R^i f_*\mathcal{O}_{\widetilde{Y}} = 0$ for $i > 0$.

In particular, a variety with rational singularities is CM by [**KKMSD73**, Ch. I, §3, Proposition]. The importance of the notion of rationality lies in the following folklore statement:

Proposition 2.1.4. *Let $Y$ be a projective variety with rational singularities. Then for any vector bundle $\mathcal{E}$ on $Y$ and any resolution of singularities $f\colon \widetilde{Y} \to Y$ we have:*

$$h^i(\widetilde{Y}, f^*\mathcal{E}) = h^i(Y, \mathcal{E}).$$

Proof. The projection formula [**Har77**, Ch. III, Ex. 8.3] gives an isomorphism for each $i$:

$$(R^i f_*\mathcal{O}_{\widetilde{Y}}) \otimes \mathcal{E} \to R^i f_*(f^*\mathcal{E}),$$

from which we obtain $\mathcal{E} = f_* f^* \mathcal{E}$ and $(R^i f_*) f^* \mathcal{E} = 0$ for $i > 0$. Consequently, the Leray spectral sequence for the locally free sheaf $f^*\mathcal{E}$ on $\widetilde{Y}$ is degenerate:

$$E_2^{pq} = H^p(Y, R^q f_*(f^*\mathcal{E})) \implies H^{p+q}(\widetilde{Y}, f^*\mathcal{E}),$$

and we are done.

□

We note that the notion of a rational singularity, although important and useful, is not well behaved. To be precise, a partial resolution of a rational singularity may not be rational, even if it stays normal, as it may cease to be CM. An example of such behaviour was given in [**Cut90**, Sect. III].

A collection of other notions emerged with the development of the Mori theory. These definitions emphasize the relation between the canonical divisors of a given variety and its (partial) resolution (so in particular we do not encounter the problem mentioned above) and are usually expressed in terms of *logarithmic pairs* $(Y, D)$. Such pair consists of a normal variety $Y$ and a formal sum $D = \sum_i a_i D_i$, where $D_i \subset Y$ are irreducible divisors and $a_i \in \mathbb{Q}_i$, $a_i > 0$. This formal sum is sometimes called *the boundary*, and any notion defined in the language of pairs can be applied to varieties by considering the pair $(Y, 0)$. We assume that *the log canonical divisor* $K_Y + D$ is $\mathbb{Q}$-Cartier, i.e. some multiple of it is a Cartier divisor. Now consider a birational morphism from a normal variety $f\colon Z \to Y$. We can write

$$K_Z = f^*(K_Y + D) + \sum_j a(E_j, Y, D) E_j,$$

where $E_j$ can either be an irreducible divisor contained in the exceptional locus, or be the birational transform of some $D_i$, i.e. $E_j = f_*^{-1} D_i$ and in this second case we put $a(E_j, Y, D) = -a_j$. The coefficients $a$ are called discrepancies, and we define the discrepancy of the pair $(Y, D)$, denoted $a(Y, D)$, to be the minimum of the $a_j$ over all possible $f\colon Z \to Y$.

Definition 2.1.5. We say that a pair $(Y, D)$ is

- terminal if $a(Y, D) > 0$,
- canonical if $a(Y, D) \geq 0$,
- Kawamata log terminal (klt) if $a(Y, D) > -1$ and $\lfloor D \rfloor = 0$,
- purely log terminal (plt) if $a(Y, D) > -1$,
- log canonical (lc) if $a(Y, D) \geq -1$.

Moreover, a divisor $E$ with discrepancy $-1$ is called a log canonical place, and its image via $f$ is *a log canonical center* (lc center for brevity). In the case of a canonical pair it can happen that for some morphism all discrepancies vanish. Such morphism is called *crepant*. An important example of a crepant morphism is *a terminalization* of a variety $Y$ having rational Gorenstein singularities: it is a birational and crepant morphism $Y' \to Y$ such that $(Y', 0)$ has only terminal singularities. The existence of a terminalization follows from [**BCHM10**, Cor. 1.4.3] and we will utilize an important property of terminal varieties – their singular locus has codimension at least 3 by [**BS95**, Lem. 1.3.1].

Frequently we will use the following relations between different classes of singularities:

Theorem 2.1.6 ([**KM98**, Thm 5.22]). *Let $(Y, D)$ be a klt pair. Then $Y$ has rational singularities.*

Theorem 2.1.7 ([**KM98**, Cor. 5.24]). *Let $Y$ be normal with locally free canonical sheaf $\omega_Y$. Then $Y$ having rational singularities is equivalent to the pair $(Y, 0)$ being canonical.*

Using the equivalence above, we are able to claim a very useful folklore lemma:

LEMMA 2.1.8. *Let $(Y, 0)$ be a canonical pair with a locally free canonical sheaf $\omega_Y$ (i.e. $Y$ is Gorenstein). Moreover let $g\colon Y' \to Y$ be its terminalization and $f\colon \widetilde{Y} \to Y$ any resolution of singularities. Then for any vector bundle $\mathcal{E}$ on $Y$ we have:*

$$h^i(Y, \mathcal{E}) = h^i(Y', g^*\mathcal{E}) = h^i(\widetilde{Y}, f^*\mathcal{E}).$$

PROOF. By Theorem 2.1.7 $Y$ has rational singularities. Its terminalization $Y'$ has terminal, so in particular canonical singularities and $\omega_{Y'} = g^*\omega_Y$ is locally free, so they remain rational. Consequently, we may choose a common resolution of singularities $g'\colon \widetilde{Y'} \to Y$ for which by Proposition 2.1.4 we have:

$$h^i(Y, \mathcal{E}) = h^i(\widetilde{Y'}, g'^*\mathcal{E}) = h^i(Y', g^*\mathcal{E}).$$

If $f\colon \widetilde{Y} \to Y$ is any other resolution of singularities, then we have $h^i(Y, \mathcal{E}) = h^i(\widetilde{Y}, f^*\mathcal{E})$, so we are done. □

## 2.2. Mori cone and extremal contractions

When defining singularities in terms of discrepancies we have assumed that for a given normal variety $Y$ the log canonical divisor $K_Y + D$ is $\mathbb{Q}$-Cartier. There are two reasons for this assumption. Namely, we use it to pull back the log canonical divisor and to define its intersection with a given proper curve $C$ contained in the same ambient variety $Y$. So, if we have a Weil divisor $D \subset Y$ such that $qD$ is a Cartier divisor $\mathcal{O}(qD)$, we define the intersection with $C$ as:

$$C \cdot D = \frac{1}{q} C \cdot \mathcal{O}(qD),$$

where the intersection between $C$ with a normalization $\eta\colon \widetilde{C} \to C \subset Y$ and a Cartier divisor $\mathcal{O}(qD)$ is

$$C \cdot \mathcal{O}(qD) = \deg_{\widetilde{C}} \eta^*(\mathcal{O}(qD)).$$

This pairing can be extended to the real vector space spanned by formal linear combinations of proper curves in $Y$. For a projective $Y$ we define the space $N_1(Y)$ as the quotient of this linear space by numerically trivial combinations of curves (i.e. those that intersect trivially with every divisor). The space $N_1(Y)$ is finite dimensional by [**BFG$^+$06**, Exp. XIII, Th. 5.1] and contains the cone of curves:

$$NE(Y) = \{\sum a_i[C_i] : a_i \in \mathbb{R}_{\geq 0}, [C_i] \text{ is a class of a proper curve in } Y\}$$

Since it may be not closed, one usually considers its closure $\overline{NE(Y)}$, that we will frequently call *the Mori cone*. There also exists a dual space $N^1(Y)$, spanned by the numerical equivalence classes of Cartier divisors, that can be obtained as a quotient of $\mathrm{Pic}(Y) \otimes \mathbb{R}$. It contains the dual cone of $\overline{NE(Y)}$, called the nef cone $\mathrm{Nef}(Y)$. One can also obtain $N^1(Y)$ by first defining the Neron-Severi group $\mathrm{NS}(Y) = \mathrm{Pic}(Y)/\mathrm{Pic}_0(Y)$, i.e. the quotient of the group of the line bundles on $Y$ by its connected component of identity and then taking the tensor product $\mathrm{NS}(Y) \otimes \mathbb{R}$.

The cornerstone of the Mori theory are theorems describing the structure of the cone of curves and relating some of the subcones to the particular morphisms from $Y$, called contractions.

DEFINITION 2.2.1. A contraction $f$ is a projective and surjective morphism between normal projective varieties $Y$ and $Z$ such that $f_*\mathcal{O}_Y = \mathcal{O}_Z$, which is equivalent to the property that the inverse image of every point is connected ([**Har77**, Ch. III, Cor. 11.3]).

We are now ready to state the following theorem, that was proved in the smooth case by Mori and later generalized to klt pairs by Kollar, Reid, Shokurov and others.

THEOREM 2.2.2 (Cone theorem, [**KM98**, Thm 3.7]). *Let $(Y, D)$ be a projective klt pair with $D$ effective. Then there exist countably many rational curves $C_i$ such that $0 < (-K_Y - D) \cdot C_i \leq 2 \dim Y$ and*

$$\overline{NE(Y)} = \overline{NE(Y)}_{(K_Y+D)\geq 0} + \sum \mathbb{R}_+[C_i],$$

*where $\overline{NE(Y)}_{(K_Y+D)\geq 0}$ denotes the part of the cone having a nonnegative intersection with $K_Y + D$. The rays generated by $C_i$ can only accumulate on the hyperplane perpendicular to $K_Y + D$. In the particular case of $(Y, 0)$ for smooth $Y$ we have a better bound: $0 < -K_Y \cdot C_i \leq \dim Y + 1$.*

*Moreover, to each $(K_Y + D)$-negative extremal face $F$ of $\overline{NE(Y)}$ we can associate a particular contraction $\mathrm{cont}_F$ from $Y$ to some normal and projective variety $Z$ such that for every proper curve $C \subset Y$ we have*

$$\mathrm{cont}_F(C) = pt \iff [C] \in F.$$

*Every contraction determined by a $(K_Y + D)$-negative extremal face is known as a Mori contraction.*

REMARK 2.2.3. A Mori contraction determined by a 1-dimensional face, i.e. a ray of the cone, is called elementary. Any Mori contraction can be decomposed into elementary ones, and they satisfy $b_2(Y) = b_2(Z) + 1$ by [**KM98**, Cor. 3.17].

Essentially, the cone theorem is useful for varieties whose canonical divisor is negative in some sense. The minimal working assumption is that $K_Y$ is not nef, and in the particular case of Fano varieties it follows that the cone $\overline{NE(Y)}$ is polyhedral.

To study Mori contractions in greater detail it is necessary to introduce schemes parametrizing rational curves. They can be constructed as a special case of schemes $\mathrm{Hom}(Z, Y)$ parametrizing morphisms between two projective schemes $Z$ and $Y$. $\mathrm{Hom}(Z, Y)$ can contain infinitely many connected components, however under some technical assumptions we can estimate its dimension at a given point:

THEOREM 2.2.4 ([**Kol96**, Ch. I, Thm 2.16]). *Suppose that $Z$ and $Y$ are projective schemes without embedded points. For a morphism $f: Z \to Y$ suppose that the image of every irreducible component of $Z$ intersects the smooth locus of $Y$. Then there is a natural isomorphism:*

$$T_{[f]} \mathrm{Hom}(Z, Y) \simeq \mathrm{Hom}_Z(f^*\Omega^1_Y, \mathcal{O}_Z)$$

*and the dimension of every irreducible component of* $\mathrm{Hom}(Z, Y)$ *at $[f]$ is at least*

$$\dim \mathrm{Hom}_Z(f^*\Omega^1_Y, \mathcal{O}_Z) - \dim \mathrm{Ext}^1_Z(f^*\Omega^1_Y, \mathcal{O}_Z)$$

There also exists a variant of this construction for when we want to keep the image of a subscheme $B$ of $Z$ fixed. Namely, for a fixed $g : B \to Y$ we can construct a scheme $\mathrm{Hom}(Z, Y; g)$ parametrizing morphisms $f: Z \to Y$ satisfying $f|_B = g$ and it is in fact a subscheme of $\mathrm{Hom}(Z, Y)$. If $B$ is a finite set of points, then we have the following bound:

$$\dim(\mathrm{Hom}(Z, Y)) - \dim(\mathrm{Hom}(Z, Y; g)) \leq |B| \cdot \dim(Y).$$

Now to obtain the scheme parametrizing rational curves on $Y$ we construct the scheme $\mathrm{Hom}(Z, Y)$ for $Z = \mathbb{P}^1$. Moreover, we can parametrize rational curves through some fixed set of points $\{y_1, ..., y_n\} \subset Y$ by using the second construction for $B = \{p_1, ..., p_n\}$ and $g$ such that $g(p_i) = y_i$. This scheme will usually be denoted by $\mathrm{Hom}(\mathbb{P}^1, Y; p_i \mapsto y_i)$.

The estimate coming from Theorem 2.2.4 take much simpler form if we moreover assume that $Y$ is smooth. In particular, we have:

THEOREM 2.2.5 ([**Kol96**, Ch. II, Thm 1.7]). *Let $Y$ be a smooth projective variety and $f$ a morphism from a smooth projective curve $C$ to $Y$. Then:*

$$\begin{aligned}\dim \operatorname{Hom}(C, Y; p_1 \mapsto x_1, ..., p_n \mapsto x_n) \geq \\ \geq h^0(C, f^*TY) - h^1(C, f^*TY) - n \dim Y = \\ = \chi(C, f^*TY) - n \dim(Y) = \\ = -K_Y \cdot f_*C + (1 - g(C) - n) \dim Y\end{aligned}$$

In applications, one usually considers an open subscheme of $\operatorname{Hom}(\mathbb{P}^1, Y)$, denoted by $\operatorname{Hom}_{bir}(\mathbb{P}^1, Y)$ which parametrizes morphisms that are birational onto their image and normalizes it to obtain $\operatorname{Hom}^n_{bir}(\mathbb{P}^1, Y)$. Moreover, usually we do not need to distinguish between morphisms from $\mathbb{P}^1$ to $Y$ that differ by an automorphism of $\mathbb{P}^1$. To that end, we can take the quotient of $\operatorname{Hom}^n_{bir}(\mathbb{P}^1, Y)$ by $\operatorname{Aut}(\mathbb{P}^1)$ and obtain a normal scheme $\operatorname{RatCurves}^n(Y)$. Let $u$ be the projection map:

$$u\colon \operatorname{Hom}^n_{bir}(\mathbb{P}^1, Y) \to \operatorname{RatCurves}^n(Y).$$

Observe that $u$ gives $\operatorname{Hom}^n_{bir}(\mathbb{P}^1, Y)$ the structure of a $\operatorname{Aut}(\mathbb{P}^1)$-bundle by [**Kol96**, Ch. II, Thm 2.15].

DEFINITION 2.2.6. Let $V$ be a closed and irreducible subvariety of $\operatorname{RatCurves}^n(Y)$ and $V' = u^{-1}(V)$. If $V$ is proper over $\mathbb{C}$ then we call it an unsplit family of rational curves (and in such situations we call $V'$ an unsplit family of morphisms). Now let us consider a map $\Pi$:

$$\Pi\colon V' \to X \times_{\mathbb{C}} X \qquad\qquad \Pi(f) = [f(0), f(\infty)].$$

We say that $V$ is a generically unsplit family of rational curves ($V'$ is a generically unsplit family of morphisms), if the fiber over a generic point of $\operatorname{im}(\Pi)$ has dimension not bigger than 1.

Intuitively, being unsplit means that elements in the family cannot be deformed to cycles with multiple components. Being generically unsplit means that there is no positive dimensional subfamily of curves passing through two generic points.

If we are given a subset $V' \subset \operatorname{Hom}(\mathbb{P}^1, Y)$ and the evaluation map $ev\colon \mathbb{P}^1 \times \operatorname{Hom}(\mathbb{P}^1, Y) \to Y$, then by locus $V'$ we will mean the image $ev(\mathbb{P}^1 \times V')$. Analogously, $\operatorname{locus}(V', y)$ denotes the evaluation of the intersection $V' \cap \operatorname{Hom}(\mathbb{P}^1, Y, 0 \mapsto y)$.

We are now ready to state the last general result that we will need – a particular form of a locus-fiber inequality. This is a powerful tool in the study of Mori contractions, however it is only applicable to smooth varieties.

THEOREM 2.2.7 ([**Kol96**, Ch IV, 2.6.1]). *Let $Y$ be a smooth and proper variety and $V' \subset \operatorname{Hom}_{bir}(\mathbb{P}^1, Y)$ a generically unsplit irreducible component. Then for a general $y \in \operatorname{locus}(V')$ and $f \in V'$*

$$\dim(Y) + \deg(f^*(-K_Y)) \leq \dim \operatorname{locus}(V') + \dim \operatorname{locus}(V', y) + 1.$$

*Moreover, if the component is unsplit, then the inequality holds for any $y \in \operatorname{locus}(V')$.*

## 2.3. Mori contractions of projective contact manifolds

As we have mentioned in the previous section, the Cone Theorem is useful for varieties for which the canonical divisor is at least not nef. Projective contact manifolds satisfy even stronger property.

THEOREM 2.3.1 ([**Dem02**, Main Theorem]). *Let $Y$ be a compact Kähler manifold with a line bundle $L$ whose dual $L^*$ is pseudoeffective. Assume that $\vartheta \in H^0(Y, \Omega_Y^p \otimes L)$ is a nonzero holomorphic section (a twisted form). Let $F \subset TY$ be a subsheaf defined by the vector fields $\xi$ for which the contraction with $\vartheta$ vanishes. Then $F$ is integrable.*

The immediate corollary of this theorem is that for a projective contact manifold $(X, F, L, \vartheta)$ of dimension $2n+1 \geq 3$ the dual of the contact line bundle $L^*$, so also the canonical divisor $K_X$, cannot be pseudoeffective, as $F$ is not integrable ([**Dem02**, Cor. 2]). Consequently, the Kodaira dimension of $X$ is negative and $K_X$ is not nef, as the nef cone is a subcone of the pseudoeffective cone [**Laz04**, I, Thm 2.2.24].

Therefore, by the Cone Theorem 2.2.2, $X$ admits a rational curve that can be contracted. The study of possible Mori contractions was conducted by Druel [**Dru99**] in the toric case and by Kebekus-Peternell-Sommese-Wiśniewski [**KPSW00**] in the general case. We can summarise their results as follows:

THEOREM 2.3.2. *Let $X$ be a projective contact manifold of dimension $2n+1 \geq 3$. Then either:*

(1) *$b_2(X) > 1$ and $(X, L) = (\mathbb{P}(TY), \mathcal{O}_{\mathbb{P}(TY)}(1))$ for some smooth variety $Y$ of dimension $n+1$. In this case $X$ is Fano only if $Y = \mathbb{P}^{n+1}$.*
(2) *$b_2(X) = 1$, $X$ is necessarily a Fano manifold and it is either a projective space $\mathbb{P}^{2n+1}$ with the contact line bundle $L = \mathcal{O}(2)$, or $L$ is the ample generator of the Picard group.*

*If moreover $X$ has the structure of a toric variety, then for each possible dimension there is exactly one representative in each family:*

(1) *the projectivization of the cotangent bundle of the product of $n$ projective lines $\mathbb{P}(T(\mathbb{P}^1 \times ... \times \mathbb{P}^1))$,*
(2) *the projective space $\mathbb{P}^{2n+1}$.*

SKETCH OF THE PROOF. We utilize arguments from [**Dru99**] and [**KPSW00**]. To begin with, let $C$ be a $K_X$-negative extremal curve $C$ which satisfies

$$0 < -K_X \cdot C \leq \dim(X) + 1 = 2n + 2$$

and whose existence is guaranteed by the Cone Theorem 2.2.2 and Theorem 2.3.1. Recall from the Definition 1.2.1 that $X$ is equipped with the contact line bundle $L$ such that $L^{\otimes n+1} = \mathcal{O}(-K_X)$ in $\mathrm{Pic}(X)$. It follows that $C \cdot L$ can be equal only to 1 or 2. In the second case, if we denote by $f$ the morphism $\mathbb{P}^1 \to C \subset X$ and by $V$ the component of $\mathrm{Hom}_{bir}(\mathbb{P}^1, X)$ containing $f$, then the locus inequality 2.2.7 gives us:

$$\dim X + \deg(f^*(-K_X)) = 4n + 3 \leq \dim \mathrm{locus}(V) + \dim \mathrm{locus}(V, x) + 1.$$

From that it follows that both $\mathrm{locus}(V)$ and $\mathrm{locus}(V, x)$ are equal to the whole $X$, so the contraction is onto a point. Since it was elementary, $b_2(X) = 1$ and $X$ is Fano. Then from $-K_X \cdot C = \dim X + 1$ it follows that $X$ is a projective space by [**Keb02**, Thm 1.1].

Now suppose that $L \cdot C = 1$. In this case the proof is more involved and it will consist of several steps.

(1) Observe that the irreducible component $V \subset \mathrm{RatCurves}(X)$ containing $C$ is an unsplit family: if it could be split then it would be possible to find a smaller family $W$ such that $\mathrm{locus}(W) = \mathrm{locus}(V)$ and for any curve $C' \in W$ $0 < C' \cdot L < C \cdot L$, but this is not possible (reasoning as in [**Kol96**, IV, Thm 2.4]).
(2) Show that for any unsplit family of rational curves $V$ containing $C$ on $X$ and a point $x \in X$ one has $\mathrm{locus}(V) = X$ and $\dim \mathrm{locus}(V, x) = n$ [**KPSW00**, Prop. 2.9], so in particular projective contact manifolds do not admit birational Mori contractions.

(3) Prove that if $X$ admits a surjective morphism to a variety $Y$ of lower, but positive dimension such that the generic fiber $X_\eta$ is Fano, then $X_\eta \simeq \mathbb{P}^n$ and $X_\eta$ is a Legendrian submanifold of $X$ [**KPSW00**, Prop. 2.11]. Observe that it still can happen that $Y$ is a point and in such case $X$ is a Fano manifold with $b_2(X) = 1$ and $L$ is not divisible in $\mathrm{Pic}(X)$, however this is not possible when $X$ is toric.
(4) Use [**Fuj87**, Lemma 2.12] to show that if $\phi\colon X \to Y$ is a Mori contraction onto a variety $Y$ of positive dimension, then $Y$ is smooth and $X = \mathbb{P}(\phi_* L)$.
(5) Show the isomorphism $\phi_* L \simeq TY$ [**KPSW00**, Thm 2.12].
(6) Observe that by [**LS94**, Cor. 4.2] if $\mathbb{P}(TY)$ is Fano, then $Y = \mathbb{P}^{n+1}$.
(7) To finish the proof in the toric case use [**OM78**, 7.6] to claim that a projectivized bundle admits the structure of a toric variety if and only if it splits as a sum of line bundles and show that the only smooth projective toric variety with a totally decomposable tangent bundle is the product of projective lines [**Dru99**, Lemme 2].

□

In particular, as both $\mathbb{P}(T\mathbb{P}^{n+1})$ and $\mathbb{P}^{2n+1}$ are coadjoint varieties for simple groups of type $A_{n+1}$ and $C_{n+1}$ respectively (we list coadjoint varieties in Section 4.2.2), Conjecture 2 remains open only in the case of prime Fano manifolds for which the contact line bundle generates the Picard group. A more detailed study of possible contractions was enough for Ye and Druel to prove Conjecture 2 in dimensions 3 [**Ye94**, Thm 2] and 5 [**Dru98**, Prop. 1] respectively (note that those results in fact precede the ones discussed here). However, further progress was obtained by developing other methods. To be precise, the conjecture in dimension 7 and 9 was proved by Buczyński, Wiśniewski and Weber [**BW22**, Thm 1.2] under the additonal assumption on the reductivity of the automorphisms group. We will discuss some aspects of this landmark result and its follow-ups and how they relate to the author's research in Chapter 4.

CHAPTER 3

# Linear systems of Fano varieties

## 3.1. Fano varieties in the singular setting

In this chapter we will temporarily set aside contact structures and discuss complex Fano varieties, sometimes in a singular setting. For the author, the problems discussed in this Chapter first emerged as motivated by the study of contact manifolds, as we will explain in Chapter 4. Nevertheless, they were classically considered on their own and for this reason we do so in the dissertation.

The Fano condition (i.e. the condition that the anticanonical divisor is ample) makes sense if we assume that a projective variety $Y$ is normal and $-K_Y$ is $\mathbb{Q}$-Cartier (i.e. $Y$ is $\mathbb{Q}$-Gorenstein). However, if we further restrict possible singularities, we can recreate some well-known properties of smooth Fano varieties. There also exists a slightly more general notion, that of a *weak Fano* variety, where we relax the assumption on the ampleness of the anticanonical divisor and only demand that it is big and nef. Recall the following classical vanishing theorem:

THEOREM 3.1.1 (Kawamata-Viehweg vanishing, [**KMM87**, Thm 1-2-3]). *Let $Y$ be a smooth projective variety. If $L$ is a big and nef line bundle on $Y$, then for all $i > 0$ we have:*

$$H^i(Y, L \otimes \mathcal{O}(K_Y)) = 0.$$

Consequently, for a big and nef line bundle $L$ on a smooth Fano variety $Y$ we have

$$H^i(Y, L) = 0$$

for $i > 0$ (as $L \otimes \mathcal{O}(-K_Y)$ is still big and nef). There is a folklore and well-known to experts generalization of this vanishing to the case of singular Fano varieties:

THEOREM 3.1.2 (Singular Kawamata-Viehweg vanishing). *Let $Y$ be a Fano variety of dimension $n$ with rational Gorenstein singularities and $L$ a big and nef line bundle on $Y$. Then $H^i(Y, L \otimes \mathcal{O}(K_Y)) = 0$ for $i > 0$. Consequently, $H^i(Y, L) = 0$ for $i > 0$.*

PROOF. For a resolution of singularities $f\colon \widetilde{Y} \to Y$ we have:

$$H^i(Y, L \otimes \mathcal{O}(K_Y)) = H^{n-i}(Y, L^*) = H^{n-i}(\widetilde{Y}, f^*L^*) = H^i(\widetilde{Y}, f^*L \otimes \mathcal{O}(K_{\widetilde{Y}})) = 0.$$

First and third equality come from Theorem 2.1.1 (Serre duality for $Y$ and $\widetilde{Y}$), second from Proposition 2.1.4 and the last one is Kawamata-Viehweg vanishing for a smooth variety $\widetilde{Y}$, as the pullback of a big and nef line bundle is big and nef. As $Y$ is Fano, the bundle $L \otimes \mathcal{O}(-K_Y)$ is big and nef, which proves the second claim. □

It also follows that for the structure sheaf $\mathcal{O}_Y$ on a Fano variety $Y$ with rational Gorenstein singularities we have:

$$\chi(Y, \mathcal{O}_Y) = h^0(Y, \mathcal{O}_Y) = 1. \tag{2}$$

The corollary of this equality on Todd genus $\chi(Y, \mathcal{O}_Y)$ is that log terminal Fano varieties are simply connected [**Zha06**, Corollary 1], as they have rational singularities by Theorem 2.1.6. This topological fact has a geometric consequence:

PROPOSITION 3.1.3 ([**IP99**, Prop. 2.1.2]). *Let $Y$ be a Fano variety with log terminal singularities. Then the Picard group of $Y$ is torsion free and the Picard number $\rho(Y)$ is equal to the second Betti number $b_2(Y)$.*

PROOF. The proof is a standard exercise in the theory of Fano manifolds, similar one can be found in cited source and we present it for the completeness of our presentation. Recall that $H^1(Y, \mathcal{O}_Y^*) \simeq \operatorname{Pic}(Y)$ and this cohomology group fits into the long exact sequence coming from the exponential sequence:

$$H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y^*) \xrightarrow{c_1} H^2(Y, \mathbb{Z}) \to H^2(Y, \mathcal{O}_Y).$$

The leftmost and rightmost terms are 0 by Equation (2), so the map denoted by $c_1$, which associates to the class of a line bundle its first Chern class, is an isomorphism. Therefore, to prove the statement on $\operatorname{Pic}(Y)$ we need to check that $H^2(Y, \mathbb{Z})$ is torsion-free. From the universal coefficient theorem we have a short exact sequence:

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H_1(Y, \mathbb{Z}), \mathbb{Z}) \to H^2(Y, \mathbb{Z}) \to \operatorname{Hom}(H_2(Y, \mathbb{Z}), \mathbb{Z}) \to 0.$$

The Ext term is 0, as the first homology group of $Y$ is the abelianization of its fundamental group, which is trivial (recall [**Zha06**, Cor. 1]), therefore $H^2(Y, \mathbb{Z})$ is isomorphic to the torsion-free Hom term.

To prove the equality between $b_2(Y) = \operatorname{rk}(H^2(Y, \mathbb{Z}))$ and $\rho(Y) = \operatorname{rk}(\operatorname{NS}(Y))$ recall that we have defined the Neron-Severi group $\operatorname{NS}(Y)$ as the quotient $\operatorname{Pic}(Y)/\operatorname{Pic}_0(Y)$, i.e. the group of connected components of $\operatorname{Pic}(Y)$. As $\operatorname{Pic}(Y) = H^2(Y, \mathbb{Z})$, it is discrete, so $\operatorname{NS}(Y) = \operatorname{Pic}(Y)$ and we have $b_2(Y) = \rho(Y)$. □

The proposition above motivates the following definition:

DEFINITION 3.1.4. We say that a Fano variety $Y$ with log terminal singularities is *prime* if $b_2(Y) = \rho(Y) = 1$.

In the case of a Fano variety with a torsion-free Picard group (but not necessarily prime) we can consider following notions:

DEFINITION 3.1.5. The index of a Fano variety $Y$ is the rational number $i(Y)$ such that:

$$i(Y) = \max\{q \in \mathbb{Q} \mid \mathcal{O}(-K_Y) \sim_{\mathbb{Q}} L^{\otimes q}, L \text{ is ample and Cartier}\}.$$

The coindex $c(Y)$ is equal to $\dim(Y) - i(Y)$. Observe that if $Y$ is Gorenstein (i.e. $-K_Y$ corresponds to a Cartier divisor) then the index is a natural number.

DEFINITION 3.1.6. For a Fano variety $Y$, the class of an ample Cartier divisor $L$ satisfying $L^{\otimes i(Y)} \sim_{\mathbb{Q}} \mathcal{O}(-K_Y)$ is called a fundamental class. Any representative of this class is called a fundamental divisor.

The index is a fundamental invariant of Fano varieties, and those for which it is big relative to dimension are well-researched. In particular, we will see that the higher the index, the more we know about the fundamental linear system and in consequence about the variety itself. Moreover, there is a relation between the index and the second Betti number: there are only few examples of varieties having $2i_Y \geq \dim Y$ and $b_2(Y) > 1$ and they are classified in a collection of papers by Wiśniewski: [**Wiś90**], [**Wiś91**], [**Wiś94**].

## 3.2. Chern classes and Hirzebruch-Riemann-Roch theorem

Our goal is to study linear systems associated to an ample (or nef and big) divisor $H$ on a (weak) Fano variety $Y$ and in particular estimate their dimensions. The state-of-the-art method to do it is via Riemann-Roch type computations, i.e. calculations of $\chi(Y, H^{\otimes t})$ in terms of a polynomial in $t$ with coefficients coming from intersection numbers and Chern classes. If we allow $Y$ to have some kind of singularities, then reasonings and constructions (e.g. of the Chern classes) require a bit of subtlety. In particular, we have the following (this is well-known, see e.g. [**Hör12**, Section 2] for a discussion in codimension 2):

CONSTRUCTION 3.2.1. Let $Y$ be a quasi-projective variety which is smooth in codimension $k$ and denote by $A^l(Y)$ the abelian group of cycles of codimension $l$ on $Y$ modulo rational equivalence. Then we can find a unique extension of every algebraic cycle of codimension at most $k$ from the smooth locus $Y_{sm}$ to the whole $Y$, so we have an isomorphism $A^l(Y_{sm}) \simeq A^l(Y)$ for $l \leq k$. Consequently, for any coherent sheaf $\mathcal{E}$ on $Y$ we can define its Chern classes for $l \leq k$ by extending $c_l(\mathcal{E}_{|Y_{sm}}) \in A^l(Y_{sm})$ to $Y$. By a slight abuse of notation, we will denote them by $c_l(\mathcal{E})$. In particular, if $Y$ is smooth in codimension 2, then there exists a uniquely defined class $c_2(TY) \in A^2(Y)$, that we will also denote by $c_2(Y)$.

To formulate the Hirzebruch-Riemann-Roch theorem, we need to recall a few definitions. In particular, for a given nonsingular projective variety $Y$ of dimension $n$ and a locally free sheaf $\mathcal{E}$ of rank $r$ on $Y$ with Chern classes $c_i(\mathcal{E})$, we can form *the Chern polynomial*:

$$c_t(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E})t ... + c_r(\mathcal{E})t^r.$$

By the splitting principle ([**Har77**, App. A, Section 3, C3]) there exists an embedding $\pi^*\colon A^*(Y) \to A^*(Y')$ for some variety $Y'$ such that the Chern polynomial can be fully decomposed: $c_t(\pi^*\mathcal{E}) = \prod_{i=1}^r (1 + a_i t)$, for some $a_i \in A^1(Y') \otimes \mathbb{Q}$. The embedding $\pi^*$ and the functoriality of the Chern classes allows us to consider $a_i$ as formal symbols that are added to $A^1(Y) \otimes \mathbb{Q}$ to ensure the decomposability of the Chern polynomial. Then we define *the Chern character* by:

$$ch(\mathcal{E}) = \sum_{i=1}^{r} e^{a_i},$$

where $e^{a_i} = \sum_{k=1}^{\infty} \frac{a_i^k}{k!}$, so that in particular $ch(\mathcal{E}) \in A(Y) \otimes \mathbb{Q}$. Analogously we can define *the Todd class* by:

$$td(\mathcal{E}) = \prod_{i=1}^{r} \frac{a_i}{1 - e^{-a_i}},$$

where we again use the power series expansion of the transcendental function $\frac{x}{1-e^{-x}}$. Finally, for an element $\gamma \in A^k(Y)$ we denote by $\int_Z \gamma$ the natural pairing with a cycle $Z$ of dimension $k$, that can be extended to the case of rational coefficients. Now we are ready to state:

THEOREM 3.2.2 (Hirzebruch-Riemann-Roch theorem [**Har77**, App. A, Thm 4.1]). *Let $\mathcal{E}$ be a locally free sheaf on a nonsingular projective variety $Y$ of dimension $n$. Then*

$$\chi(Y, \mathcal{E}) = \int_Y ch(\mathcal{E}) \cdot td(TY).$$

Observe that although the expression under the integral contains elements of $A^k(Y)$ for different $k$, only those from $A^n(Y)$ have a nonzero evaluation on $Y$. Now suppose that we have a singular projective variety $Y$, such that singularities are rational. Recall that by Proposition 2.1.4 for any line bundle $L$ on $Y$ we can compute its cohomology using $f^*L$ for any resolution of singularities $f\colon \widetilde{Y} \to Y$. By the functoriality of the Chern character and the projection formula we have:

$$f_*(td(T\widetilde{Y}) \cdot ch(f^*L)) = f_*(td(T\widetilde{Y}) \cdot f^*(ch(L)) = f_*(td(T\widetilde{Y})) \cdot ch(L) \in A^*(Y) \otimes \mathbb{Q}.$$

This in particular allows us to deduce the highest coefficients of the Hilbert polynomial of $L$ (for brevity we identify line bundles and Weil divisors with their first Chern class):

$$p(t) = \chi(Y, L^{\otimes t}) = \frac{L^n}{n!}t^n + \frac{-K_Y \cdot L^{n-1}}{2(n-1)!}t^{n-1} + r(t) \tag{3}$$

where $r(t)$ is a polynomial of degree $n-2$ such that $r(0) = \chi(Y, \mathcal{O}_Y)$. If moreover $Y$ is smooth in codimension two (for instance the singularities are terminal) then by the argument from the beginning of the section, we can define the second Chern class of $Y$ by extending it from the smooth locus, and that gives us the next coefficient of the Hilbert polynomial:

$$p(t) = \chi(Y, L^{\otimes t}) = \frac{L^n}{n!}t^n + \frac{-K_Y \cdot L^{n-1}}{2(n-1)!}t^{n-1} + \frac{((-K_Y)^2 + c_2(Y)) \cdot L^{n-2}}{12 \cdot (n-2)!}t^{n-2} + r(t), \tag{4}$$

where $r(t)$ has degree $n-3$ and $r(0) = \chi(Y, \mathcal{O}_Y)$.

For a variety $Y$ which is Gorenstein and a line bundle $L$ such that $L^{\otimes j} = \mathcal{O}_Y(-K_Y)$ for some $j \in \mathbb{N}$ (e.g. $Y$ is Gorenstein-Fano, $L$ is the fundamental divisor and $j = i_Y$ is the index) we have the following (anti)symmetry of the Hilbert polynomial:

$$p(t) = \chi(Y, L^{\otimes t}) = \sum_{i=0}^{\dim Y} (-1)^i h^i(Y, L^{\otimes t}) =$$

$$= \sum_{i=0}^{\dim Y} (-1)^i h^{\dim Y - i}(Y, L^{\otimes -t-j}) = (-1)^{\dim Y} p(-t-j). \tag{5}$$

This (anti)symmetry will allow us to express the whole Hilbert polynomial in terms of lowest Chern classes. To make use of it, we still need to estimate the terms containing $c_2(Y)$. The most general result that we will use was proved by Ou:

THEOREM 3.2.3. [**Ou17**, Corollary 1.5] *Let $Y$ be a normal projective variety of dimension $n$ with $\mathbb{Q}$-factorial log canonical singularities, smooth in codimension 2 and with the anticanonical class $-K_Y$ that is nef. Then for any nef divisors $H_1, H_2, ..., H_{n-2}$ we have:*

$$c_2(Y) \cdot H_1 \cdot ... \cdot H_{n-2} \geq 0. \tag{6}$$

For smooth varieties with semistable tangent bundles we have the following Bogomolov's inequality:

THEOREM 3.2.4. [**Lan04**, Thm 0.1] *Let $E$ be a semistable, torsion free sheaf on a smooth projective variety $Y$ of dimension $n$ and $L$ be an ample line bundle. Then:*

$$2\,\mathrm{rk}(E) c_2(E) \cdot L^{n-2} \geq (\mathrm{rk}\, E - 1) c_1^2(E) \cdot L^{n-2}. \tag{7}$$

Consequently, if we assume that $E$ is the semistable tangent bundle of a Fano manifold, we obtain a stronger bound on $c_2(TY)$ than the one coming from inequality (6). However, the semistability is known only in some cases. As smooth Fano varieties which allow nontrivial Mori contractions can have unstable tangent bundles, one usually adds the assumption that the second Betti number $b_2$ is equal to 1. In particular, [**PW95**] and [**Hwa98**] showed (semi)stability in dimensions up to 6. [**Hwa01**] proved stability in any dimension $n$ in the case where the index is high, i.e. $i_Y > \frac{n+1}{2}$. On the other side of the spectrum, [**Rei78**] showed the stability in the case of $i_Y = 1$. However, the folklore conjecture on the semistability of tangent bundles for smooth and prime Fano varieties was recently disproved by Kanemitsu [**Kan21**, Thm 0.4], who showed that some of two-orbit varieties studied by Pasquier [**Pas09**, Thm 0.2] provide a counterexample.

The good news is that Liu proved a weaker inequality on $c_2$ that holds even if we cannot assume the semistability:

THEOREM 3.2.5. [**Liu19**, Thm 1.1.] *Let $Y$ be a smooth Fano variety having dimension $n \geq 7$, $b_2(Y) = 1$, index $i_Y$ and fundamental divisor $L$.*

- *If $i_Y = 2$, then*
$$c_2(Y) \cdot L^{n-2} \geq \frac{11n - 16}{6n - 6} L^n.$$
- *If $2 < i_Y \leq n$, then*
$$c_2(Y) \cdot L^{n-2} \geq \left( \frac{i_Y(i_Y - 1)}{2} + \frac{2n(i_Y - 1) - i_Y^2}{2(n-1)(i_Y - 1)} \right) L^n.$$

## 3.3. The existence of sections for Gorenstein-Fano varieties

The setting in this section is as follows: we start with $Y$, which is an $n$-dimensional Fano variety with canonical Gorenstein singularities. Let $L$ be the fundamental class of $Y$ and $i_Y$ be the index. We can replace $Y$ by its terminalization, that exists by [**BCHM10**, Cor. 1.4.4]. By Lemma 2.1.8 this operation does not change the cohomology and we may assume that $Y$ is smooth in codimension 2, so that the Hilbert polynomial $p(t) = \chi(Y, tL)$ has form as in Equation (4). The price for terminalizing $Y$ is that $\mathcal{O}(-K_Y)$ is no longer ample, but only big and nef, however this will not affect the computations. Finally, let $f \colon \widetilde{Y} \to Y$ be a resolution of singularities. By the Vanishing Theorem 3.1.1, Serre duality 2.1.1 and the rationality of singularities 2.1.4 we have for all $j \in \{0, ...n\}$ and $t \in \{1, ..., i_Y - 1\}$:
$$0 = H^j(\widetilde{Y}, f^*(L^{\otimes t}) \otimes \mathcal{O}(K_{\widetilde{Y}})) = H^{n-j}(\widetilde{Y}, f^*(L^{\otimes -t})) = H^{n-j}(Y, L^{\otimes -t}).$$
It follows that $p(t) = 0$ for $t \in \{-1, -2, ..., -i_Y + 1\}$. As the degree of the Hilbert polynomial $p(t)$ is equal to $n$, it follows that the index is bounded by $n+1$. We also can see the following:

OBSERVATION 3.3.1. If the coindex $n - i_Y$ is an even number, then the Hilbert polynomial of $L$ has an additional root, which is equal to $\frac{-i_Y}{2}$. In the case when $n$ ($i_Y$) is even, it means that $\frac{-i_Y}{2}$ is a double root.

PROOF. Take the equality coming from Serre duality (5) and for odd $n$ evaluate it at $\frac{-i_Y}{2}$ to see that it must be a nonintegral root. For even $n$, evaluation at $\frac{-i_Y}{2} + \epsilon$ yields that $p(t)$ is symmetric around $\frac{-i_Y}{2}$, but in this case $\frac{-i_Y}{2}$ is an integer greater than $-i_Y + 1$, i.e. we know that $p(\frac{-i_Y}{2}) = 0$, so the symmetry condition implies that it is an even root. ☐

This observation does not give any new information on the Hilbert polynomial – it is implicit in the conditions imposed by the Serre duality, however it makes the considered expressions easier to manage.

Now we will combine Equation (4) with the knowledge of some of the roots to calculate the value of $p$ at 1, which by Singular Vanishing Theorem 3.1.2 is equal to $\dim H^0(Y, L)$. Recall the following classical conjecture, attributed to Kawamata and Ambro:

CONJECTURE 3 ([**Kaw00**, Conj. 2.1], Effective non-vanishing). Let $Z$ be a normal and complete variety with an effective $\mathbb{R}$-divisor $\Delta$ such that the pair $(Z, \Delta)$ is klt. Moreover, let $H$ be a nef Cartier divisor on $Z$ such that $H - (K_Z + \Delta)$ is big and nef. Then $H^0(Z, H)$ is nonzero.

This chapter is in essence a discussion of a special case of this general conjecture, where we put $Z = Y$, $\Delta = 0$, $H = L$ and our claim is that $h^0(Y, L) > 0$ irrespectively of the index. The method that we present utilizes Hilbert polynomial calculations, so it depends mainly on the value of the index. Consequently, it is quite general, but it allows us to obtain positive results only when this value is big enough. On the other hand, for special families of Fano varieties one can determine $h^0(Y, L)$ by other means, and in particular there are examples of Fano varieties with $h^0(Y, L) \neq 0$ with arbitrarily small value of the index (we give some smooth examples in Section 3.6).

The computations for the first four considered cases are classical, and the results can be found in [**IP99**, Cor. 2.1.14].

**3.3.1.** $i_Y = \dim Y + 1$**.** In that case we obtain the complete decomposition of the Hilbert polynomial:

$$p(t) = \frac{L^n}{n!}(t+1)(t+2)...(t+n). \tag{8}$$

Moreover, the condition that $p(0) = 1$ translates to $L^n = 1$. We conclude that

$$h^0(Y, L) = p(1) = n + 1. \tag{9}$$

**3.3.2.** $i_Y = \dim Y$**.** Now we start with one less root of the Hilbert polynomial, however Observation 3.3.1 gives us a full decomposition:

$$p(t) = \frac{L^n}{n!}(t+1)(t+2)...(t+\left\lfloor\frac{n}{2}\right\rfloor)(t+\frac{n}{2})(t+\left\lfloor\frac{n}{2}\right\rfloor+1)...(t+n-1). \tag{10}$$

Again, the value of $p(t)$ at 0 gives us $L^n = 2$ and in consequence:

$$h^0(Y, L) = p(1) = n + 2. \tag{11}$$

Remark 3.3.2. In fact, we can say much more about Fano varieties with $i_Y \geq n$, as Fujita [**Fuj90**, Ch. I, Thm 5.10 and Thm 5.15] classified polarized pairs $(V, \mathcal{L})$ (not necessarily Fano) with $\Delta(V, \mathcal{L}) = 0$, where we put $\Delta(V, \mathcal{L}) = \dim V + \mathcal{L}^{\dim V} - h^0(V, \mathcal{L})$ (see also [**IP99**, Thm 3.1.14] for a presentation focused on the Fano case). In particular, if $V$ is a smooth Fano variety, then it can be either a projective space or a smooth quadric, what was also shown earlier by Kobayashi and Ochiai [**KO73**]. If we allow rational Gorenstein singularities, then the only examples are provided by singular quadrics, i.e. cones over smooth irreducible quadrics of lower dimensions.

**3.3.3.** $i_Y = \dim Y - 1$**.** We know $n-2$ roots of the Hilbert polynomial:

$$p(t) = \frac{L^n}{n!}(t+1)(t+2)...(t+n-2)(t^2 + at + b). \tag{12}$$

Again, we use Serre duality (5) and $p(0) = 1$ to obtain:

$$a = n - 1, \qquad b = \frac{n(n-1)}{L^n}.$$

We conclude that:

$$h^0(Y, L) = p(1) = L^n + n - 1 \geq n. \tag{13}$$

Remark 3.3.3. If we assume that $Y$ is smooth then again we have a classification provided by Fujita in a series of papers [**Fuj80**], [**Fuj81**], [**Fuj84**] according to the auxiliary invariant $L^n$ which is an integer between 1 and 8.

**3.3.4.** $i_Y = \dim Y - 2$**.** In this case we can write the Hilbert polynomial as:

$$p(t) = \frac{L^n}{n!}(t+1)(t+2)...(t+n-3)(t^3 + at^2 + bt + c) \tag{14}$$

As before, we can calculate the coefficients $a, b, c$ via Serre duality and the evaluation at 0:

$$a = \frac{3}{2}(n-2), \qquad b = \frac{2n(n-1)}{L^n} + \frac{(n-2)^2}{2}, \qquad c = \frac{n(n-1)(n-2)}{L^n}.$$

It follows that:

$$h^0(Y, L) = p(1) = n + \frac{L^n}{2} \geq n + 1. \tag{15}$$

We obtained the desired inequality by observing that $p(t)$ is an integer-valued polynomial, so in particular $L^n$ has to be an even number.

REMARK 3.3.4. Again, Fano varieties having $i_Y = n-2$ that are smooth are classified by Mukai [**Muk89**], under an additional assumption on the fundamental linear system proved by Mella [**Mel99**], we will briefly discuss it and analogous results in Section 3.4.

**3.3.5.** $i_Y = \dim Y - 3$**.** This case is where the classification is not available, and we can only provide statements on the fundamental linear system. It requires dealing with another (after the dimension and the top self-intersection $L^n$) invariant of a variety, namely $c_2(Y) \cdot L^{n-2}$. The analysis of this case is thanks to Floris [**Flo13**], however in dimensions 6 and 7 an additional assumption on semistability was needed to conclude. Then Liu proved his analogue of the Bogomolov's inequality (Theorem 3.2.5) and was able to get rid of this additional assumption, claiming Theorem 3.3.5.

As before, we decompose the Hilbert polynomial in known linear terms:

$$p(t) = \frac{L^n}{n!}(t+1)(t+2)...(t+n-4)(t^4+at^3+bt^2+ct+d). \tag{16}$$

On the other hand recall that we know the highest coefficients of $p(t)$ in monomial basis (4) and if we combine it with Serre duality and value of $p(t)$ at 0 we obtain:

$$\begin{aligned} a &= 2(n-3), \\ b &= \frac{-n^4+8n^3+9n^2-160n+264}{24} + \frac{n(n-1)c_2(Y)\cdot L^{n-2}}{12L^n}, \\ c &= (n-3)(b-(n-3)^2), \\ d &= \frac{n(n-1)(n-2)(n-3)}{L^n}. \end{aligned}$$

In consequence:

$$h^0(Y,L) = p(1) = \frac{L^n}{24}(-n^2+7n-8) + \frac{c_2(Y)\cdot L^{n-2}}{12} + n - 3 \tag{17}$$

Observe that, contrary to the previous cases, we cannot claim the existence of sections independently of dimension. Nevertheless, the polynomial $-n^2+7n-8$ takes positive values for $n = 4, 5$, and the term with $c_2$ is nonnegative by (6), so for these two dimensions we have:

$$h^0(Y,L) \geq n-3+1 = n-2. \tag{18}$$

We are ready to state and prove the following theorem:

THEOREM 3.3.5 ([**Liu19**, Thm 1.2]). *Let $Y$ be a smooth Fano variety of dimension $n$ and index $i_Y = n-3$. Then we have $h^0(Y,L) \geq n-2$.*

PROOF. The proof that we present is essentially the same as in original expositions, but we treat all of the cases together and do not skip the verification of all examples with $\rho(Y) \geq 2$.
*Case $n \geq 8$.* A Fano manifold of dimension at least 8 and index $\dim(Y) - 3$ has by the result of Wiśniewski [**Wiś90**, Thm B] Picard number $\rho(Y) = 1$ unless it is isomorphic to $\mathbb{P}^4 \times \mathbb{P}^4$, and in that particular case a simple calculation yields ($H$ denotes the class of the hyperplane in the projective space):

$$h^0(\mathbb{P}^4 \times \mathbb{P}^4, L) = h^0(\mathbb{P}^4, H)^2 = 25 > n-2 = 6.$$

If $\rho(Y) = 1$ then by [**Hwa01**, Thm 2.11] the tangent bundle of $Y$ is stable, as $i_Y = n-3 > \frac{n+1}{2}$, so we can use the inequality (7), which in this case becomes:

$$c_2(Y) \cdot L^{n-2} \geq \frac{n-1}{2n} i_Y^2 L^n.$$

It follows that:

$$\begin{aligned} h^0(Y,L) =& \frac{L^n}{24}(-n^2+7n-8)+\frac{c_2(Y)\cdot L^{n-2}}{12}+n-3 \\ \geq& \frac{L^n}{24}(-n^2+7n-8+\frac{n-1}{n}(n-3)^2)+n-3= \\ =& \frac{L^n}{24}\cdot\frac{7n-9}{n}+n-3. \end{aligned}$$

Since the fraction is positive and $h^0(Y,L)$ is necessarily an integer, we obtain

$$h^0(Y,L)\geq n-2.$$

*Case* $n=7$. We cannot assume the semistability if $\rho(Y)=1$, however we can use Liu's Theorem 3.2.5 instead. It reads:

$$c_2(Y)L^5\geq\frac{121}{18}L^7.$$

In consequence we get:

$$h^0(Y,L)\geq-\frac{L^7}{3}+\frac{121}{216}L^7+4=\frac{49}{216}L^7+4,$$

and therefore

$$h^0(Y,L)\geq 5,$$

so the claim follows. Now if $\rho(Y)\geq 2$, then [**Wiś91**, Theorem] has classified such varieties. Each of them is a projectivization of a vector bundle $\mathcal{E}$ on a lower dimensional variety $Z$, such that for the natural projection $\pi\colon Y\to Z$ we have $\mathcal{E}=\pi_*L$, so that $h^0(Y,L)=h^0(Z,\mathcal{E})$. The list of possible cases is as follows ($Q^n$ denotes an $n$-dimensional quadric):

| $Z$ | $\mathcal{E}$ | $h^0(Y,L)$ |
|---|---|---|
| $Q^4$ | $\mathcal{O}(1)^{\oplus 4}$ | 24 |
| $\mathbb{P}^4$ | $T\mathbb{P}^4$ | 24 |
| $\mathbb{P}^4$ | $\mathcal{O}(2)\oplus\mathcal{O}(1)^{\oplus 3}$ | 30 |

As the claim holds in all possible cases, we are done.

*Case* $n=6$. We argue as before: if $\rho(Y)=1$, then by [**Hwa98**, Thm 3] the tangent bundle is semistable, so we have Bogomolov's inequality (7):

$$c_2(Y)\cdot L^4\geq\frac{15}{4}\cdot L^6.$$

Then the expression for $h^0(Y,L)$ can be estimated by:

$$h^0(Y,L)\geq-\frac{7}{24}L^6+\frac{15}{48}L^6+3=\frac{1}{48}L^6+3.$$

Consequently,

$$h^0(Y,L)\geq 4.$$

In the case where $\rho(Y)>1$ we again use existing classification results [**Wiś94**, Prop. 3.3] and check case by case. The maximal possible value of $\rho(Y)$ is equal to 3, and it is attained only if $Y=\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2$. Then we have $h^0(Y,L)=3^3$. There are three families of six dimensional Fano manifolds with index 3 and Picard number 2. In the simplest case, the situation is analogous to dimension 7, i.e. $Y$ is a projectivized bundle $\pi\colon\mathbb{P}(\mathcal{E})\to Z$, where $Z$ is a lower dimensional variety with a locally free sheaf $\mathcal{E}$ equal to $\pi_*L$. Moreover, $Z$ can be either $\mathbb{P}^4$ or $Q^4$ or $V_d$, i.e. a 4-dimensional del Pezzo manifold of degree $d\in\{1,...,5\}$. The table below lists all representatives of the first family:

| $Z$ | $\mathcal{E}$ | $h^0(Y,L)$ |
|---|---|---|
| $V_d$ | $\mathcal{O}_{V_d}(1)^{\oplus 3}$ | $3 \cdot (d+2)$ |
| $\mathbb{P}^4$ | $\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1)$ | 35 |
| $\mathbb{P}^4$ | $\mathcal{O}(3) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ | 110 |
| $Q^4$ | $\mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$ | 32 |
| $Q^4$ | $S(1) \oplus \mathcal{O}(1)$ | 10 |

By $S$ we denote the spinor bundle over the quadric [**Ott88**, Def. 1.3].

The second family also consists of projectivized bundles $\mathbb{P}(\mathcal{E})$ over a smooth, lower dimensional base $Z$, however this time $\mathcal{E}$ is not locally free, but there is a locally free extension $\mathcal{F}$ of $\mathcal{E}$ by $\mathcal{O}_Z$, i.e. we have a short exact sequence:

$$0 \to \mathcal{O}_Z \to \mathcal{F} \to \mathcal{E} \to 0.$$

Moreover, $Z$ is either $\mathbb{P}^4$ or $Q^4$, so in particular it is a Fano manifold, and by the long exact sequence in cohomology we have that $h^0(Z,\mathcal{E}) = h^0(Z,\mathcal{F}) - 1$, and both of those numbers are equal to $h^0(Y,L)$. There are 4 such cases:

| $Z$ | $\mathcal{F}$ | $h^0(Y,L)$ |
|---|---|---|
| $\mathbb{P}^4$ | $T\mathbb{P}^4$ | 23 |
| $\mathbb{P}^4$ | $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 3}$ | 29 |
| $\mathbb{P}^4$ | $\mathcal{G}$ | 20 |
| $Q^4$ | $\mathcal{O}(1)^{\oplus(4)}$ | 23 |

$\mathcal{G}$ is a spanned locally free sheaf such that $-K_{\mathbb{P}(\mathcal{G})} = \mathcal{O}_{\mathbb{P}(\mathcal{G})}(3)$ is nef and the map associated to $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(3)$ contracts a section of the projective bundle over a hyperplane in $\mathbb{P}^4$ to a point.

Finally, $Y$ can have the structure of a quadric bundle over $Z$. It follows that $Y$ is a divisor of relative degree 2 in $\mathbb{P}(\mathcal{E})$ over $Z$, i.e. $Y \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \pi^*\mathcal{O}_Z(-K_Z - \det \mathcal{E})|$. The table below lists all possibilities:

| $Z$ | $\mathcal{E}$ | $h^0(Y,L)$ |
|---|---|---|
| $\mathbb{P}^3$ | $T\mathbb{P}^3 \oplus \mathcal{O}(1)^{\oplus 2}$ | 23 |
| $\mathbb{P}^3$ | $\mathcal{O} \oplus \mathcal{O}(1)^{\oplus 4}$ | 17 |
| $\mathbb{P}^3$ | $\mathcal{O}(1)^{\oplus 5}$ | 20 |
| $\mathbb{P}^3$ | $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 4}$ | 26 |
| $Q^3$ | $\mathcal{O}(1)^{\oplus(5)}$ | 25 |

In all possible cases, the space of sections has dimension greater than 4, so we are done.

*Case* $n = 4, 5$. We have already observed that for the two smallest possible dimensions of $Y$ the estimate holds even in the Gorenstein-Fano case by inequality (18). ☐

**3.3.6.** $i_Y = \dim Y - 4$**.** The analysis of this value of the index is – according to author's knowledge – nonexistent in literature, with the exception of the particular case where $\dim Y = 5$, which was earlier treated by Höring in a collaboration with the author in [**HŚ20**].

Utilizing Observation 3.3.1, we write the Hilbert polynomial as:

$$p(t) = \frac{L^n}{n!}(t+1)(t+2)\dots(t+n-5)\left(t + \frac{n-4}{2}\right)\left(t^4 + at^3 + bt^2 + ct + d\right).$$

Again, we use the value at 0, Serre duality and the expression (4) to find the unknown coefficients:

$$
\begin{aligned}
a &= 2(n-4)\\
b &= \frac{n(n-1)\cdot c_2(Y)\cdot L^{n-2}}{12\cdot L^n} - \frac{1}{12}(n-4)(n^3-6n^2-21n+114)\\
c &= (n-4)(b-(n-4)^2)\\
d &= \frac{2\cdot n!}{L^n\cdot (n-4)!}.
\end{aligned}
$$

Evaluation at 1 yields:

$$
p(1) = n-2+\frac{c_2(Y)\cdot L^{n-2}}{24} - \frac{L^n}{48}(n-3)(n-6), \tag{19}
$$

which again is not enough to reach conclusion in all dimensions in full generality. The term with $c_2$ is nonnegative by Ou's inequality (6) and the last term is positive for $n=5$ and nonnegative for $n=6$, so we have $p(1)\geq 4$ for these dimensions. What we have proven so far is enough to claim the following useful result, also observed in a weaker form (nonvanishing) in [**HŚ20**, Cor. 1.3]:

COROLLARY 3.3.6. *Let $Y$ be a Gorenstein-Fano variety of dimension $\leq 5$ with canonical singularities that is different from a point. Then*

$$h^0(Y,L)\geq 2.$$

Moreover, we are able to retrieve the first part of the main result of Höring and the author [**HŚ20**, Thm 1.1a]:

THEOREM 3.3.7. *Let $Y$ be a Gorenstein-Fano variety of dimension 5 with canonical singularities. Then we have*

$$h^0(Y,-K_Y)\geq 4.$$

PROOF. The dimension of $H^0(Y,-K_Y)$ is equal to $p(t)$ evaluated at $i_Y$ for the suitable Hilbert polynomial (notice that we need to consider Hilbert polynomials calculated for all studied cases) and for $n=5$. We have just done it for $i_Y=1$. For $i_Y=2$ take the Equation (16) and expressions for the coefficients of the Hilbert polynomial and evaluate at $t=2$ to obtain:

$$p(2) = 3+\frac{8}{45}L^5+\frac{2}{9}c_2(Y)\cdot L^3 \geq 4.$$

Note that the considered expression is not an estimation, but a precise value, so in particular it is an integer. We conclude by the postivity of $L^5$ and the nonnegativity of $c_2(Y)\cdot L^3$ coming from Ou's inequality (6).

Finally, if the index is greater than 2, then we know already that $p(1)>5$ by Equations (9), (11), (13) or (15). To conclude, simply observe that in our setting the Hilbert polynomial is an increasing function for positive integers. Indeed, if $i_Y\geq 5$ then we have the full decomposition into linear terms (Equation (8) and Equation (10)) and all the roots are negative. If $i_Y\in\{3,4\}$ then the Hilbert polynomials (12) and (14) decompose into some linear terms that correspond to negative roots and a quadratic or cubic term whose coefficients are positive, so in both cases they are increasing for positive integers. ☐

Now we would like to have an analogue of Theorem 3.3.5, but we cannot claim the nonvanishing in the case when $b_2(Y)>1$, $n=7$, $i_Y=3$, as for such manifolds we do not have sufficiently strong inequalities, nor are they classified. Nevertheless, we can prove the following new result:

THEOREM 3.3.8. *Let $Y$ be a smooth Fano variety of dimension $n$ and index $i_Y = n-4$ with $b_2(Y) = 1$. Then we have:*

$$h^0(Y, L) \geq n - 1.$$

PROOF. If $n > 9$ then the tangent bundle of $Y$ is semistable by [**Hwa01**, Thm 2.11] or [**Hwa98**, Thm 3], so we are able to use the Bogomolov's inequality (7) to estimate:

$$p(1) \geq n - 2 + \frac{3n-8}{24n} \cdot L^n,$$

and as $p(t)$ in an integer valued polynomial and the fraction is positive, we obtain $p(1) \geq n-1$. If $n \in \{7, 8, 9\}$ then we apply Liu's inequality instead:

$$p(1) \geq n - 2 + \frac{3n^2 - 14n - 6}{(n-5)(n-1)} \cdot \frac{L^n}{48}$$

and we reach the conclusion by the same argument as before.

If $n = 6$, then recall from our earlier discussion that the tangent bundle is stable by [**Hwa98**], so Bogomolov's inequality implies the positivity of the second Chern class, and the evaluation of Equation (19) yields the result.

Finally, in the case where $n = 5$ we apply Theorem 3.3.7. □

**3.3.7. Low dimensional, log canonical Fano varieties.** Up to this point, all results presented in Section 3.3 relied heavily on the rationality of singularities, so that we were able to use the Hilbert polynomial and the Hirzebruch-Riemann-Roch theorem. In the case when singularities are worse, for instance log canonical, the notion of the fundamental divisor is no longer meaningful as the Picard group may have a nontrivial torsion. Nevertheless, one can still study the anticanonical linear system. In such a setting Höring with the author ([**HŚ20**, Prop. 1.4]) proved the following:

THEOREM 3.3.9. *Let $Y$ be a normal Gorenstein-Fano variety with log-canonical singularities of dimension at most 5. Then $H^0(Y, -K_Y) \neq 0$.*

We will make use of the following lemma:

LEMMA 3.3.10 ([**HŚ20**, Prop. 2.1, Cor. 2.2]). *Let $Z$ be a normal projective threefold such that $(Z, \Delta)$ is klt and $-(K_Z + \Delta)$ is pseudoeffective for some effective $\mathbb{R}$-divisor $\Delta$. If $D$ is a nef and big Cartier divisor on $Z$, then $H^0(Z, D) > 0$.*

PROOF. The assumption on the log anticanonical divisor means that $-K_Z$ is numerically equivalent to a sum of an effective and a pseudoeffective divisor, so that it is generically nef, i.e. we have $-K_Z \cdot H^2 \geq 0$ for any nef Cartier divisor $H$. Moreover, we can assume that $Z$ is $\mathbb{Q}$-factorial, as we can use the projection formula for the $\mathbb{Q}$-factorial modification [**Kol13**, Cor. 1.37]. By [**Deb01**, Thm 3.10] there are two families of varieties with a generically nef anticanonical divisor: either they are uniruled or $-K_Z$ is in fact numerically trivial.

In the first case, we use the Hilbert polynomial. To that end, observe that $Z$ has rational singularities by Theorem 2.1.6, so $h^i(Z, \mathcal{O}_Z) = h^i(\widetilde{Z}, \mathcal{O}_{\widetilde{Z}})$ for any resolution of singularities $\widetilde{Z}$. $\widetilde{Z}$ is also uniruled, therefore by Serre duality 2.1.1 $h^3(\widetilde{Z}, \mathcal{O}_{\widetilde{Z}}) = h^0(\widetilde{Z}, \mathcal{O}(K_{\widetilde{Z}}))$ and the canonical bundle of a uniruled variety does not have sections by [**Kol96**, Ch. IV, Cor. 1.11], so $h^3(\widetilde{Z}, \mathcal{O}_{\widetilde{Z}}) = 0$. Moreover, we can also assume that $h^1(Y, \mathcal{O}_Z) = 0$, as otherwise the lemma was proved in [**Xie09**, Cor. 4]. In consequence, we get that $\chi(Z, \mathcal{O}_Z) = h^0(Z, \mathcal{O}_Z) + h^2(Z, \mathcal{O}_Z) \geq 1$.

Now consider the Hilbert polynomial:

$$p(t) = \chi(Z, tD) = \frac{D^3}{3!}(t^3 + at^2 + bt + c)$$

and observe that by a standard application of Kawamata-Viehweg vanishing 3.1.2 we have $\chi(Z, D) = h^0(Z, D)$. As $D$ is a nef and big divisor on a variety with rational singularities, cohomologies of $-D$, excluding the top ones, vanish, i.e. $\chi(Z, -D) = -h^3(Z, -D) \leq 0$. In effect, we can get rid of odd terms in the polynomial:

$$h^0(Z, D) = \chi(Z, D) \geq \chi(Z, D) + \chi(Z, -D) = \frac{D^3}{3}(a + c).$$

To conclude, we need to show that $a+c > 0$, as $D^3 > 0$. Recall the Riemann-Roch formula (3) and compare it with our expression for the Hilbert polynomial of $tD$ to obtain:

$$a\frac{D^3}{3!} = \frac{-K_Z \cdot D^2}{4} > 0, \qquad\qquad c\frac{D^3}{3!} = \chi(Y, \mathcal{O}_Z) > 0.$$

Therefore, we conclude that $h^0(Z, D) > 0$.

Now consider the second possibility, i.e. assume that $K_Z$ is numerically trivial. If the singularities of $Z$ are not canonical, then consider its canonical modification $Z'$, existing by [**Kol13**, Thm 1.31]. Then the divisor $-K_{Z'}$ is numerically equivalent to a non-zero, effective $\mathbb{Q}$-divisor, so $Z'$ is uniruled, and we may apply the reasoning above. Finally, if the singularities of $Z$ are canonical, then the nonvanishing was shown in [**Kaw00**, Prop. 4.1] by the means of crepant blowups and Riemann-Roch computations. □

Proof of Theorem 3.3.9. We may assume that there exists a minimal (with respect to inclusion) lc center $Z$ of the pair $(Y, 0)$, because otherwise singularities would be canonical, as $Y$ is Gorenstein. $Y$ is normal of dimension at most 5, so its minimal lc center $Z$ is of dimension at most 3. The restriction map

$$H^0(Y, -K_Y) \to H^0(Z, -K_{Y|Z})$$

is surjective by [**Fuj11**], as $-K_Y - (K_Y + 0) = -2K_Y$ is ample. Moreover, by the Kawamata subadjunction [**Kaw98**, Thm 1] there exists an effective $\mathbb{Q}$-divisor $\Delta_Z$ such that the pair $(Z, \Delta_Z)$ is klt and log Fano, as $K_Z + \Delta_Z \sim_{\mathbb{Q}} K_{Y|Z}$. Therefore, it is enough to show the nonvanishing of $H^0(Z, -K_{Y|Z})$, and that follows from [**Kaw00**, Thm 3.1] if $\dim Z \leq 2$ and Lemma 3.3.10 if $\dim Z = 3$, as minimal lc centers are normal by [**Amb99**, Thm 1.6]. □

## 3.4. Structure of the general element

Suppose we are given a Fano variety $Y$ with the prescribed index $i_Y$ and the class of singularities. Besides the dimension of the linear system, one can also ask if a general element $Z \in |L|$ is *good*, i.e. whether its singularities are not worse than those of the ambient variety $Y$. If this condition is satisfied, then it can serve to construct a ladder:

$$Y = Y_0 \supset Y_1 \supset Y_2 \supset ... \supset Y_{i_y},$$

where $Y_{j+1}$ is an element in the fundamental linear system of $Y_j$, and all $Y_j$ have equal coindices and the same class of singularities. Such ladders can serve as a basis for induction arguments. Although we will not use similar arguments in the next chapters, for the completeness of the presentation we will discuss existing results concerning the goodness of the fundamental divisor.

Alexeev [**Ale91**] proved the existence of a good divisor for log terminal Fano varieties with index $i_Y > \dim(Y) - 2$. For the case of index $i_Y = \dim(Y) - 2$ it was shown by Mella [**Mel99**], in the case of smooth and canonical variety. Floris [**Flo13**] proved the result for canonical Fano varieties of index $\dim(Y) - 3$ and this is the last general case. Höring with the author [**HŚ20**, Thm 1.1.2] proved the following:

Theorem 3.4.1. *Let $Y$ be a Fano variety of dimension 5 with canonical Gorenstein singularities. If a general element $D \in |-K_Y|$ is reduced, then it has canonical singularities.*

We will now present a proof for this theorem. The results of Alexeev, Mella and Floris mentioned above were proved using similar techniques (i.e. the inversion of adjunction and the Kawamata subadjunction) and reasoning.

Proof. To prove that $D$ has canonical singularities, we will use the inversion of adjunction [**Kol97**, Thm 7.5] and show that the pair $(Y, D)$ is plt. Suppose that this is not the case, i.e there exists a $0 < c \leq 1$ such that $(Y, cD)$ is log canonical and not plt for $c = 1$. The base locus of $-K_Y$ contains a minimal lc center $Z$ of $(Y, cD)$ by [**Amb99**, Lem. 5.1], so it follows that the map:

$$H^0(Y, -K_Y) \to H^0(Z, -K_{Y|Z})$$

must be zero. Simultaneously it is surjective by [**Fuj11**, Thm 2.2], as the divisor class $-K_Y - (K_Y + cD) = (2 - c)(-K_Y)$ is ample. We conclude that $H^0(Z, -K_{Y|Z}) = 0$.

To arrive at a contradiction, consider the klt pair $(Z, \Delta_Z)$, where $\Delta_Z$ exists by the Kawamata subadjunction [**Kaw98**]. We have:

$$K_Z + \Delta_Z \sim_{\mathbb{Q}} (K_Y + cD)_{|Z} \sim_{\mathbb{Q}} (1 - c)K_{Y|Z},$$

so the pair $(Z, \Delta_Z)$ is either log Fano or log Calabi-Yau.

Now, the dimension of $Z$ is at most 3, as none of the irreducible components of $D$ can be a minimal lc center for $(Y, cD)$ as we have assumed that it is reduced. If $\dim(Z) = 3$ then the nonvanishing showed in Lemma 3.3.10 yields the contradiction as the minimal lc center is normal. If $\dim(Z) < 3$, then we can use [**Kaw00**, Thm 3.1]. □

## 3.5. Contact Fano case

The goal of this section is to give a lower estimate on the dimension of $h^0(X, L)$ for $X$ which is a contact Fano manifold with a contact line bundle $L$ generating $\mathrm{Pic}(X)$ and of dimension up to 9. In fact, such estimates were obtained earlier by other researchers (see [**BW22**, Thm 6.1] for dimensions 7 and 9), but we obtain them as a corollary of the computations conducted throughout this chapter, so we do not use the stability of the tangent bundle, nor the existence of the contact structure (besides the value of the index). Therefore, our result is stated in a more general setting of a prime and smooth Fano manifold $Y$ of odd dimension $2n + 1$ and index $n + 1$.

In the next chapter we will see that $H^0(X, L)$ is isomorphic to the Lie algebra of contactomorphisms of $X$, and in conseqence the nonvanishing of $H^0(X, L)$ allows us to claim that $X$ has nontrivial automorphisms. We may also observe that our assumptions do not exclude any interesting case: by Theorem 2.3.2 if a Fano contact manifold does not satisfy our assumptions then it is either $\mathbb{P}^{2n+1}$ or $\mathbb{P}(T\mathbb{P}^n)$ and for both these cases the fundamental line bundle is very ample. If $\dim(X) = 3$ then by [**Ye94**, Thm 2] then the only prime Fano contact manifold is $\mathbb{P}^3$.

Corollary 3.5.1. *Let $Y$ be a smooth and prime Fano manifold with fundamental divisor $L$ of dimension $2n + 1$ and index $n + 1$ (e.g. $Y = X$ is a contact Fano manifold different from $\mathbb{P}^{2n+1}$ and $\mathbb{P}(T\mathbb{P}^{n+1})$ with the contact line bundle $L$).*

- *If $Y$ is of dimension 5, then $h^0(Y, L) \geq 6$.*
- *If $Y$ is of dimension 7 then $h^0(Y, L) \geq 5$.*
- *If $Y$ is of dimension 9 then $h^0(Y, L) \geq 8$.*

Proof. As the index is equal to $n + 1$, we know $n$ roots of the Hilbert polynomial (and Observation 3.3.1 gives us one additional if $n$ is even). Consequently, if $\dim(Y) = 5$ then $i(Y) = 3$ and we apply Equation 15:

$$h^0(Y, L) = 5 + \frac{L^3}{2} \geq 6.$$

If $\dim(Y) = 7$, then $i(Y) = 4$ then it is enough to use Theorem 3.3.5 to obtain the desired bound.

We are left with the last case, i.e. $\dim(Y) = 9$, for which we use Theorem 3.3.8 and obtain $h^0(Y, L) \geq 8$. □

## 3.6. Conclusion and another conjecture

As we have seen in this chapter, there is some evidence for the nonemptiness of the fundamental linear system of Fano varieties. This evidence encouraged the author to consider the following conjecture:

CONJECTURE 4. Let $Y$ be a smooth Fano variety of dimension $n \geq 1$ with $b_2(Y) = 1$ and fundamental line bundle $L$. Then $h^0(Y, L) \geq 2$.

Clearly, this conjecture is essentially a variant of Conjecture 3 of Kawamata and Ambro, where we restrict ourselves to prime and smooth Fano varieties without the boundary divisor, but on the other hand demand $h^0(Y, L) > 1$ instead of $h^0(Y, L) > 0$. The main motivation for introducing such conjecture is that proving it would have tremendous consequences for the LeBrun-Salamon conjecture, as we will show in the next chapter.

Besides the situation discussed in this chapter, i.e. when $i(Y) \geq n - 4$, the conjecture also holds when the fundamental divisor is very ample. The most natural examples are provided by the toric theory – any ample divisor on a smooth projective toric variety is very ample [**CLS11**, Thm 6.1.15]. This is also the case for smooth projectivized orbit closures – besides adjoint varieties we can mention two-orbit varieties studied by Pasquier [**Pas09**]. They are smooth and prime Fano varieties that decompose into precisely two orbits (open and closed) under the natural action of the automorphism group and the same holds for the blowup of such variety along the closed orbit. Their example is additionally interesting, as the very ampleness holds irrespectively of the semistability of the tangent bundle or the lack of it that we have mentioned earlier. In particular, the first family on Pasquier's list, denoted by $(B_n, \omega_{n-1}, \omega_n)$, has dimension $\frac{n(n+3)}{2}$, index $n+2$ and its tangent bundle is not semistable. Consequently, as $n$ grows, we obtain a family of examples where the difference between dimension and index has quadratic order in $n$, the tangent bundle is unstable, and yet the fundamental divisor has many sections.

Another rich source of computable examples is provided by complete intersections in weighted projective spaces. In this case one has an explicit description of the cohomology in terms of weights that allows to calculate the dimension of $h^0(Y, L)$. In particular, Conjecture 4 holds also in this case, see [**Ovc23**, Prop. 2.40] and the references therein.

CHAPTER 4

# Linear systems and projective contact manifolds

## 4.1. Introduction

We return now to projective contact manifolds and our goal is to discuss how the study of linear systems of Fano varieties can be applied in the work on the LeBrun-Salamon conjecture. In fact, the significance of having many sections of the contact line bundle was already observed by Beauville in the form of the following theorem:

THEOREM 4.1.1 ([**Bea98**, Thm 0.1]). *Let $(X, L)$ be a Fano contact manifold. If the group of automorphisms of $X$ is reductive and the rational map defined by the linear system of $L$ is generically finite, then $X$ is the adjoint variety associated to a simple group.*

As we have discussed in Section 1.3, the assumption on the reductivity of the group of automorphisms is redundant if we are considering only contact Fano manifolds equipped with a Kähler-Einstein metric, i.e. those coming from the twistor construction. However, as we have seen in Chapter 3, known methods are not sufficient to claim that $h^0(X, L)$ is big enough even in dimensions 7 and 9.

Nevertheless, the homogeneity was shown in dimension up to 9 (under the assumption on the reductivity of the automorphisms group) by Buczyński, Weber and Wiśniewski [**BW22**]. The methods proposed in [**BW22**], based on the analysis of the action of algebraic tori and BB decomposition, were extended by Occhetta, Romano, Solà Conde and Wiśniewski, resulting in the following theorem:

THEOREM 4.1.2 ([**ORCW21**, Thm 6.1]). *Suppose $(X, L)$ is a Fano contact manifold of dimension $2n+1$ with $\mathrm{Pic}(X) = \mathbb{Z} \cdot L$ and the group of automorphisms $G$ such that its identity component is reductive of rank $r \geq \max(2, \frac{n-3}{2})$. Then $G$ is a simple group of type $B_r$ $(r \geq 3)$, $D_r$ $(r \geq 4)$, $E_6$, $E_7$, $F_4$ or $G_2$ and $X$ is the associated adjoint variety.*

If one analyzes thoroughly the proof of the theorem above, one finds that the bound on $r$ is directly related to known results on the dimension of the fundamental linear system of a smooth Fano variety, and in particular Corollary 3.3.6. Therefore one may wonder what is the dependence of this and related theorems on results concerning nonvanishing and how improving them would influence our understanding of LeBrun-Salamon conjecture. The author is able to provide an answer in the form of the following theorem:

THEOREM 4.1.3. *Let $(X, L)$ be a Fano contact manifold of dimension $2n+1$ with a reductive group of automorphisms $G$. Assume that:*

- *$h^0(X, L) \neq 0$, i.e. the effective nonvanishing holds for $X$,*
- *Conjecture 4 holds up to dimension $n$, i.e. for every smooth Fano variety $Y$ of dimension at most $n$ with $b_2(Y) = 1$ we have $h^0(Y, \mathcal{L}) \geq 2$, where $\mathcal{L}$ is the fundamental divisor on $Y$.*

*Then $G$ is a simple group and either:*

1. *$X$ is the adjoint variety associated with $G$;*
2. *$G = \mathrm{SL}(2)$, $h^0(X, L) = 3$, the maximal torus $\mathbb{C}^* \subset G$ acts on $X$ in such a way that the source and the sink of the action are isolated points and $\dim(X) \geq 11$.*

We first note that the second case is the undesirable one, that – if one wants to prove the conjecture of LeBrun and Salamon – should be somehow excluded. From the statement itself we can see that to do it, it would be enough to show that it is not possible for a contact Fano manifold of dimension at least 11 to have $h^0(X, L) = 3$. Alternative way of proceeding would be to take a closer look at the postulated $SL(2)$ action.

We also note that although this theorem is an original contribution of the author and may be of interest to the community, the proof contains no new ideas and just boils down to verifying that numerous lemmas and propositions used to develop Theorem 4.1.2 work with our assumptions. The goal of this chapter is therefore to show how the estimates on the dimension of the fundamental linear system in the general and the contact Fano case, in particular Corollary 3.3.6 and Corollary 3.5.1, are necessary for the proof of Theorem 4.1.2 to work and how replacing them by our nonvanishing assumptions allows to claim Theorem 4.1.3. As eventually we want to discuss homogeneous varieties, we first take a brief detour to recall the fundamental notions of Lie theory for the purpose of fixing notation and definitions for this and subsequent chapters.

## 4.2. Lie theory basics

**4.2.1. Representations, weights and roots.** The most general type of (complex) Lie groups that we are be interested in are reductive Lie groups. As we are working in the complex category, we can assume that a group being reductive means that any finite-dimensional representation decomposes into a direct sum of irreducible ones.

Let $G$ be a reductive group, $\mathfrak{g}$ an associated Lie algebra and $T$ a maximal torus of $G$. The *(reductive) rank* $r$ of the group is the dimension of $T$. Every representation of $T$ splits into the sum of 1 dimensional representations, which can be identified with elements of $\mathrm{Hom}(T, \mathbb{C}^*)$ (known as characters). The choice of coordinates on $T$ allows further identification of the group of characters $\mathrm{Hom}(T, \mathbb{C}^*)$ with the lattice $\mathbb{Z}^r$, whose elements are called *weights*, traditionally denoted by $M$.

In particular, for any $G$, if we denote the inner automorphism associated with $g \in G$ by $\Psi_g$ then we can define adjoint representations of group and of algebra ($X, Y \in \mathfrak{g}$):

$$\begin{aligned} \mathrm{Ad}\colon G &\to \mathrm{GL}(\mathfrak{g}), & \mathrm{ad}\colon \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}) \\ \mathrm{Ad}_g(X) &= (d\Psi_g)_{id} X, & \mathrm{ad}_X(Y) &= [X, Y]. \end{aligned}$$

If moreover we denote by $\mathfrak{g}^*$ the dual vector space of the algebra, we can also define coadjoint representations ($f \in \mathfrak{g}^*$):

$$\begin{aligned} \mathrm{Ad}^*\colon G &\to \mathrm{Aut}(\mathfrak{g}^*), & \mathrm{ad}^*\colon \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}^*) \\ \mathrm{Ad}^*_g(f)(X) &= f(\mathrm{Ad}_{g^{-1}}(X)), & \mathrm{ad}^*_X f(Y) &= -f(\mathrm{ad}_X Y) \end{aligned}$$

The adjoint and coadjoint representation of a group induce the representation of the maximal torus, that in particular allows us to decompose the Lie algebra as:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{t}$ is the part of the algebra having weight 0 (and being precisely the Lie algebra of the torus, called *the Cartan subalgebra*) and the sum of 1-dimensional subspaces indexed by the set of nonzero weights $\Phi$ called roots. We can pick a hyperplane dividing the set of roots into two disjoint subsets, denoted $\Phi_+$ and $\Phi_-$ (positive and negative roots), in such a way that $\Phi_- = \{\alpha \in \Phi : -\alpha \in \Phi_+\}$. Among positive roots, we can distinguish those that are not a sum of any two other positive roots and call them *simple*. The convex hull of $\Phi \subset M \otimes \mathbb{R}$ is called the *root polytope*, denoted by $\Delta(G)$. Its dimension is equal to the number of simple roots, as they are linearly independent by [**Hal03**, Thm 8.16].

Weights admit a partial ordering: we say that $m_1$ *is higher* than $m_2$ if $m_1 - m_2$ can be expressed as a linear combination of positive roots with nonnegative coefficients. For a given subset of weights we say that a weight $m$ is *the highest weight* if it is higher than any other weight. There is a bijective correspondence between isomorphism classes of irreducible representations of semisimple complex Lie algebras and their highest weights [**FH91**, Prop. 14.13]. Moreover, every weight that can arise as the highest weight of a representation is a nonnegative, integral combination of *fundamental weights* [**FH91**, Thm 14.18].

Example 4.2.1. For the sake of future use, we will work out one particular example in detail. Namely, we start with a simple Lie group $\mathrm{Sp}(6)$ which acts naturally on $F$, a symplectic vector space of dimension 6. We pick a base $e_1, ..., e_6$ of $F$ and a dual base $x_1, ..., x_6$ of $F^*$ such that the symplectic form $\omega$ is equal to $x_1 \wedge x_4 + x_2 \wedge x_5 + x_3 \wedge x_6$ and $\mathrm{Sp}(6)$ consists of matrices preserving it. The maximal torus $T$ has dimension 3 and consists of matrices of the form $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, -\lambda_1, -\lambda_2, -\lambda_3)$ where $\lambda_i \neq 0$ for all $i$. Consider characters $h_i$ for $i = 1, 2, 3$ that act on $t \in T$ by extracting $(i, i)$-th coordinate of the matrix. By abuse of notation, we can think of $h_i$ as weights of the torus action. Denote by $E_{i,j}$ an elementary matrix having 1 in position $i, j$. Then the Lie algebra $\mathfrak{sp}(6)$ is spanned by:

- $E_{1,4}$, $E_{2,5}$, $E_{3,6}$ that are eigenvectors associated to the long roots $2h_1$, $2h_2$ and $2h_3$ respectively;
- $E_{1,2} - E_{5,4}$, $E_{2,3} - E_{6,5}$, $E_{1,3} - E_{6,4}$ associated to the short roots $h_1 - h_2$, $h_2 - h_3$ and $h_1 - h_3$ respectively;
- $E_{1,5} + E_{2,4}$, $E_{2,6} + E_{3,5}$, $E_{1,6} + E_{3,4}$ associated to the short roots $h_1 + h_2$, $h_2 + h_3$ and $h_1 + h_3$ respectively;
- the transposes of all of the above that correspond to the negatives of the roots;
- the elements from the Cartan subalgebra.

Consequently, we have made a choice of the set of the positive roots. It is generated by three simple roots $h_1 - h_2$, $h_2 - h_3$ and $2h_3$. The root polytope is a regular octahedron with vertices being the long roots.

The fundamental weights are $h_1$, $h_1 + h_2$ and $h_1 + h_2 + h_3$ and there exists an irreducible representation with the highest weight $(a + b + c)h_1 + (b + c)h_2 + ch_3$ for every $a, b, c \in \mathbb{N}$, denoted by $W_{(a,b,c)}$. In particular, $W_{(1,0,0)} = F$ and $W_{(2,0,0)} = \mathfrak{sp}(6) = S^2F^*$. Moreover, the symplectic form on $F$ gives us an isomorphism $S^2F^* \simeq S^2F$.

**4.2.2. Nilpotent orbits and their projectivizations.** We can also think of $\mathfrak{g}$ and its dual as varieties equipped with an action of $G$. In this way, they can be decomposed into orbits of this action, called adjoint and coadjoint orbits. The Killing form gives a $G$-invariant isomorphism between these vector spaces, that allows us to identify both types of orbits. In particular, we may say that a coadjoint orbit has some property, if only corresponding adjoint orbit has it.

Among orbits of $\mathfrak{g}$ we may distinguish orbits that are semisimple, nilpotent or neither (sometimes called mixed). Recall that every endomorphism of a finite dimensional vector space has a Jordan decomposition into the sum of semisimple and nilpotent part. We say that an element $X \in \mathfrak{g}$ is semisimple (nilpotent) if and only if the corresponding endomorphism $\mathrm{ad}_X$ is. Then any adjoint orbit containing a semisimple (nilpotent) element is called semisimple (nilpotent) and we may also define these types for coadjoint orbits via the Killing form isomorphism.

On every coadjoint orbit we can construct a nondegenerate 2-form giving it the structure of a (smooth) symplectic variety. Namely, let $f \in \mathfrak{g}^*$ and denote by $\mathcal{O}_f$ the orbit through $f$, which can be identified with the quotient $G/G_f$, where $G_f = \{g \in G : \mathrm{Ad}^*_g(f) =$

$f\}$ is the stabilizer subgroup. Its Lie algebra is equal to $\mathfrak{g}_f = \{X \in \mathfrak{g} : \mathrm{ad}^*_X(f) = 0\}$. Now define a 2-form:

$$\omega_f \colon \mathfrak{g} \times \mathfrak{g} \ni (X, Y) \mapsto f([X, Y]) \in \mathbb{C}.$$

It is degenerate precisely on $\mathfrak{g}_f$ and we can identify $T_f\mathcal{O}_f$ with $\mathfrak{g}/\mathfrak{g}_f$. The resulting symplectic form on $\mathcal{O}_f$ is called the Kostant-Kirillov form ([**Bea98**, Section 2]), which in the case of nilpotent orbits is homogeneous of weight 1 with respect to the natural action of $\mathbb{C}^*$ ([**Bea98**, Prop. 2.2]). Equivalently, we could observe that the Lie bracket on $\mathfrak{g}$ defines the structure of a holomorphic Poisson scheme on $\mathfrak{g}^*$ (see Section 5.2.3 for more detail) and coadjoint orbits are symplectic leaves of this structure.

The set of (co)adjoint orbits is equipped with a partial ordering: $\mathcal{O}_1 \leq \mathcal{O}_2 \iff \mathcal{O}_1 \subset \overline{\mathcal{O}_2}$. The minimal elements (i.e. closed orbits) are semisimple orbits and the zero orbit. Nilpotent orbits can be characterized precisely as those orbits that are comparable with the zero orbit in the closure ordering. Moreover, the set of nilpotent orbits posseses a unique maximal element – called the regular or principal orbit. Then the structure gets more complicated, however nilpotent orbits are finite in number and they can be classified by combinatorial objects [**CM93**, Thm 3.5.4]. If we assume that the group $G$ is simple then we have two more unique orbits, namely the minimal that has only 0 in its boundary and the subregular that is dominated only by the principal one.

EXAMPLE 4.2.2. In the case of $\mathrm{Sp}(2n)$ nilpotent orbits correspond bijectively to partitions of $2n$ where every odd part occurs with even multiplicity by [**CM93**, Thm 5.1.3]. For $2n = 6$ we have the following diagram of nonzero nilpotent orbits as presented in [**CM93**, Example 6.2.6] (we write $\mathcal{O}_1 \to \mathcal{O}_2$ if $\mathcal{O}_2 \leq \overline{\mathcal{O}_1}$).

$$\begin{tikzcd}
\mathcal{O}_{[6]} \arrow[r] & \mathcal{O}_{[4,2]} \arrow[r] \arrow[dr] & \mathcal{O}_{[4,1^2]} \arrow[r] & \mathcal{O}_{[2^3]} \arrow[r] & \mathcal{O}_{[2^2,1^2]} \arrow[r] & \mathcal{O}_{[2,1^4]} \\
& & \mathcal{O}_{[3^2]} \arrow[ur] & & &
\end{tikzcd}$$

$\dim = 18$ $\dim = 16$ $\dim = 14$ $\dim = 12$ $\dim = 10$ $\dim = 6$

The significance of the minimal orbit becomes clear when we observe that every nilpotent orbit is a cone, i.e. it is preserved by the natural action of the multiplicative group on $\mathfrak{g}$ (or equivalently $\mathfrak{g}^*$). Then the minimal orbit becomes the unique closed orbit in $\mathbb{P}(\mathfrak{g})$ ($\mathbb{P}(\mathfrak{g}^*)$), i.e. a closed, homogeneous subvariety of the projective space, *the (co)adjoint variety.* They are listed below:

| G | root system | coadjoint variety $X$ |
|---|---|---|
| SL(n+2) | $A_{n+1}$ | $\mathbb{P}(T\mathbb{P}^{n+1})$ |
| SO(n+4) | $B_{\frac{n+3}{2}}$ | Grassmannian of projective lines on $Q^{n+2}$ |
| Sp(2n+2) | $C_{n+1}$ | $v_2(\mathbb{P}^{2n+1})$ |
| SO(n+4) | $D_{\frac{n+4}{2}}$ | Grassmannian of projective lines on $Q^{n+2}$ |
| $E_6$ group | $E_6$ | $E_6$ variety |
| $E_7$ group | $E_7$ | $E_7$ variety |
| $E_8$ group | $E_8$ | $E_8$ variety |
| $F_4$ group | $F_4$ | $F_4$ variety |
| $G_2$ group | $G_2$ | Grassmannian of special lines on $Q^5$ |

The algebraic version of the LeBrun-Salamon conjecture states precisely that the list above contains all possible contact Fano manifolds.

We may also observe that although all (projectivized) orbits are smooth, they are not closed, and their closures are singular (they may even fail to be normal). This is a model example of the symplectic/contact stratification presented in Section 5.2.3.

## 4.3. Fixed point components and polytopes

Even though our description of the machinery developed to prove Theorem 4.1.2 is partial and focused on those elements of the reasoning that utilize the existence of sections, we still need to discuss in greater detail fixed point components of a torus action and introduce some combinatorial objects. We can do it in the setting of a smooth polarized variety, i.e. a smooth projective variety $Y$ equipped with a choice of an ample line bundle $L$. We moreover assume that $Y$ is equipped with an effective action of an algebraic torus $T \approx (\mathbb{C}^*)^r$ of some rank $r$. Recall from Section 4.2 that we can associate to $T$ a rank $r$ lattice $M$, whose elements are called weights. The will often use the following notion:

Definition 4.3.1. We say that a vector bundle $\mathcal{E}$ on a variety $Y$ equipped with an action of a group $G$ is *$G$-linearized* (or simply *linearized* if the choice of the group is clear) if there exists a lift of the action to the total space of $\mathcal{E}$ commuting with the projection onto the base variety $Y$. Sometimes we will instead say that $\mathcal{E}$ is a $G$-vector bundle and we will denote it by $\mu\colon \mathcal{E} \times G \to \mathcal{E}$.

We are interested in a situation where the ample line bundle $L$ admits a $T$-linearization, so we always assume that it exists. Then, the vector space $H^0(Y, L)$ can be decomposed into eigenspaces of the action of $T$, indexed by weights $m$, i.e.

$$H^0(Y, L) = \bigoplus_{m \in M} H^0(Y, L)_m.$$

Let $Y^T$ denote the set of fixed points. It has a finite number of connected components: $Y^T = Z_1 \sqcup \ldots \sqcup Z_l$. For each $y \in Y^T$ we have an action of $T$ on the fiber $L_y$ via some weight $\mu(y) \in M$, which is constant on every connected component, therefore we can consider $\mu(Z_i) \in M$, i.e. the weight associated to the fixed point component.

In the particular case of a rank 1 torus we can identify the set of weights with integers. In consequence, we have two distinguished fixed point components with minimal and maximal weight, called the sink ($Z_+$) and the source ($Z_-$).

We are ready to define two lattice polytopes living in $M$:

Definition 4.3.2. Consider the set of the weights $\{\mu(Z_i)\}$, where $Z_i$ is the component of $Y^T$. The convex hull of this finite set will be denoted by $\Delta$ and will be called the polytope of fixed points. Observe that it depends on $Y$, $L$, $T$ and $\mu$ and sometimes we will indicate that dependence by notation $\Delta(Y, L, T, \mu)$. Moreover, the components $Z_i$ whose weights are vertices of $\Delta$ will be called *extremal components*.

Definition 4.3.3. Consider the finite set of weights $m \in M$ such that for the eigendecomposition of $H^0(Y, L)$ we have $H^0(Y, L)_m \neq 0$. Its convex hull will be denoted by $\Gamma$ and will be called the polytope of sections. Again, sometimes we will point out the dependence on the setting by using notation $\Gamma(Y, L, T, \mu)$.

These two polytopes are not unrelated – one can show that $\Gamma(Y, L, T, \mu) \subset \Delta(Y, L, T, \mu)$ ([**BW22**, Lem. 2.4]) and showing that they both coincide is a crucial step in the proof of Theorem 4.1.2 and Theorem 4.1.3, that allows to claim the (semi)simplicity of the automorphism group of the contact manifold.

Considering extremal fixed point components, we can sum up their properties that are relevant to our discussion by the following lemma (although not explicitly stated in this form, this is essentially a combination of arguments from [**BW22**, Lemma 3.4, Lemma 3.6, Prop. 3.9]):

Lemma 4.3.4. *Let $Y$ be a Fano manifold with the fundamental line bundle $L$ that generates the Picard group (i.e. $b_2(Y) = 1$) and $T$ be a torus acting effectively on $Y$. Let $Z$ be an extremal fixed point component. Then $Z$ is a Fano manifold with the Picard group generated by the restriction of $L$, unless it is a point. In both cases, the restriction $H^0(Y, L) \to H^0(Z, L_{|Z})$ is surjective and in fact $H^0(Y, L)_{\mu(Z)} = H^0(Z, L_{|Z})$.*

Proof. The statement on $Z$ was shown in the case of a $\mathbb{C}^*$-action in [**BW22**, Lemma 3.4]. If the torus has rank $r > 1$, one can argue by a standard flag reasoning, frequently used in the context of torus actions, see e.g. [**BW22**, Prop. 3.9]. To be precise, we pick a 1-parameter subgroup $T_r$ of $T$ in such a way that its sink $Z_r$ contains $Z$ and that the quotient $T/T_r$ is again a torus (of rank $r-1$) . We apply [**BW22**, Lemma 3.4] to this 1-parameter action and see that $Z_r$ is a prime Fano manifold with the fundamental line bundle equal to the restriction of $L$ and that it is acted upon by the quotient torus $T/T_r$. We can now repeat the argument $r$ times and obtain a strictly decreasing sequence of subvarieties of $Y$:

$$Z_r \supset Z_{r-1} \supset \ldots \supset Z_1 = Z,$$

such that every one of them is a prime Fano manifold and fundamental line bundles are obtained by consequent restrictions.

Statements concerning global sections are proved in [**BW22**, Lemma 3.6]. □

## 4.4. Discussion of proofs

In this section we will describe main ideas leading to the proof of Theorem 4.1.2 and Theorem 4.1.3.

**Reduction to the prime and fundamental case.** By Theorem 2.3.2 we may restrict ourselves to the following setting: $X$ is a contact Fano manifold with $b_2(X) = 1$ and the contact line bundle $L$ that generates the Picard group. Moreover, we assume that $G$ is the group of automorphisms of $X$ that is reductive.

**Showing the relation between sections of $L$ and contactomorphisms.** The significance of the space of sections of the fundamental divisor is first illustrated by the following well-known lemma:

Lemma 4.4.1. *Suppose that we have setting as above. Then the contact distribution $F$ on $X$ is unique, so every automorphism of $X$ preserves $F$. Moreover, we have $\mathrm{Lie}(G) \simeq H^0(X, TX) \simeq H^0(X, L)$ as representations of $G = \mathrm{Aut}(X)$.*

Proof. The first statement is shown in [**Keb01**, Thm 4.4] and the second in [**Bea98**, Thm 1.1]. □

In particular, the contact line bundle $L$ is $G$-linearized, as a quotient of $G$-line bundles $TX$ and $F$. Recall that as the group $G$ is reductive, it has a maximal torus $T$. In both considered cases this torus is nontrivial: for the proof of Theorem 4.1.2 we explicitly assume that $T$ has rank $r \geq \max(2, \frac{n-3}{2})$. For Theorem 4.1.3, the nontriviality of $G$ that follows from the nonvanishing of $h^0(X, L)$ only allows us to claim that $r \geq 1$. Consequently, in both cases the action of $T$ on $X$ has extremal fixed points components $Z_i$ and we can construct polytopes $\Delta(X, L, T, \mu)$ and $\Gamma(X, L, T, \mu)$ using the linearization of the torus action $\mu$ coming from the $G$-linearization.

**Showing the equality of polytopes.** The next step is to again use our knowledge of fundamental linear systems, this time for extremal fixed point components and their lifting to the ambient variety to show that the polytopes of sections, fixed points and roots are all equal. To that end, recall that by Lemma 4.3.4 every extremal fixed point component $Z_i$ is either a point or a prime Fano manifold with $\mathrm{Pic}(Z_i) = \mathbb{Z} \cdot L_{|Z_i}$. Moreover, by [**ORCW21**, Lemma 4.6] it is an isotropic subvariety of $X$, so in particular $\dim(Z_i) \leq n$. Then, the second assumption of Theorem 4.1.3 says precisely that $h^0(Z_i, L_{|Z_i}) \geq 2$ if $Z_i$

is different from a point (and the space of sections over a point has dimension 1). By [**ORCW21**, Lemma 5.3] we obtain the desired equality $\Delta(X, L, T) = \Gamma(X, L, T) = \Delta(G)$. If we are instead in the situation of Theorem 4.1.2, then we cannot simply claim the existence of sections if some $Z_i$ have positive dimension. This is where the assumption on the reductive rank $r$ of $G$ comes into play: we demand that $r \geq \frac{n-3}{2} > 0$ in order to ensure that $\forall_i \dim(Z_i) \leq 5$ via a flag argument. Then, by known results on fundamental systems, i.e. Corollary 3.3.6 we again conclude that extremal fixed point components have sections, we can distinguish points from nontrivial ones by the dimension of $H^0(Z_i, L_{|Z_i})$ and that we have an equality of polytopes $\Delta(X, L, T) = \Gamma(X, L, T) = \Delta(G)$. For details, see [**ORCW21**, Lemma 6.4].

**Proving the triviality of extremal fixed point components and the simplicity of the group.** The equality of three lattice polytopes has two immediate consequences: first, it implies that extremal fixed point components are in fact points, as first argued in [**BW22**, Lemma 4.7]. Second, it allows us to prove that the group $G$ is in fact a semisimple one [**BW22**, Lemma 4.6]. With more work, one can show that $G$ cannot be a nontrivial product, i.e. it is a simple group [**BW22**, Prop. 4.8].

**Concluding both proofs.** We can list all the possibilites for $G$ and observe that either its reductive rank $r$ is at least 2 or it is a simple group of type $A_1$, i.e. $G = \mathrm{SL}(2)$ and $r = 1$. The second case is the odd one in Theorem 4.1.3, where the current state of the art is not enough to show that the variety is adjoint. However, it is clear that one way to exclude it would be to show that the contact line bundle admits even more sections, i.e. that $h^0(X, L) \geq 4$, so in particular by Corollary 3.5.1 we see that this situation is not possible in dimensions up to 9.

In both cases, we sweep most of the heavy lifting needed to finish proofs under the rug by invoking the following theorem:

THEOREM 4.4.2 ([**ORCW21**, Thm 5.1]). *Let $(X, L)$ be a projective contact manifold of dimension $2n+1$. Assume that $X$ is a prime Fano variety with the Picard group generated by the contact line bundle. Moreover, let $G$ be the identity component of a reductive group of automorphisms of $X$ such that $\operatorname{rk} G \geq 2$ and the action of the maximal torus $T \subset G$ has only isolated points as extremal fixed point components. Then $G$ is a simple group of type $B_r$ ($r \geq 3$), $D_r$ ($r \geq 4$), $E_6$, $E_7$, $E_8$, $F_4$ or $G_2$ and $X$ is the associated adjoint variety.*

The theorem above concludes the proofs of both Theorem 4.1.2 and Theorem 4.1.3, and so it closes the discussion conducted in this chapter. In particular, by Corollary 3.3.6 and Corollary 3.5.1 assumptions of both theorems are satisfied when $X$ is a contact Fano manifold of dimension at most $2n + 1 = 9$ with a reductive group of automorphisms, so Conjecture 1 holds in dimensions up to $4n = 16$.

CHAPTER 5

# Symplectic and contact varieties in the singular setting

## 5.1. Introduction

As we have seen in the previous chapters, projective manifolds admitting a contact structure are quite rare. This lack of examples is expected, but it makes studying them more difficult. In particular, neither birational nor finite maps preserve the contact structure. Therefore, it would be beneficial to develop less restrictive notions and study their relations with usual contact manifolds. In this chapter we focus on one particular generalization, that of *a singular contact variety.* It constitutes an original contribution of the author and these results will be published separately [**Śmi23**], however we note the exploration of similar ideas in works of others. Moreover, we observe that sometimes resolutions of singularities of contact varieties produce manifolds that only admit contact structures on some open subset. Such manifolds are called *generically contact* and they provide an alternative approach to weakening the notion of a contact manifold.

## 5.2. Symplectic singularities and varieties

The starting point for our discussion is the following classical construction, that provides a link between symplectic and contact manifolds:

Construction 5.2.1 (Symplectization). For a given variety $Y$ with a line bundle $\mathcal{L}$ over it, we can consider the total space of $\mathcal{L}^*$ and remove the image of zero section to obtain a $\mathbb{C}^*$-bundle over $Y$, that will be denoted by $\mathcal{L}^\bullet$. If we apply this construction to a contact manifold $X$ with the contact line bundle $L$ and the twisted form $\vartheta$ then the resulting $L^\bullet$ has the structure of a symplectic manifold with the 2-form coming from $\vartheta$ ([**Buc09**, Thm E.6]). In such situation, $L^\bullet$ is called *the symplectization* of $X$.

This well-established relation, along with a widely accepted and researched generalization of the notion of a symplectic manifold to a singular setting provide both an additional motivation and the guide to explore analogous ideas in the contact case. Therefore, now we present a short survey of some results from the singular symplectic theory that have contact analogues provided by the author's work.

### 5.2.1. Definitions and basic properties.

Definition 5.2.2. [**Bea00**, Def. 1.1] We say that a normal variety has a symplectic singularity at a point $p$ if there is an open neighbourhood $U \ni p$ such that its smooth part admits a symplectic 2-form $\omega$ and the pullback of that form to any resolution of singularities extends to a holomorphic 2-form on the whole resolution.

Similarly, a normal variety equipped with a symplectic form on its smooth locus whose pullback to any resolution extends to a holomorphic 2-form on the resolving variety is called a symplectic variety.

It is easy to see that symplectic singularities are rational and Gorenstein, as the top wedge power of the symplectic form $\omega$ is the trivializing section of the canonical bundle over the smooth locus that can be extended to the whole variety by normality. The inverse statement is a nontrivial theorem of Namikawa:

THEOREM 5.2.3. [**Nam01**, Thm 6] *A normal variety is symplectic if and only if it has rational Gorenstein singularities and its smooth part admits a holomorphic symplectic form.*

We may observe that in particular symplectic singularities are examples of canonical singularities.

A natural question that can be asked is: are resolutions of singularities of symplectic varieties symplectic manifolds? The answer is negative in general, however the following folklore result gives a necessary and sufficient condition for that:

PROPOSITION 5.2.4. *For a symplectic variety $Y$ equipped with a form $\omega$ and any resolution of singularities $f\colon \widetilde{Y} \to Y$ the following are equivalent:*

(1) *$f$ is a crepant morphism,*
(2) *$\widetilde{Y}$ is a symplectic manifold (we say that the resolution $f$ is symplectic).*

PROOF. We have already observed that the canonical divisor of $Y$ is trivial. If $f$ is crepant, it means that $\widetilde{Y}$ also has a trivial canonical divisor, which outside of the exceptional locus is trivialized by the top wedge power of $f^*\omega$. This trivializing section extends over the exceptional divisor and is nonzero there, but this means precisely that the top power of $f^*\omega$ is nowhere degenerate, so $\widetilde{Y}$ is symplectic.

Going the other way around, if both $\widetilde{Y}$ and $Y$ have symplectic structures, it means that $K_{\widetilde{Y}}$ and $K_Y$ are trivial, so the resolution is crepant. □

**5.2.2. Examples of symplectic varieties.** The fundamental family of examples of symplectic varieties comes from the Lie theory. We have already explained in Section 4.2.2 that every (co)adjoint orbit for a semisimple group $G$ is equipped with a symplectic form. Closures of orbits may fail to be normal, but their normalizations are examples of symplectic varieties, as [**Pan91**, Thm 1] observed that they have rational Gorenstein singularities. Moreover, Beauville proved:

THEOREM 5.2.5. [**Bea00**, Theorem, p. 1] *Let $Y$ be a symplectic variety with an isolated singularity $p$ such that its projective tangent cone is smooth. Then $(Y, p)$ is analytically isomorphic to the closure of the minimal nilpotent orbit for some simple complex Lie algebra with the singular point being mapped to $0$.*

Having some examples we can construct new ones by taking quotients:

PROPOSITION 5.2.6 ([**Bea00**, Prop. 2.4]). *Let $Y$ be a symplectic variety and $G$ a finite subgroup of symplectomorphisms (i.e. automorphisms of $Y$ that preserve the symplectic form). Then the quotient $Y/G$ is a symplectic variety.*

We will present the original proof of the proposition, so that the reader may later compare it with an analogous result for contact varieties (Theorem 5.3.24), where the twist of the form slightly complicates the situation. Before that, we will state and prove the folklore result that appears without a reference in the original exposition.

LEMMA 5.2.7. *Let $Y$ be a smooth symplectic variety equipped with a symplectic form $\omega$ and $G$ a finite group of symplectomorphisms. For $g \in G \setminus \{1_G\}$ let $F_g$ denote the locus of points fixed by it. Then every irreducible component $Z$ of $F_g$ is a symplectic subvariety.*

PROOF. Denote by $H$ the subgroup of $G$ generated by $g$. The symplectic form $\omega$ defines an isomorphism between $TY$ and $\Omega^1_Y$, which is $G$-equivariant. At every point $z \in Z$ the tangent bundle can be decomposed as $T_zZ \oplus N_z$ and the first summand is equal to $(T_zY)^H$ by an easy consequence of the Luna slice theorem [**Dré04**, Thm 5.4]. We have an analogous decomposition for the cotangent bundle $(\Omega^1_Y)_z = (\Omega^1_Z)_z \oplus N_z^*$ where again we can identify

$(\Omega^1_Z)_z$ with $((\Omega^1_Y)_z)^H$. Therefore, the isomorphism defined by $\omega$ maps $T_zZ$ isomorphically onto $(\Omega^1_Z)_z$ and consequently $Z$ is symplectic. $\square$

Proof of Proposition 5.2.6. First, we observe that the normality of $Y$ implies the normality of $Y/G$. Indeed, let $(Y/G)^n$ be the normalization. By the universal property of the normalization, we have a map $Y \to (Y/G)^n$ and it is $G$-invariant, so it factorizes through the quotient map $Y \to Y/G$. The resulting map $Y/G \to (Y/G)^n$ is the inverse of the normalization, so it is an isomorphism.

To show the existence of a symplectic form on $Y/G$ that pulls back to a resolution, consider the commutative diagram:

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\phi"] \arrow[d, "\widetilde{\pi}"] & Y \arrow[d, "\pi"] \\ \widetilde{Y/G} \arrow[r, "\psi"] & Y/G \end{tikzcd}$$

where $\widetilde{Y/G}$ is a resolution of singularities of $Y/G$ and $\widetilde{Y}$ is a resolution for $Y$ making the diagram commutative. For any $g \in G \setminus \{1_G\}$ let $F_g$ be the locus of points fixed by $g$ in $Y_{sm}$. $F_g$ is of codimension $\geq 2$ in $Y_{sm}$, as its every component is a symplectic subvariety by Lemma 5.2.7. Define the open subset $Y_0 = Y_{sm} \setminus \bigcup_{g \neq 1_G} F_g$. Over $Y_0$ the symplectic form descends to the quotient $Y_0/G$, and can be extended to $(Y/G)_{sm}$ because the difference $(Y/G)_{sm} \setminus (Y_0/G)$ has codimension $\geq 2$. Denote the symplectic form on $Y/G$ by $\omega$. To finish the proof we need to show that the meromorphic form $\psi^*\omega$ is actually holomorphic, but this follows as the form $\phi^*\pi^*\omega$ is holomorphic by the symplecticity of $Y$ and by the commutativity of the diagram it is equal to $\widetilde{\pi}^*\psi^*\omega$. $\square$

**5.2.3. Kaledin's stratification.** To end our survey of relevant symplectic results, we discuss the existence of the canonical stratification for symplectic varieties, proved by Kaledin (see [**Kal06**] for a definitive reference on which this section is based). The reason for presenting it in greater detail is that later we will want to show that the construction behaves well with respect to a possible $\mathbb{C}^*$-bundle structure on a symplectic variety. Kaledin was aware of this possible modification of his theorems [**Kal06**, Remark 2.6], it also was stated without proof for a particular case in [**MOSC$^+$15**, Proposition 5.9].

The basic idea is to switch focus from the symplectic structure to the Poisson structure:

Definition 5.2.8. Let $Y$ be a complex scheme. We say that it is Poisson if $\mathcal{O}_Y$ has the structure of the Poisson algebra, i.e. it is equipped with the skew-linear Poisson bracket $\{\cdot,\cdot\}\colon \mathcal{O}_Y \times \mathcal{O}_Y \to \mathcal{O}_Y$ such that:

$$\begin{aligned} \{a, bc\} =& \{a, b\}c + \{a, c\}b && \text{(Leibniz rule)}, \\ 0 =& \{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} && \text{(Jacobi identity)} \end{aligned}$$

for any $a, b, c \in \mathcal{O}_Y$. An ideal $I$ such that $\{a, i\} \in I$ for any $a \in \mathcal{O}_Y$ and $i \in I$ is called a Poisson ideal. A subscheme locally defined by a Poisson ideal is a Poisson subscheme.

Observe that for any $f \in \mathcal{O}_Y$ $\{f, \cdot\}$ is a vector field, which will be denoted by $H_f$ and called a Hamiltonian vector field. Any such vector field preserves Poisson subschemes. As the Poisson bracket is a derivation in both arguments, we can define the *Poisson bivector* $\Theta$, i.e. a map:

$$\begin{aligned} \Theta\colon \Omega^1_Y \wedge \Omega^1_Y \to \mathcal{O}_Y \\ \{f, g\} = \Theta(df \wedge dg), \end{aligned}$$

such that $[\Theta, \Theta] = 0$ for the induced bracket on multivector fields (this condition is equivalent to the Jacobi identity).

The classical proposition below relates a Poisson structure to a symplectic one. We present its proof, as it illustrates how to define the Poisson bracket on a symplectic variety and how to calculate weights in the situation of a $\mathbb{C}^*$-action.

PROPOSITION 5.2.9 ([**Kal06**, Discussion in Section 2], [**Buc09**, Lemma D.15]). *A symplectic variety $Y$ has the structure of a Poisson scheme. If moreover $Y$ is equipped with a $\mathbb{C}^*$ action for which the symplectic form $\omega$ is homogeneous of weight $k$, then the Poisson bracket is homogeneous of weight $-k$.*

PROOF. We need to define the bracket for any two functions $f, g \in \mathcal{O}_Y$. As $Y$ is necessarily normal, the codimension of the singular locus is at least 2, so it is enough to define $\{f, g\}$ on the smooth locus and its extension to $Y_{sing}$ will be unique.

On $Y_{sm}$ we have a nondegenerate 2-form $\omega$ giving an isomorphism $\Omega^1_{Y_{sm}} \simeq TY_{sm}$. This identification in turn defines the bivector $\Theta$. Now, $\Theta$ is a Poisson bivector if and only if the bracket satisfies the Jacobi identity, and this is equivalent to $\omega$ being a closed form.

To prove the second statement, observe that we have $df = \omega(H_f, \cdot)$ and if we denote weight by $w$, we have $w(df) = w(f) = w(\omega) + w(H_f)$. We also have $\{f, g\} = df(H_g)$ so $w(\{f, g\}) = w(f) + w(g) - w(\omega)$. □

We are now ready to state the main result:

THEOREM 5.2.10 ([**Kal06**, Thm 2.3]). *Let $Y$ be a normal symplectic variety. Then it admits a finite stratification $Y = Y_0 \supset Y_1 \supset ... \supset Y_k$ by Poisson subschemes such that:*

(1) *$Y_{r+1}$ is the singular locus of $Y_r$.*
(2) *The normalization of every irreducible component of each stratum is a symplectic variety.*

*Moreover, the Poisson bracket that exists on $Y$ allows to define Poisson structures on strata that are compatible with the symplectic forms.*

We will not present the proof of this theorem. Instead, we note that the switch to Poisson structure is crucial and that in the original exposition the statement does not contain the claim on the compatibility, however it is an essential part of the proof [**Kal06**, Section 2.3, Step 2].

## 5.3. Singular contact varieties

Having surveyed all relevant results from the theory of symplectic singularities, we can finally discuss the notion of a singular contact variety along with its various properties. Both the definition and every presented result is a contribution of the author, although the reader can compare our proofs with their symplectic analogues to see both similarities and differences. Moreover, later we focus on the projective case, where we are able to determine the Kodaira dimension of a projective contact variety and give a classification result in dimension 3.

### 5.3.1. Definitions and basic properties.

DEFINITION 5.3.1. A contact variety is an algebraic variety $X$ of odd dimension $2n+1$ ($n \geq 0$) with rational singularities and a globally defined line bundle $L$ such that on the smooth locus $X_{sm}$ we have an exact sequence of vector bundles:

$$0 \to F \to TX_{sm} \xrightarrow{\vartheta} L_{|X_{sm}} \to 0$$

which defines the contact structure on $X_{sm}$, i.e. $d\vartheta \colon \Lambda^2 F \to L_{|X_{sm}}$ is nowhere degenerate. Equivalently, one can demand that $\vartheta \wedge (d\vartheta)^{\wedge n}$ as an element of $H^0(X_{sm}, \Omega^{2n+1}_{|X_{sm}} \otimes L^{n+1}_{|X_{sm}})$ has no zeroes.

REMARK 5.3.2. We recall our discussion on the admissibility of the one dimensional case conducted in Remark 1.2.2 and uphold the convention established therein, i.e. we allow it. As singular contact varieties are normal, it follows one dimensional singular contact varieties are in fact smooth, so they are the classical contact manifolds. In particular, Definition 5.3.1 is satisfied by any smooth curve.

Immediately from the definition we obtain:

PROPOSITION 5.3.3. *For a (singular) contact variety $X$, $-K_X$ is a Cartier divisor and we have $\mathcal{O}(-K_X) = L^{\otimes n+1}$ in $\mathrm{Pic}(X)$. Therefore, singularities of $X$ are canonical and Gorenstein.*

PROOF. $\vartheta \wedge (d\vartheta)^{\wedge n}$ gives an equality of $(n+1)L_{|X_{sm}}$ and $-K_{X_{sm}}$ in the class group of $X_{sm}$. Since $X$ is normal as its singularities are rational, we can take unique closures of both divisors and prolong considered equality to the whole $X$. As $L$ is in fact a Cartier divisor, the canonical class also is. Then the equivalence provided by Theorem 2.1.7 implies that $X$ has canonical singularities. Moreover, rational singularities are in particular CM by [**KKMSD73**, Ch. I, §3, Proposition], so local freeness of $\omega_X$ implies that $X$ is Gorenstein. $\square$

REMARK 5.3.4. The idea of defining and studying contact structures in the singular complex setting first appeared in [**CF02**], where authors give the definition of a *contact singularity.* Their definition is local and they only demand the existence of a 1-form $\vartheta$ (on the regular locus) such that $\vartheta \wedge (d\vartheta)^n$ is nonzero there. These assumptions are enough to prove that the singularity is quasi-Gorenstein, i.e. the canonical sheaf is locally free ([**CF02**, Lemma 3.1]) and cannot be isolated ([**CF02**, Thm 3.5]). However, Campana and Flenner do not demand in their definition that a contact form extends over a resolution, only that it is normal. This is a stark difference with the notions proposed by Beauville and by the author. We assume that singularities of our varieties are rational, which in particular implies the extension property – we discuss it in Lemma 5.3.12. Moreover, at the end of the chapter we will be sufficiently equipped to construct an example of a threefold that has contact singularities in the sense of Campana and Flenner but does not satisfy the definition proposed by the author, for details see Example 5.4.11.

Similar ideas were also explored by Namikawa in e.g. [**Nam16**, Section 2] in the form of *contact orbifolds*, that were used to study symplectic varieties with a good $\mathbb{C}^*$-action. As the name indicates, they are varieties that can locally be presented as quotients by finite and commutative groups and that admit contact structures on the smooth locus. On the other hand, the assumption on the global existence of the line bundle is relaxed – Namikawa only assumes the local existence in the orbifold charts, so that some multiple of such *orbifold line bundle* is an honest line bundle.

Now recall that in Construction 5.2.1 we have sketched how to associate a symplectic manifold to a given contact manifold. As we have already indicated, we have defined contact varieties in such a way to be able to extend this classical construction:

THEOREM 5.3.5. *If $X$ is a contact variety, then $L^\bullet$ is a symplectic variety such that the symplectic 2-form $\omega$ is homogeneous of weight 1 with respect to the $\mathbb{C}^*$-action. Going the other way around, if $Y$ is a symplectic variety having the structure of a principal $\mathbb{C}^*$-bundle over some base $Z$ such that the symplectic form $\omega$ is homogeneous of weight 1, then the base space $Z$ is a contact variety.*

In fact, the presence of the singularities does not change the situation much and one can quickly reduce the proof of the theorem to the smooth case. Nevertheless, it will be useful for us to understand the relation between symplectic and contact forms, so we sketch the classical reasoning.

Proof. For both statements, let $\pi$ denote the natural projection from the total space of the principal bundle onto the base.

To prove that $L^\bullet$ is symplectic, we use the characterization provided by Theorem 5.2.3. As singularities of $X$ are rational Gorenstein by Proposition 5.3.3, so are those of $L^\bullet$ and we just need to provide the symplectic form on the smooth locus of $L^\bullet$. Consider the following composition of maps:

$$TL^\bullet_{sm} \xrightarrow{D\pi} \pi^* TX_{sm} \xrightarrow{\pi^*\vartheta} \pi^* L_{|X_{sm}} \xrightarrow{\iota} \mathcal{O}_{L^\bullet_{sm}}.$$

The rightmost arrow is an isomorphism coming from the pairing between $L$ and $L^*$, and the 1-form (not twisted) being the composition of all three maps will be denoted by $\vartheta^\bullet$. Then one can show that nondegeneracy of $\vartheta$ implies the nondegeneracy of $d\vartheta^\bullet$, for details see [**Buc09**, Section C.5].

Now we will prove the second statement. On $Z$ we have a line bundle $\mathcal{L}$ which is the dual of a line bundle associated to the principal bundle $Y$. Now, to show the (local) existence of a twisted 1-form on $Z_{sm}$ we consider some open subset of $Z_{sm}$ on which $Y$ (and $\mathcal{L}$) are trivial. $\omega$ is closed, so we can assume that it is exact and of the form $\omega = d(z\phi + fdz)$ (we separate $z$ – the coordinate on the fiber). Using the properties of the differentiation operator we can write it as $\omega = d(z(\phi - df))$ and put $\vartheta = \phi - df$. It is a unique homogeneous 1-form of weight 0 such that locally $\omega = d(z\vartheta)$ and one can check that it glues to a global twisted 1-form with values in $\mathcal{L}$. To show that $\vartheta$ defined in such a way is nondegenerate, we again reason precisely as in the smooth case, proved in [**Buc09**, Prop. C.16]. ☐

As our construction is a natural generalization of the classical one, we will call $(L^\bullet, \omega)$ obtained from a contact variety $(X, L, \vartheta)$ *the symplectization* of $X$. Before presenting its applications, we will discuss what happens if we additionally assume that there is a group $G$ acting on a possibly singular contact variety $X$. First, recall from the discussion after Lemma 4.4.1 that we have already observed that in the smooth case the contact line bundle is equipped with a canonical linearization. In the singular case we need a more subtle argument:

Proposition 5.3.6. *Let $(X, F, L)$ be a singular contact variety and $G$ a group of automorphisms preserving the contact distribution $F$. Then $L$ is equipped with the canonical $G$-linearization.*

Proof. Consider the product $G \times X$ with two morphisms to $X$: $\alpha$ corresponding to the action and $\pi_2$ – projection from the product. Then by [**Bri15**, Lemma 2.9] the bundle $L$ is linearized if and only if $\pi_2^* L \simeq \alpha^* L$. On the smooth locus we have the canonical action of $G$ on $L$, as it is a quotient of $TX_{sm}$ by $F$ there. Moreover, $G$ acts on $TX_{sm}$ preserving $F$, so we have the above-mentioned isomorphism of pullback bundles on $X_{sm}$. As the singular locus has codimension at least 2 by normality, this isomorphism extends to the whole $X$. Indeed, on any open $U \subset X$ trivializing both bundles such isomorphism is given by a nowhere vanishing regular function $f$ defined on $U \cap X_{sm}$. Any such $f$ can be extended to a nowhere vanishing regular function on the whole $U$ by the normality of $X$. The resulting linearization is canonical, as the unique extension of the canonically defined linearization on the smooth locus. ☐

We note that in our setting we immediately obtain the following lifting property:

Corollary 5.3.7. *In the setting of Proposition 5.3.6, any action of a group preserving the contact distribution lifts to the symplectization. If moreover the contact form is invariant with respect to the $G$-action, then the symplectic form is invariant with respect to the lifted action.*

PROOF. The symplectization of $X$ was defined as the total space of the dual of $L$ with removed zero section, so the existence of the linearization clearly implies that we have an action of $G$ on $L^\bullet$. If we assume that $\vartheta$ is preserved by $G$, then $\vartheta^\bullet$ is $G$-invariant as a composition of $G$-invariant maps. Indeed, any group element $g$ acts on differential forms via the pullback, which is commutative with taking the exterior derivative, so $\omega$, a derivative of $\vartheta^\bullet$, is $G$-invariant. □

Unfortunately, this observation does not go both ways: starting from $G$-invariant $\omega$ one does not necessarily obtain an invariant twisted form downstairs. As locally $\omega = d(z\vartheta)$, the action on both the fiber coordinate $z$ and $\vartheta$ can be nontrivial and yet give an invariant $\omega$.

The first application of the symplectization is to show the existence of the stratification, analogous to the one discussed for symplectic varieties in Theorem 5.2.10.

THEOREM 5.3.8. *Let $(X, L, \vartheta)$ be a contact variety. Then there exists a finite stratification $X = X_0 \supset X_1 \supset \dots \supset X_k$ such that:*

(1) *$X_{i+1}$ is the singular part of $X_i$.*
(2) *The normalization of every irreducible component of each stratum is a contact variety.*

PROOF. Let $(L^\bullet, \omega)$ be the symplectization of $(X, L, \vartheta)$ and denote by $\pi$ the natural projection. Consider the symplectic stratification of $L^\bullet$ existing by Theorem 5.2.10. To prove our theorem it is enough to show that all strata and their normalizations have a structure of a principal $\mathbb{C}^*$-bundle induced from the one on $L^\bullet$ and that induced forms are homogeneous of weight $wt(\omega)$.

First, observe that the singular locus of $L^\bullet$ must necessarily be preserved by the $\mathbb{C}^*$-action, if $Z$ is an irreducible component of $L^\bullet_{sing}$, then it still admit the structure of a principal $\mathbb{C}^*$-bundle. Then, the action can be lifted to the normalization $Z^n$. To see it, consider the diagram:

$$\begin{tikzcd}
\mathbb{C}^* \times Z^n \arrow[r, dashed] \arrow[d, "\mathrm{id} \times \eta"] & Z^n \arrow[d, "\eta"] \\
\mathbb{C}^* \times Z \arrow[r] & Z,
\end{tikzcd}$$

where $\eta$ is the normalization morphism and the lower horizontal arrow comes from the action. The existence of the map denoted by the dashed arrow comes from the universal property of the normalization and it defines the $\mathbb{C}^*$-action on the normalization. Moreover, $Z^n$ also admits the structure of a principal $\mathbb{C}^*$-bundle, as it is equal to the pullback (via the normalization morphism of $\pi(Z)$) of the principal bundle on $\pi(Z)$.

Now we need that induced forms have the same weight as $\omega$. To that end, recall that symplectic forms on components of strata come from the Poisson structure. If we start with the symplectic form on $Y_{sm}$ which is homogeneous of weight $k$, then by Proposition 5.2.9 the Poisson bracket is homogeneous of weight $-k$. Now, the bracket extends over the singular locus and it is still homogeneous of weight $-k$, so the induced forms on components of strata, which have to agree with the bracket, have weight $k$. □

The existence of the stratification will come in handy in Section 5.4, where we will study projective contact threefolds. Moreover, we can immediately observe the following:

COROLLARY 5.3.9. *Let $X$ be a singular contact variety. Then any component of the singular locus has even codimension in $X$.*

**5.3.2. Birational morphisms.** Now we will discuss some birational maps of contact varieties and manifolds, in vein of Proposition 5.2.4. To begin with, it is well-known that projective contact manifolds do not admit birational Mori contractions ([**KPSW00**, Lem. 2.10], we have already mentioned it when sketching the proof of Theorem 2.3.2). This result can be easily strengthened via an application of the negativity lemma:

THEOREM 5.3.10. *Let $f\colon \widetilde{X} \to X$ be a birational morphism from a projective contact* ***manifold*** *$\widetilde{X}$ to a variety $X$ with canonical and Gorenstein singularities (for instance $X$ is a projective contact variety and $\widetilde{X}$ a resolution by a contact manifold). Then $f$ is crepant.*

PROOF. By the assumptions on $X$, we can write

$$K_{\widetilde{X}} = f^*K_X + \sum_i a_i E_i,$$

where $a_i \geq 0$. Then $D = \sum a_i E_i$ corresponds to an effective Cartier divisor and $f_*(-D) = 0$, so $f_*(-D)$ is also effective. Suppose that $D$ is $f$-nef. In such case we obtain by the negativity lemma [**KM98**, Lem. 3.39] that $-D$ is effective, so $D = 0$ and we are done.

Now assume that $D$ is not $f$-nef, so there exists a curve $C$ contracted by $f$ that intersects $D$ negatively. We can write

$$K_{\widetilde{X}} \cdot C = f^*K_X \cdot C + D \cdot C = 0 + D \cdot C.$$

It follows that $K_{\widetilde{X}} \cdot C < 0$. Then the morphism contracting $C$ is a Mori contraction, however by [**KPSW00**, Lem. 2.10] such maps cannot be birational. We reached a contradiction, so $D$ must necessarily be $f$-nef. As we have already shown, in this case $D = 0$ and it means precisely that $f$ is crepant. □

To provide a converse for this statement, we need to discuss the notion of sheaf reflexivity. A coherent sheaf is *reflexive* if it is isomorphic to its second dual. Hartshorne showed ([**Har80**, Prop. 1.6]) that a sheaf $\mathcal{F}$ on a normal and integral scheme $Y$ is reflexive if and only if it is torsion free and satisfies the Barth normality condition, i.e. for any open $U \subset Y$ and closed $Z \subset U$ of codimension at least 2 the restriction $\mathcal{F}(U) \to \mathcal{F}(U \setminus Z)$ is bijective, that is every section defined outside of $Z$ possesses a unique extension. To prove our main theorem, we need the following result concerning the reflexivity of differentials:

THEOREM 5.3.11 ([**GKKP11**, Thm 1.4]). *Let $Y$ be a quasi-projective variety with canonical singularities and a resolution of singularities $f\colon \widetilde{Y} \to Y$. Then the sheaf $f_*\Omega^p_{\widetilde{Y}}$ is reflexive for $p \leq \dim Y$.*

We note an easy consequence of Theorem 5.3.11:

LEMMA 5.3.12. *Let $Y$ be a quasi-projective variety with canonical singularities and a resolution of singularities $f\colon \widetilde{Y} \to Y$. Assume that $Y$ is equipped with a line bundle $\mathcal{L}$ and a twisted form $\theta$ with values in $\mathcal{L}$ defined on the smooth locus $\theta\colon TY_{sm} \to \mathcal{L}_{|Y_{sm}}$. Then there exists a unique twisted form $\widetilde{\theta}$ on the whole $\widetilde{Y}$ with values in $f^*\mathcal{L}$ extending $\theta$.*

PROOF. Consider the sheaf $(f_*\Omega^1_{\widetilde{Y}}) \otimes \mathcal{L}$ on $Y$. It is reflexive, as a product of the line bundle $\mathcal{L}$ with a sheaf that is reflexive by Theorem 5.3.11. Moreover, by the projection formula this product sheaf is isomorphic to the pushforward sheaf $f_*(\Omega^1_{\widetilde{Y}} \otimes f^*\mathcal{L})$. Now put $U = Y_{sm}$ and let $Z$ be the singular locus of $Y$, which has codimension at least 2. The reflexivity of $f_*(\Omega^1_{\widetilde{Y}} \otimes f^*\mathcal{L})$ implies that any section defined on $U$ extends to the whole $Y$. As we may consider $\theta$ as a section of $(f_*\Omega^1_{\widetilde{Y}}) \otimes \mathcal{L}$ defined over $U$, it extends uniquely to a globally defined section. Finally, recall that from the definition of the pushforward sheaf we have $f_*(\Omega^1_{\widetilde{Y}} \otimes f^*\mathcal{L})(Y) = (\Omega^1_{\widetilde{Y}} \otimes f^*\mathcal{L})(f^{-1}(Y))$, i.e. we have a globally defined twisted form on the resolution with values in $f^*\mathcal{L}$. □

We are now ready to state and prove the main result of this section:

THEOREM 5.3.13. *Let $(X, F, L, \vartheta)$ be a quasi-projective contact variety. Suppose that $f\colon X' \to X$ is a birational and crepant morphism such that $f(\mathrm{Exc}(f)) \subset X_{sing}$. Then $X'$ is again a contact variety with the distribution $F'$ such that $f_*F' = F$ on the smooth locus of $X$ and the contact line bundle is $f^*L$. In particular, a terminalization of a contact variety is contact and a crepant resolution of singularities produces a classical contact manifold.*

PROOF. Let $g\colon \widetilde{X} \to X'$ be a resolution of singularities of $X'$, which composed with $f$ becomes a resolution of singularities of $X$. By Lemma 5.3.12 $\widetilde{X}$ is equipped with a twisted form $\widetilde{\vartheta}$ with values in $(fg)^*L$ (note however that we cannot show that it is epimorphic nor nondegenerate on $\widetilde{X}$). As $\widetilde{X}$ is a resolution of singularities of $X'$, we have an isomorphism $\widetilde{X} \setminus \mathrm{Exc}(g) \simeq X'_{sm}$, which allows us to define $\vartheta'$ on the smooth locus of $X'$.

Now recall from Proposition 5.3.3 that $\vartheta \wedge (d\vartheta)^{\wedge n}$ can be extended to the whole $X$ and it is a nowhere vanishing section of $\mathcal{O}(K_X) \otimes L^{\otimes(n+1)}$. We pull back this section to $X'$, where it defines an isomorphism between $\mathcal{O}(-K_{X'})$ and $f^*L^{\otimes(n+1)}$. Moreover, on the intersection of $X'_{sm}$ and any open $U \subset X'$ trivializing $f^*L$ this section agrees with $\vartheta' \wedge (d\vartheta')^{\wedge n}$.

Finally, observe that $\vartheta'$ is surjective on $X'_{sm}$. If this were not the case at some point $x \in X'_{sm}$, then $\vartheta' \wedge (d\vartheta')^{\wedge n}$ would be zero at $x$, which is absurd. We define $F'$ to be the kernel of $\vartheta'$ and we clearly have $f_*F' = F$ on $X_{sm}$, as the exceptional locus of $f$ is mapped to $X_{sing}$. □

COROLLARY 5.3.14. *Let $X$ be a projective contact variety. Then its resolution of singularities is a contact manifold if and only if the resolution morphism is crepant.*

Now it is natural to wonder how much is broken by noncrepant resolutions, for instance whether a general resolution preserves the existence of the contact distribution or the surjectivity of the contact form. In general, this is not the case, but to discuss it let us first introduce a more general class of manifolds:

DEFINITION 5.3.15. A smooth variety (or a holomorphic manifold) $X$ of dimension $2n+1$ equipped with an exact sequence of vector bundles:

$$0 \to F \to TX \xrightarrow{\vartheta} L \to 0 \tag{20}$$

such that $L$ is a line bundle is called *a generically contact manifold* if $d\vartheta_{|F}$ is nondegenerate on some open dense subset $U$ of $X$. Equivalently, one can demand that $(d\vartheta)^{\wedge n} \wedge \vartheta \in H^0(X, \Omega_X^{2n+1} \otimes L^{\otimes n+1})$ is nonzero on $U$. The locus along which $d\vartheta_{|F}$ is degenerate is necessarily an effective divisor, called the degeneracy divisor.

REMARK 5.3.16. The notion of a generically contact manifold is relatively unexplored in literature – the author is only aware of the existence of two works dealing with it. To be precise, authors of [**BKK22**] observed that some of their results concerning singular lines on contact manifolds still hold if we only assume that the contact structure is nondegenerate generically. Initial study of generically contact manifolds was conducted in an unpublished MSc degree thesis [**Mro18**], where in particular some examples and nonexamples were provided.

We can prove that the resolution of a contact variety is generically contact only in a very special case, as in general the twisted form on the resolution needs not to be surjective.

PROPOSITION 5.3.17. *Let $(X, L, \vartheta)$ be a quasi-projective contact variety with the singular locus consisting of disjoint, smooth subvarieties $Z_i$. Moreover, assume that the exceptional locus $E$ of the resolution $f\colon \widetilde{X} \to X$ consists of disjoint, smooth varieties $E_i$, each mapping onto corresponding $Z_i$. Then $\widetilde{X}$ is a generically contact manifold.*

Proof. By Lemma 5.3.12 $\widetilde{X}$ is equipped with a twisted form $\widetilde{\vartheta}$ with values in $f^*L$. To claim the surjectivity of $\widetilde{\vartheta}$, we observe that every $Z_i$ has the induced (smooth) contact structure by Theorem 5.3.8, so in particular we have a surjection $TZ_i \twoheadrightarrow L_{|Z_i}$ for every $Z_i$. Consider the following diagram:

$$\begin{tikzcd} TE_i \arrow[r] \arrow[dr, dashed] & f^*_{|E_i} TZ_i \arrow[r] \arrow[d] & 0 \\ & f^*_{|E_i}(L_{|Z_i}), & \end{tikzcd}$$

where the horizontal epimorphism comes from the relative tangent sequence. $\widetilde{\vartheta}$ is surjective over any component of the exceptional locus as a composition of surjections. We therefore define the contact distribution $\widetilde{F}$ as the kernel of $\widetilde{\vartheta}$. To finish, observe that by our assumptions $d\widetilde{\vartheta}_{|\widetilde{F}}$ is nondegenerate outside of $E$. $\square$

We will see one example of a projective contact variety whose resolution is generically contact in Example 5.3.28. The same reasoning can be used to show that the blow-up of a contact manifold along a smooth subvariety $Z$ such that $\vartheta_{|TZ}$ is surjective has a generically contact structure[1], so further blow-ups along such subvarieties are still generically contact. It is also clear from the proof that blow-ups along isotropic subvarieties force the twisted form to be zero on the exceptional divisor. Consequently, one can easily modify any resolution of singularities by such blow-up, so that the resulting variety does not even admit an exact sequence as in Definition 5.3.15 and consequently, there is no globally defined contact distribution.

We finish our discussion by noting that the seemingly small relaxation given by allowing $d\vartheta$ to degenerate makes the notion of a generically contact manifold quite unwieldy. To be precise, the simple relation between the anticanonical divisor and the contact line bundle, $\mathcal{O}(-K_X) = L^{\otimes(n+1)}$ that holds for both smooth contact manifolds and singular contact varieties gets replaced by $\mathcal{O}(-K_X) = L^{\otimes(n+1)} \otimes \mathcal{O}(-B)$, where by $B$ we denote the degeneracy divisor (that can have components and nontrivial positive multiplicities) [**Mro18**, Stw. 1.6]. Consequently, we cannot argue on the positivity of the anticanonical bundle from our knowledge of $L$. Therefore, the notion of a generically contact manifold may be too weak to obtain interesting results.

**5.3.3. Quotients.** In this section we will consider an action of a (finite) group $G$ on a possibly singular contact variety $X$ preserving the contact distribution and our purpose is to prove an analogue of Proposition 5.2.6. Since we are interested in varieties equipped with some special vector bundles, we need to discuss the behaviour of these objects with respect to taking quotients by group actions. In particular, we introduce the notion of a bundle descent:

Definition 5.3.18. Let $Y$ be a $G$-variety with a good quotient $\pi\colon Y \to Y/G$. The pullback of any vector bundle on $Y/G$ has a natural structure of a $G$-vector bundle on $Y$. We say that a $G$-vector bundle $\mathcal{E}$ on $Y$ *descends* if there exists a vector bundle $\mathcal{E}'$ on $Y/G$ such that we have a $G$-equivariant isomorphism $\mathcal{E} \simeq_G \pi^*\mathcal{E}'$. This vector bundle, if it exists, is unique and equal to $(\pi_*\mathcal{E})^G$. We will frequently call it *the descent* of $\mathcal{E}$.

The question on the existence of the descent was completely answered by Kempf:

[1]This result appears in Mroczek's thesis [**Mro18**] only as a conjecture supported by examples, although according to our common advisor, it was also proved by her.

LEMMA 5.3.19 (Kempf's lemma, [**DN89**, Théorème 2.3]). *Let $Y$ be an algebraic variety equipped with an algebraic action of a reductive group $G$ such that there exists a good quotient $\pi\colon Y \to Y/G$. Let $\mathcal{E}$ be a $G$-vector bundle. Then $\mathcal{E}$ descends to a vector bundle on the quotient if and only if for all closed points $y \in Y$ such that the orbit $G \cdot y$ is closed, the stabilizer $G_y$ of $y$ acts trivially on the fiber $\mathcal{E}_y$.*

It will be useful to have estimates on the dimension of subschemes stabilized by some group elements, analogous to Lemma 5.2.7:

LEMMA 5.3.20. *Let $(X, F, L)$ be a smooth contact manifold of dimension $2n + 1 \geq 3$, $G$ a finite subgroup of automorphisms preserving the contact distribution $F$ and $Z^g$ the subscheme of points of $X$ stabilized by some $g \in G \setminus \{\mathrm{id}\}$ (it can have multiple components). Then $Z^g$ does not have any divisorial component, i.e. its codimension in $X$ is at least 2.*

PROOF. Let us pick a smooth point $x$ in some component $Y \subset Z^g$. From the Luna slice theorem [**Dré04**, Thm 5.4] we have $T_xY \simeq (T_xX)^G$, so to calculate the dimension of $Y$ at each point it is enough to determine the dimension of the invariant subspace of the tangent space. To that end, define $H$ to be the subgroup of $G$ generated by $g$ and observe that $(T_xX)^G \subset (T_xX)^H$, so we will consider this bigger subspace. Denote by $\phi_g$ the automorphism of $T_xX$ induced by $g$. By the definition of $G$ we have $\phi_g(F) \subset F$ and the invertibility of $\phi_g$ implies that $\phi_g(F) = F$. The short exact sequence appearing in Definition 1.2.1 implies the existence of a direct sum decomposition $T_xX = L_x \oplus F_x$ that in our case is in the category of $G$-modules. $\phi_g$ is an invertible linear map such that $\phi_g^k = \mathrm{id}$ for some $k \in \mathbb{Z}_{>0}$, so by picking a diagonal base for $F$ we obtain $\phi_g = \mathrm{diag}(\xi_0, \xi_1, ..., \xi_{2n})$, where $\xi_i$ are roots of unity of degree dividing $k$ and let $\xi_0$ be the root acting on $L_x$. The nondegeneracy of the contact form implies that there is an isomorphism $F \simeq F^* \otimes L$. This isomorphism gives us the following equality of sets: $\{\xi_1, ..., \xi_{2n}\} = \{\xi_1^{-1} \cdot \xi_0, ..., \xi_{2n}^{-1} \cdot \xi_0\}$.

Now, if $\xi_0 \neq 1$ then it follows that it is not possible for all other roots to be equal to 1, so $(T_xX)^H$ has codimension at least 2 in $T_xX$, so we conclude. If $\xi_0 = 1$, then the equality of sets above is not enough to conclude. Recall that we have $\mathcal{O}_X(-K_X) = L^{\otimes(n+1)}$, so in this case the action of $\phi_g$ on $\mathcal{O}_X(-K_X)_x$ is also trivial. But this implies that $\xi_0 \cdot \xi_1 \cdot ... \cdot \xi_{2n} = 1$, so it is not possible that only one root differs from 1 and we conclude as before. □

From the proof above one can also deduce a statement on the singularities of the quotient:

COROLLARY 5.3.21. *In the setting of Lemma 5.3.20, if $\pi\colon X \to X/G$ is the quotient map, then the variety $X/G$ is singular along $\pi(Z^g)$ for every $g \in G \setminus \{\mathrm{id}\}$.*

PROOF. Recall the classical theorem attributed to Chevalley-Shephard-Todd [**Ben93**, Ch. 6]: the quotient is smooth if and only if the stabilizer of each point is generated by pseudoreflections, that is elements that fix pointwise a codimension 1 subvariety containing $x$. But we have just seen in the proof of Lemma 5.3.20 that subvarieties preserved by any element of $G$ have codimension at least 2. Therefore, the image of any point with a nontrivial stabilizer is singular. □

We understand now which smooth points of $X$ become singular in the quotient, but it is also possible that some singular points of $X$ get mapped to smooth ones. Such situation poses a significant obstacle, as then it is not clear how to define the contact distribution on those unexpectedly smooth points. To avoid this difficulty, we will explicitly assume $\pi(X_{sing}) \subset (X/G)_{sing}$. Let us put $\mathcal{X} = X_{sm} \setminus \bigcup_{g \in G \setminus \{\mathrm{id}\}} Z^g$ and observe that with our assumption $\mathcal{X}$ is precisely the preimage of $(X/G)_{sm}$. Moreover, if we restrict ourselves to $\mathcal{X}$, then every $G$-linearized vector bundle on $\mathcal{X}$ has a descent (to a vector bundle on $(X/G)_{sm}$) and the morphism $\pi$ is étale, i.e. flat and unramified. It follows that we have an isomorphism of tangent spaces $T\mathcal{X} \simeq \pi^* T(X/G)_{sm}$ [**Mil13**, Ch. I, Prop. 2.9] and the tangent bundle of $\mathcal{X}$ descends to the tangent bundle of $(X/G)_{sm}$. This discussion guides us to propose the following definition:

Definition 5.3.22. Let $(X, F, L, \vartheta)$ be a contact variety of dimension $2n+1 \geq 3$ and $G$ a finite group of automorphisms preserving $F$. Moreover, assume that the quotient map $\pi$ satisfies $\pi(X_{sing}) \subset (X/G)_{sing}$. Let us take the contact exact sequence of $X$ restricted to $\mathcal{X}$:

$$0 \to F_{|\mathcal{X}} \to T\mathcal{X} \xrightarrow{\vartheta_{|\mathcal{X}}} L_{|\mathcal{X}} \to 0. \tag{21}$$

Consequently, we have a sequence of vector bundles on $(X/G)_{sm}$:

$$0 \to (\pi_* F_{|\mathcal{X}})^G \to T(X/G)_{sm} \xrightarrow{\vartheta'} (\pi_* L_{|\mathcal{X}})^G, \tag{22}$$

that is exact, as it comes from the application of two left exact functors, $\pi_*$ and $(\cdot)^G$. We say that $X/G$ has *the induced contact structure* if the above sequence of vector bundles on $(X/G)_{sm}$ gives a contact structure on $X/G$. Precisely, we demand the surjectivity of $\vartheta'$, the existence of the line bundle $L'$, extending $(\pi_* L_{|\mathcal{X}})^G$ and the nondegeneracy of $d\vartheta'$.

Our goal now is to discuss conditions that imply the existence of the induced contact structure on the quotient. First observe that clearly the existence of the descent of $L$ is a necessary condition, however the following example illustrates that it is not sufficient:

Example 5.3.23. Let us consider $\mathbb{C}^4 \ni (x_0, x_1, x_2, x_3)$ with the symplectic form $\omega = dx_0 \wedge dx_2 + dx_1 \wedge dx_3$ and the action of $\mathbb{Z}_2$ with generator $g$, where $g \cdot (x_0, x_1, x_2, x_3) = (x_0, x_1, -x_2, -x_3)$. Then the projective space $\mathbb{P}^3 = \mathbb{P}(\mathbb{C}^4)$ has the (standard) contact structure $(F, L, \vartheta)$ and an action of $\mathbb{Z}_2$. The symplectic and contact forms are not fixed by the action: in fact we have $g \cdot \omega = -\omega$, $g \cdot \vartheta = -\vartheta$. At the same time, the line bundle $\mathcal{O}(-1)$ admits a linearization coming from the action on $\mathbb{C}^4$ and as a result, the action of the stabilizer of any point $x$ on $L_x = \mathcal{O}(2)_x$ is trivial. Moreover, the kernel of $\vartheta$, i.e. $F$ is preserved by the action. Consequently, the quotient $\mathbb{P}^3/\mathbb{Z}_2$ admits a globally defined line bundle descended from $L$, a rank 2 distribution defined on its smooth locus that is induced by $F$, yet they do not give the structure of a contact variety, as the map induced from $\vartheta$ is 0.

We are now ready to state our criterion:

Theorem 5.3.24. *Let $(X, F, L, \vartheta)$ be a contact variety of dimension $2n+1 \geq 3$ and $G$ a finite group of automorphisms preserving $F$. Suppose that the quotient map $\pi$ satisfies $\pi(X_{sing}) \subset (X/G)_{sing}$. The quotient $X/G$ has the induced contact structure $(F', L', \vartheta')$ if the following two conditions are satisfied:*

(1) $\forall_{x \in X_{sing}}$ $\mathrm{Stab}(x)$ *acts trivially on* $L_x$.
(2) $\forall_{g \in G}$ $g^*\vartheta = \vartheta$,

As the proof of the theorem above utilizes euclidean topology, the following folklore argument will allow us to go back to the algebraic category:

Lemma 5.3.25. *Let $Y$ be a smooth algebraic variety and suppose that we have a morphism of locally free sheaves*

$$\phi \colon \mathcal{T} \to \mathcal{L},$$

*where $\mathcal{L}$ has rank 1 and $\mathcal{T}$ has rank $r \geq 1$. Assume that on the analytification $Y^{an}$ (i.e. $Y$ equipped with the euclidean topology) the induced morphism $\phi^{an}$ is a surjection of sheaves. Then $\phi$ also is a surjection of sheaves.*

PROOF. The surjectivity of $\phi^{an}$ is equivalent to the surjectivity on the stalks. Consequently, for any point $y \in Y^{an}$ we have a surjection:

$$\phi_y^{an} \colon \mathcal{T}_y^{an} = \bigoplus_r \mathcal{O}_{y,Y}^{an} \to \mathcal{O}_{y,Y}^{an} = \mathcal{L}_y^{an}.$$

Now let us consider the completion of the analytic local ring, $\widehat{\mathcal{O}}_{y,Y}^{an}$ that remains local and is canonically isomorphic to $\widehat{\mathcal{O}}_{y,Y}$. The map $\phi_y^{an}$ is determined by some holomorphic function, so by taking its Taylor expansion we obtain the map:

$$\bigoplus_r \widehat{\mathcal{O}}_{y,Y} \to \widehat{\mathcal{O}}_{y,Y}$$

that is still surjective, as the completion functor is exact. Now we can take the quotient of both local $\widehat{\mathcal{O}}_{y,Y}$-modules by the maximal ideal of $\widehat{\mathcal{O}}_{y,Y}$ to obtain fibres of associated vector bundles, that will be denoted by $\mathcal{T}(y)$ and $\mathcal{L}(y)$. We have the following commutative diagram (note that this time the stalks are algebraic and $y$ is a closed point):

$$\begin{tikzcd} \mathcal{T}_y \arrow[r] \arrow[d] & \mathcal{L}_y \arrow[d] \\ \mathcal{T}(y) \arrow[r] & \mathcal{L}(y). \end{tikzcd}$$

The lower horizontal arrow is surjective, as it comes from a surjective map of modules, so the basis element $\ell$ spanning $\mathcal{L}(y)$ has a preimage $\tau$ in $\mathcal{T}(y)$. By the Nakayama Lemma we can lift $\tau$ to an element of $\mathcal{T}_y$ that gets mapped to a generator of $\mathcal{L}_y$, so the map of algebraic stalks over $y$ is surjective for any closed point $y \in Y$. But $Y$ is a smooth algebraic variety, so in particular as a scheme it is noetherian and locally of finite type over $\mathbb{C}$, consequently any open subset of $Y$ containing all closed points is the whole $Y$. Therefore, we can conclude that surjection on stalks over closed points implies surjection over stalks on any point by [**AM69**, Prop. 3.9]. $\square$

PROOF OF THEOREM 5.3.24. To begin with, we recall the result of Boutot [**Bou87**, Corollaire, p.2] who showed that any quotient of a variety with rational singularities by a reductive group still has rational singularities. Let $\pi \colon X \to X/G$ be the quotient map. As before, $Z^g$ is the locus of points of $X_{sm}$ fixed by $g \in G \setminus \{\mathrm{id}\}$ and let $Z = \bigcup_{g \in G \setminus \{\mathrm{id}\}} Z^g$. Any component of $Z$ has codimension at least 2 by Lemma 5.3.20.

Observe now that the listed conditions imply that $L$ descends to the quotient. Indeed, let $x \in X$ be any point with a nontrivial stabilizer $\mathrm{Stab}(x) \subset G$. If $x \in X_{sm}$ then the invariance of the form implies that the action of $\mathrm{Stab}(x)$ on $L_x$ is trivial. Indeed, we have a direct sum decomposition of $G$-modules: $T_xX = F_x \oplus L_x$ and $\vartheta_x$ is the projection onto the second summand. If there were $g \in \mathrm{Stab}(x)$ acting nontrivially on $L_x$, then the projection from $T_xX$ onto $L_x$ (i.e. $\vartheta$) would not commute with the action of $g$ and this is not the case.

If $x \in X_{sing}$, then the triviality of the action of $\mathrm{Stab}(x)$ on $L_x$ is directly assumed in the first listed condition and we claim the existence of the descent by Lemma 5.3.19. The resulting line bundle $L'$ will be the contact line bundle for $X/G$.

The second listed condition implies that the map of the descended bundles induced from $\vartheta$ is an epimorphism. Recall that in our setting $\mathcal{X}$ is precisely the preimage of $(X/G)_{sm}$. To prove the surjectivity, it is enough to see that the map $(\pi_*T\mathcal{X})^G \to (\pi_*L)^G$ is surjective on stalks over the points of $(X/G)_{sm}$. First, we will show it in the analytic category, where the stalk over $x$ consists of pairs $(s, G \cdot U)$, where $U$ is an open (in the euclidean topology) subset of $X$, $U$ contains some preimage of $x$ and $s$ is an invariant section defined on $G \cdot U$. Without losing generality, we can assume that $U$ is connected and $G \cdot U$ is a disjoint union of $|G|$ copies of $U$, each one containing a single preimage of $x$.

Now if $\sigma$ is a section of $L$ defined on $U$, then $g^*\sigma$ is a section of $L$ defined on $g \cdot U$ (recall that the action on sections is given by $g^* \cdot \sigma(x) = \sigma(g \cdot x)$). Therefore, by translating a given $\sigma$ by every $g \in G$ we obtain a $G$-invariant section of $L$ defined on $G \cdot U$ (note that the finiteness of $G$ is crucial for our construction to work). By a slight abuse of notation we will denote such $G$-invariant section $\sum_{g \in G} g^*\sigma$. Note that in this way we can obtain every $G$-invariant section defined on $U$. By the surjectivity of $\vartheta$, there is a section $\tau$ of $T\mathcal{X}$ defined on $U$ such that $\vartheta(\tau) = \sigma$. We construct an associated $G$-invariant section defined on $G \cdot U$ for $\tau$, that we again denote by $\sum_{g \in G} g^*\tau$. By the invariance of $\vartheta$ it gets mapped to $\sum_{g \in G} g^*\sigma$, so $\vartheta'$ is indeed surjective for the analytic topology. To move back to algebraic category, apply Lemma 5.3.25.

To see that $d\vartheta'$ is nowhere degenerate on $(X/G)_{sm}$, we use ideas that appeared in our discussion of generically contact manifolds in the previous subsection. Namely, suppose that $\vartheta'$ is degenerate along some nontrivial degeneracy divisor $B \geq 0$ (the effectiveness of $B$ is shown in [**Mro18**, Stw. 1.6]), then we have $\mathcal{O}_{(X/G)_{sm}}(-K_{(X/G)_{sm}}) = (L'_{|(X/G_{sm})})^{\otimes(n+1)} \otimes \mathcal{O}_{(X/G)_{sm}}(-B)$ in the Picard group of $(X/G)_{sm}$. We pull back this relation to $\mathcal{X}$, but it can hold only if $\pi^*\mathcal{O}_{(X/G)_{sm}}(B) = \mathcal{O}_{\mathcal{X}}$, as $\mathcal{X}$ is a smooth subset of a contact variety, whose complement has codimension at least 2. Triviality of pullback implies that $\mathcal{O}_{(X/G)_{sm}}(-B)$ is a torsion element of the Picard group. To see it, observe that as $\pi$ is finite and flat of degree $d$ (on $\mathcal{X}$), we can define a covariant map $\overline{\pi}_*$ between Picard groups by:

$$\overline{\pi}_*(L_{|\mathcal{X}}) = (\bigwedge^d \pi_* L_{|\mathcal{X}}) \otimes (\bigwedge^d \pi_* \mathcal{O}_{\mathcal{X}})^{-1}.$$

The composition $\overline{\pi}_* \circ \pi^*$ is multiplication by $d$, so if an element of the Picard group of the target is in the kernel of $\pi^*$, it is in the kernel of the multiplication, i.e. it is a torsion element. But a nontrivial effective Weil divisor cannot give a torsion element in the Picard group, so we must have $B = 0$. Consequently, $d\vartheta'$ is nowhere degenerate on the smooth locus and the proof is concluded. □

This criterion will be used to construct two examples of contact varieties, namely Example 5.3.27 and Example 5.3.28.

REMARK 5.3.26. There are some differences between contact and symplectic cases. First one is that Lemma 5.2.7 shows that components of loci stabilized by some group element are symplectic, while Lemma 5.3.20 only shows that their codimension is big enough. We discuss it a little further after showing additional examples in Remark 5.3.29.

The second stark difference is that to prove Theorem 5.3.24 it was not enough to assume the invariance of the form and we needed an additional assumption on the triviality of the action on the fibers over the singular locus. Clearly, we use it to see that the contact line bundle descends to the quotient. One could hope that it is possible to get rid of this assumption by the use of the stratification theorem, if the induced contact forms on the strata were $G$-invariant. However, as we have explained in the discussion after Corollary 5.3.7, there is no reason to believe that this is the case and the twist of the contact form constitutes an unavoidable complication in the study of the contact varieties.

### 5.3.4. Examples – toric quotients and projectivized orbits.

EXAMPLE 5.3.27. To see applications of the results just presented, we will work out in detail one particular example which is especially accessible as it is toric (for the reference on toric varieties see [**CLS11**]). Start with an affine space with fixed even dimension $\mathbb{C}^{2n+2}$ and a symplectic form: $dx_0 \wedge dx_{n+1} + ... + dx_n \wedge dx_{2n+1}$. Consider the finite group $\widehat{G}$ of symplectomorphisms generated by $\xi_i$ for $i = 1, ..., n$, where $\xi_i$ acts on a vector by multiplying its $i$-th and $(i + n + 1)$-th coordinate by $(-1)$, i.e.:

$$\xi_i(x_0, ..., x_{2n+1}) = (x_0, ..., x_{i-1}, -x_i, x_{i+1}, ..., x_{i+n}, -x_{i+n+1}, x_{i+n+2}, ..., x_{2n+1}).$$

Note that $\widehat{G}$ is a subgroup of $T^{2n+2}$, the torus giving $\mathbb{C}^{2n+2}$ the structure of a toric variety. Now take the associated projective space $\mathbb{P}^{2n+1}$ equipped with an induced contact form $\vartheta = \sum_{i=0}^{n+1}(x_i dx_{i+n+1} - x_{i+n+1}dx_i)$. Observe that the action of $\widehat{G}$ descends to the projective space, it is a subgroup of the torus $T^{2n+1} \subset \mathbb{P}^{2n+1}$ and it preserves the twisted form. Denote the resulting group of automorphisms of $\mathbb{P}^{2n+1}$ preserving the contact structure by $G \simeq \mathbb{Z}_2^n$ and consider the quotient $X = \mathbb{P}^{2n+1}/G$. By Theorem 5.3.24 $X$ is a contact variety, as for all $g \in G$ we have $g^*\vartheta = \vartheta$.

To describe $X$ explicitly as a toric variety, let $M = \mathrm{Hom}(T^{2n+1}, \mathbb{C}^*)$ be the lattice of characters and $N \simeq \mathbb{Z}^{2n+1}$ the dual lattice spanned by $e_1, ..., e_{2n+1}$. Then $\mathbb{P}^{2n+1}$ is described by $\Sigma \subset N_{\mathbb{R}}$. It is a complete fan having $\rho_1 = e_1, ..., \rho_{2n+1} = e_{2n+1}, \rho_0 = -e_1 - ... - e_{2n+1}$ as rays and such that every strict subset of rays spans a cone. Now, finite subgroups of the torus correspond bijectively to finite index sublattices $\iota\colon M' \hookrightarrow M$, so dually we have an inclusion $N \to N'$ with a finite cokernel (equal to $G$), so that $N_{\mathbb{R}} = N'_{\mathbb{R}}$. Therefore, to obtain the fan $\Sigma'$ of the quotient variety, take the image of $\Sigma$ via the adjoint $\pi$ of the inclusion $\iota$. Our choice of generators of $\iota(M')$ will be: $w_i = e_i^* + e_{i+n+1}^*$ and $w_{n+i+1} = e_i^* - e_{i+n+1}^*$ for $n = 1, ..., n$ and $w_{n+1} = e_{n+1}^*$, so that the matrix of $\iota$ is symmetric and in consequence it also describes $\pi$.

The fan of $X$ is spanned by the following $2n+2$ rays: $\rho'_i = e_i + e_{i+n+1}$ and $\rho'_{n+i+1} = e_i - e_{i+n+1}$ for $n = 1, ..., n$, $\rho'_{n+1} = e_{n+1}$ and $\rho'_0 = -2 \cdot (e_1 + ... + e_n) - e_{n+1}$ and any strict subset of rays forms a cone. We denote the cone of the form $\mathrm{Cone}(\rho'_i, \rho'_{n+i+1})$ for $i = 0, ..., n$ (mind the case $i = 0$) by $\sigma_i$. Every such cone is singular as its generators cannot be extended to a basis of the whole lattice, and it corresponds to a codimension 2 singular subvariety of $X$, that is the image of $\mathbb{P}^{2n-1} = \{[x_0 : ... : x_{2n+1}] \mid x_i = x_{i+n+1} = 0\}$ via the quotient map. On the other hand, every cone that does not contain any of the $\sigma_i$ as a subcone is smooth. The columns of the matrix below are precisely rays of the fan of $X$ for $\dim(X) = 5$. Observe that $i$-th and $(i+3)$-th column span the singular cone $\sigma_i$:

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}.$$

Now we will describe a certain (non-standard) resolution of singularities of $X$. Toric resolutions of singularities are provided by particular refinements of the fans that do not change smooth cones and subdivide singular ones. Our resolution works in $n+1$ sequential steps, indexed by $i \in \{0, ..., n\}$. Put $\Sigma'_{-1} = \Sigma'$ and $X_{-1} = X$. Suppose now we are given a fan $\Sigma'_{i-1}$. Define a new ray $\rho'_{E_i} = \frac{1}{2}(\rho'_i + \rho'_{i+n+1})$ (every such added ray will be called *exceptional*) and refine $\Sigma'_{i-1}$ by dividing $\sigma_i$ onto two (smooth) cones: $\sigma'_i = \mathrm{Cone}(\rho'_i, \rho'_{E_i})$ and $\sigma''_i = \mathrm{Cone}(\rho'_{n+i+1}, \rho'_{E_i})$. Analogously, divide every cone that contained $\sigma_i$ into two new cones. The fan that we have obtained by this refinement is $\Sigma'_i$ and the corresponding toric variety is $X_i$. Such an operation is a particular example of a generalized star subdivision (see [**CLS11**, §11.1] for details) and in particular it corresponds to a projective morphism [**CLS11**, Prop. 11.1.6]. By construction, this morphism is birational and it is an isomorphism on the smooth locus of $X_{i-1}$.

Now observe that the final fan $\Sigma'_n$ is smooth, as we have divided every singular cone of $\Sigma'$ into subcones that are eventually smooth. Columns of the matrix below are rays of the resolution for $\dim(X) = 5$, where exceptional rays are in the last columns of the matrix. By the construction, cones of the fan are formed by any subset of rays, subject to two rules:

(1) there are no cones containing both $\rho'_i$ and $\rho'_{n+i+1}$ for any $i$ (as they were divided in some step),
(2) if a cone contains an exceptional ray $\rho'_{E_i}$ then it must contain either $\rho'_i$ or $\rho'_{i+n+1}$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ -2 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

We remark that this resolution of singularities is not standard: the algorithm described for example in [**CLS11**, Thm 11.1.9] in each step picks and divides the cone with the biggest multiplicity. In our case, it would be one of the higher dimensional cones, containing all but one $\sigma_i$. However such resolution, that blows up centers of higher codimension first, would not be crepant.

To see how the (anti)canonical divisor behaves with respect to our resolution, recall first that for any toric variety given by a fan, the class of the anticanonical divisor is equal to the class of the sum of all torus-invariant divisors (corresponding to rays) by [**CLS11**, Thm 8.2.3]. In our case, $-K_X$ is Cartier and can be described in terms of a support function $\phi$ whose domain is $N'_{\mathbb{R}}$ and that is linear on each cone [**CLS11**, Thm 4.2.11]. In the case of the anticanonical, the value of $\phi$ on each ray generator is $-1$. Pulling back this support function to any $\Sigma'_i$ we see that it takes value $-1$ also on the generator of any exceptional ray. This however implies that it is equal to the support function of the anticanonical of $X_i$, i.e. the maps $X_i \to X$ are all crepant and so are $X_i \to X_j$ for $i > j$.

Now, by Theorem 2.3.2 and Theorem 5.3.13 $X_n$ has to be isomorphic to $\mathbb{P}(T(\mathbb{P}^1 \times ... \times \mathbb{P}^1))$. It can also be seen directly, as described in [**OM78**, Prop. 7.3 and (7.6')]. We take the lattice $N'$ and project it by restricting to last $n+1$ coordinates. Then the image of $\Sigma'_n$ is the product fan for $n+1$ copies of $\mathbb{P}^1$, so $X_n$ is equipped with the projection onto $\mathbb{P}^1 \times ... \times \mathbb{P}^1$. The kernel of the lattice projection contains the standard fan of $\mathbb{P}^n$, and this is the fiber of this projection. Toric computations show that this bundle is in fact the projectivization of $\mathcal{O}(2,0,...,0) \times ... \times \mathcal{O}(0,...,0,2) = T\mathbb{P}^1 \times ... \times T\mathbb{P}^1$. The diagram below ilustrates our example: the vertical map is the finite quotient and horizontal ones are sequential blow-ups.

$$\begin{tikzcd} & & & & \mathbb{P}^{2n+1} \arrow[d] \\ \mathbb{P}(T(\mathbb{P}^1 \times ... \times \mathbb{P}^1)) = X_n \arrow[r] & X_{n-1} \arrow[r] & ... \arrow[r] & X_0 \arrow[r] & X \end{tikzcd}$$

The significance of this example comes from the fact that it allows us to provide a link between $\mathbb{P}^{2n+1}$ and $\mathbb{P}(T(\mathbb{P}^1 \times ... \times \mathbb{P}^1))$, i.e. two distinct families of contact manifolds via the quotient operation and a carefully chosen resolution, such that the intermediate varieties are contact in the sense of our Definition 5.3.1. It is an interesting question whether other contact manifolds can be linked similarly.

Example 5.3.28. Now take $X = \mathbb{P}^5$ with the contact and toric structure as described in the previous example and let $\xi$ be the generator of $\mathbb{Z}_2$. Assume that $\mathbb{Z}_2$ acts on $X$ by: $\xi \cdot [x_0 : x_1 : x_2 : x_3 : x_4 : x_5] = [x_0 : -x_1 : x_2 : x_3 : -x_4 : x_5]$ and consider the quotient of $X$ by this action. By Theorem 5.3.24 it is a contact variety. The fixed point locus has two connected components, described by $x_1 = x_4 = 0$ ($\mathbb{P}^3$) and $x_0 = x_2 = x_3 = x_5 = 0$ ($\mathbb{P}^1$), which get mapped isomorphically onto two components of the singular locus in the quotient. The (toric) resolution is provided by two disjoint blow-ups centered at those components. One can check that the resulting smooth variety has no chance of being contact, as its canonical divisor is not divisible by 3 in the class group. Nevertheless, the partial resolution obtained by blowing up just the bigger component is crepant, so it is another projective contact variety, call it $X'$. Now, assumptions of Proposition 5.3.17 are satisfied, as the resolution of $X'$ is provided by a blow-up along the singular locus isomorphic to $\mathbb{P}^1$. Consequently, the resulting variety is generically contact in the sense of Definition 5.3.15.

REMARK 5.3.29. In both of those examples, as in earlier Example 5.3.23, every component $Y$ of the locus of points stabilized by some nontrivial $g$ (denoted by $Z^g$) is a contact submanifold of some projective space. Indeed, in all three cases the group action is first defined on the affine space, so we have a linearization of $\mathcal{O}(-1)$. Moreover, every group element has order 2, so the stabilizers necessarily act trivially on respective fibers of $\mathcal{O}(2)$. Consequently, at each point $y \in Y$ we have $L_y \subset T_Y Y$ and the restriction of the twisted form to $Y$ gives it a contact structure.

It is equally easy to give an example of the action for which components of $Z^g$ are isotropic. To that end, take $\mathbb{C}^4$ with the symplectic form as before and let $\mathbb{Z}_4$ act on it via $(x_0, x_1, x_2, x_3) \mapsto (i \cdot x_0, -i \cdot x_1, i \cdot x_2, -i \cdot x_3)$. The resulting projective space with an action of $\mathbb{Z}_2$ is isomorphic as a $G$-variety (without taking into account the contact structure) to the one described in Example 5.3.27 with $n = 1$. However, the linearization is different, and for instance if $Y$ is given by $x_1 = x_3 = 0$, then the stabilizer of each point $y \in Y$ acts nontrivially on $L_y = \mathcal{O}(2)_y$. Consequently, $TY \subset F$ and $Y$ is an isotropic subvariety. Note that the quotient by this action does not result in a variety with the induced contact structure, as $\mathcal{O}(2)$ does not descend (and the contact form is not $G$-invariant).

However, in general we cannot claim that components of the locus stabilized by some $g \in G$ are isotropic or contact, as we are dealing with finite groups and our reasonings are pointwise.

As in the case of symplectic varieties, the second source of examples after quotients of known objects are (projectivized) orbits.

PROPOSITION 5.3.30. *For a semisimple group $G$ with a Lie algebra $\mathfrak{g}$ consider a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^*$ and its projectivization $\mathbb{P}(\mathcal{O}) \subset \mathbb{P}(\mathfrak{g}^*)$. Then the normalization of its closure $\overline{\mathbb{P}(\mathcal{O})}^n$ is a contact variety.*

PROOF. As we have already seen in Section 5.2.2, the normalization of a nilpotent orbit closure is a symplectic variety. Moreover, the natural $\mathbb{C}^*$-action lifts to the normalization and the Kostant-Kirillov form is homogeneous of weight 1. As the normalization commutes with the projectivization, we conclude by the Symplectization Theorem 5.3.5. Note that the contact line bundle comes from the restriction of $\mathcal{O}_{\mathbb{P}(\mathfrak{g}^*)}(1)$. $\square$

The idea of studying projectivizations of closures of nilpotent orbits already appeared in the literature, we mention in particular the unpublished preprint [**Fu06**], whose author proves some properties of their resolutions. Let us also point out that contrary to the smooth case, where for each type of a simple group we only have one manifold, corresponding to the projectivization of the minimal nilpotent orbit, we can obtain singular examples from any nilpotent orbit for a **semi**simple group, so even those specific examples are more frequent.

REMARK 5.3.31. The language of nilpotent orbits allows us to give an alternate description of both described examples. To begin with, consider the simple group SL(2). Its

Lie algebra $\mathfrak{sl}(2)$ is 3 dimensional and has the 2-dimensional nilpotent cone:

$$N = \left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \mid x^2 + yz = 0 \right\} = \mathrm{SL}(2) \cdot E_{1,2},$$

that – when we pass to the projectivization of the algebra – gets mapped to $\mathbb{P}^1$, that is simultaneously the minimal and the principal orbit. Now take a product of $(n+1)$ copies of $\mathrm{SL}(2)$. It is a semisimple group whose algebra is a direct sum of $(n+1)$ copies of $\mathfrak{sl}(2)$ and the adjoint action is component-wise. In particular, the nilpotent cone is the sum of nilpotent cones of the components. Every nilpotent orbit has generator of the form $(m_0, m_1, ..., m_n)$ where each $m_i$ is either the zero matrix or the elementary matrix $E_{1,2}$. Clearly, the element $(E_{1,2}, ..., E_{1,2})$ belongs to the principal orbit. Moreover, for an orbit given by a generator we can obtain generators of orbits lying in its closure by replacing some nonzero $m_i$'s by zero matrices.

To see that the projectivization of the nilpotent cone coincides with the toric quotient variety described in Example 5.3.27, let us define a map:

$$\mathbb{C}^{2n+2} \to \mathfrak{sl}(2) \oplus ... \oplus \mathfrak{sl}(2),$$

$$(x_0, x_1, ..., x_{2n+1}) \mapsto \left( \begin{bmatrix} x_0 \cdot x_{n+1} & x_0^2 \\ -x_{n+1}^2 & -x_0 \cdot x_{n+1} \end{bmatrix}, ..., \begin{bmatrix} x_n \cdot x_{2n+1} & x_n^2 \\ -x_{2n+1}^2 & -x_n \cdot x_{2n+1} \end{bmatrix} \right).$$

Now observe that the image of the affine space is precisely the nilpotent cone and that the map descends to the morphism between the projectivizations (of the affine space and of the algebra). Moreover, recall that in Example 5.3.27 we have defined an action of the group $\mathbb{Z}_2 \times ... \times \mathbb{Z}_2$ on $\mathbb{P}^{2n+1}$ and we see that the projectivization of our map is constant on orbits of this action. Finally, by a direct computation we may verify that the symplectic forms agree and in this way we rediscover our toric quotient as a projectivized nilpotent cone. Moreover, the resolution that we have described is the projectivized version of *the Springer resolution* (see [**Gin98**, Section 6] or [**Fu03**] for a reference).

Example 5.3.28 is a union of 3 orbits in the projectivization of the algebra $\mathfrak{sp}(4) \oplus \mathfrak{sl}(2)$. Recall from Section 4.2.2 that for the symplectic group the adjoint variety is the projective space, embedded via the Veronese map. We put:

$$\mathbb{C}^6 \to \mathfrak{sp}(4) \oplus \mathfrak{sl}(2),$$

$$(x_0, x_1, x_2, x_3, x_4, x_5) \mapsto \left( \begin{bmatrix} x_0x_5 & x_0x_3 & x_0^2 & x_0x_2 \\ x_2x_5 & x_2x_3 & x_0x_2 & x_2^2 \\ x_3^2 & x_3x_5 & -x_0x_5 & -x_2x_5 \\ x_3x_5 & x_5^2 & -x_0x_3 & -x_2x_3 \end{bmatrix}, \begin{bmatrix} x_1x_4 & x_1^2 \\ -x_4^2 & -x_1x_4 \end{bmatrix} \right).$$

One can see that the $4 \times 4$ matrix is indeed an element of the algebra $\mathfrak{sp}(4)$: the off-diagonal $2 \times 2$ blocks are symmetric and the lower diagonal block is the negative transpose of the upper diagonal block. Moreover, it is traceless and of rank 1, so it is nilpotent and more precisely belongs to the minimal (nilpotent) orbit $\mathcal{O}_{[2,1,1]}$. As before, we pass to the map between projective spaces and observe that it is constant on $\mathbb{Z}_2$-orbits to conclude. The two components of the singular locus described in Example 5.3.28 are precisely the projectivizations of the two minimal orbits of $\mathfrak{sp}(4)$ and $\mathfrak{sl}(2)$.

**5.3.5. Existence of rational curves.** It would be desirable to have the full classification of possible Mori contractions for singular contact varieties, analogous to Theorem 2.3.2. Unfortunately, it is currently out of reach, but we can recreate initial steps. To begin with, Theorem 2.3.1 of Demailly was very recently generalized by Cao and Höring, so that we can claim:

PROPOSITION 5.3.32. *Let $(X, F, L, \vartheta)$ be a projective contact variety. Then $L^*$ is not pseudoeffective. Consequently, $X$ is uniruled and it admits a Mori contraction.*

Proof. We first need to verify that we are indeed in the setting of [**CH22**, Thm 1.2], i.e. that $X$ is a normal compact Kähler space (in the sense of [**Uen83**, Def. 1.1]) with klt singularities and that $L^*$ is a reflexive sheaf of rank 1. The second claim is clear, as locally free sheaves are reflexive. $X$ is clearly compact and has klt (in fact canonical) singularities.

To see that $X$ is a Kähler space, observe that it has an embedding in a projective space $\mathbb{P}^N$ and we may consider the $(1,1)$ form $\omega$ coming from the restriction of the curvature of the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^N}(1)$. The form $\omega$ gives a Kähler metric on $X_{sm}$. Moreover, the local embeddings from the definition of the Kähler space are provided by the global embedding $X \hookrightarrow \mathbb{P}^N$.

Consequently, we may apply [**CH22**] and observe that if $L^*$ were pseudoeffective, then the kernel $F$ of the twisted form $\vartheta \in H^0(X, (\Omega^1_X)^{**} \otimes L)$ would define a foliation, but it is absurd, as $F$ is non-integrable.

Now, as $L^*$ is not pseudoeffective, the same thing holds for $\mathcal{O}(K_X)$ and for $\mathcal{O}(K_{\widetilde{X}})$, where $\widetilde{X}$ is a resolution of singularities (we use the fact that $X$ has canonical singularities). Consequently, [**BDPP13**, Thm 2.2] implies that $\widetilde{X}$ admits a moving family of rational curves intersecting negatively with $K_{\widetilde{X}}$, so in particular the Mori cone has a nontrivial $K_{\widetilde{X}}$-negative part. The images (via the resolution map) of curves from this family cover $X$ and they intersect negatively with $K_X$. □

In the singular case the Cone Theorem 2.2.2 states that any curve $C$ spanning an extremal ray has bounded intersection with $-K_X$. It can be translated to the condition that $L \cdot C \in \{1, 2, 3\}$ for the contact line bundle $L$. Unfortunately we cannot further mimic the proof of Theorem 2.3.2 as for the rational curves on singular varieties we cannot present the bound on dimension of the space of morphisms from Theorem 2.2.4 in terms of cohomology. Consequently, the locus-fiber inequality from Theorem 2.2.7 does not take an easily computable form, so we cannot compute the dimension of fibers, nor claim that there are no birational Mori contractions. The situation is clearer in the cases where the projective contact variety $X$ admits a crepant resolution.

Proposition 5.3.33. *Let $(X, L)$ be a singular projective contact variety admiting a crepant resolution. Then it has a unique $K_X$-negative ray in its Mori cone and the rational curve $C$ generating it satisfies $L \cdot C = 1$. Moreover, the induced contraction is of fiber type (i.e. it is onto a variety of lower dimension).*

Proof. Let $f \colon \widetilde{X} \to X$ be the resolution morphism. By Theorem 5.3.13 and Theorem 2.3.2 we have $(\widetilde{X}, f^*L) = (\mathbb{P}(TM), \mathcal{O}_{\mathbb{P}(TM)}(1))$ for some manifold $M$ and there is a unique $K_{\mathbb{P}(TM)}$-negative ray in the Mori cone of $\mathbb{P}(TM)$, generated by the rational curve $\ell$ in the fiber of the natural contraction $\mathbb{P}(TM) \to M$. Moreover, $\mathcal{O}_{\mathbb{P}(TM)}(1) \cdot \ell = 1$. The morphism $f$ is crepant and surjective, therefore the induced map $f_*$ on 1-cycles is surjective and for any 1-cycle $\gamma$ on $\mathbb{P}(TM)$ we have $\gamma \cdot K_{\mathbb{P}(TM)} = f_*\gamma \cdot K_X$, so we also have $\gamma \cdot f^*L = f_*\gamma \cdot L$. Therefore, the curve $f_*\ell$ generates the unique $K_X$-negative ray and $f_*\ell \cdot L = 1$. Finally, the surjectivity of $f$ implies that the locus of the contraction associated to $f_*\ell$ is the whole $X$. □

The stark difference with the smooth case is that we cannot determine the dimension of the variety obtained via the Mori contraction. We can illustrate it using the diagram of commuting contractions coming from Example 5.3.27 for $\dim(X) = 5$:

$$\begin{array}{ccccccc}
\mathbb{P}(T(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)) & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1 & \longrightarrow & \mathbb{P}^1\times\mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 & \longrightarrow & *
\end{array}$$

Upper horizontal arrows are crepant blow-ups, while the vertical ones are Mori contractions: even though the dimesions of contact varieties do not change, they get mapped onto varieties of decrementally smaller dimensions.

On the other hand, even though the variety described in Example 5.3.28 does not admit a crepant resolution of singularities, it is also a Fano variety with $b_2(X) = 1$. Consequently, its only Mori contraction is onto a point.

## 5.4. Projective singular contact threefolds

As a conclusion of our study we give the full classification of strictly singular projective threefolds equipped with the contact structure. It turns out that they all can be constructed from ruled surfaces, therefore we begin with fixing the notation.

Setting 5.4.1. In this section $S$ denotes a ruled surface in the sense of [**Har77**, V.2], i.e. a smooth projective surface equipped with a surjective morphism $p\colon S \to B$ to a smooth projective curve $B$ of genus $g$ such that every fiber is isomorphic to $\mathbb{P}^1$. We will denote (any) such fiber by $\ell$, remembering that it can also be considered as an effective divisor on $S$. With these assumptions, there exists a (non-unique) locally free sheaf $\mathcal{E}$ of rank 2 on $B$ such that $S \simeq \mathbb{P}(\mathcal{E})$ [**Har77**, V, Prop. 2.2]. Moreover, to normalize the construction, one can demand that $H^0(B, \mathcal{E}) \neq 0$ but $H^0(B, \mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on $B$ of negative degree. Then $e = -\deg \mathcal{E}$ is an invariant of $S$ and there exists a section $s\colon B \to S$ such that the (divisor) class of its image, $B_0$ is equal to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ [**Har77**, V, Prop. 2.8]. The space $N_1(S)$ as well as $N^1(S)$ is spanned by the classes of $\ell$ and $B_0$.

In the particular case when $B = \mathbb{P}^1$, there are countably many isomorphism classes of ruled surfaces, determined by the integer $e \geq 1$, that will be denoted by $\mathcal{H}_e :=$ $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$. They are known as *the Hirzebruch surfaces*. Every Hirzebruch surface admits a divisorial contraction of the section $B_0$ (it is called *the minimal section*, as any other section has larger self-intersection) and the resulting variety is a cone that we will denote by $\mathcal{C}_e$.

Finally, we will consider the projectivized tangent bundle $\mathbb{P}(TS)$ over $S$, along with the natural projection $\pi\colon \mathbb{P}(TS) \to S$ and the fiber $C_\pi$. For the brevity of notation, we will sometimes denote by $\xi$ the class of $\mathcal{O}_{\mathbb{P}(TS)}(1)$. In our setting, $N^1(\mathbb{P}(TS))$ is spanned by classes of $\xi$, $\pi^*\ell$ and $\pi^*C_0$ [**Har77**, II, Ex. 7.9]. Note that although in general $TS$ could also be twisted by some line bundle and the resulting projectivized bundle would be isomorphic to $\mathbb{P}(TS)$, we do not do it, as we want to keep the natural surjective morphism $\pi^*TS \to \mathcal{O}_{\mathbb{P}(TS)}(1)$.

Remark 5.4.2. There exists a unique ruled surface admitting two distinct rulings, namely $\mathbb{P}^1 \times \mathbb{P}^1$. It will provide a special case in our classification and it will require some additional care in our reasonings.

Our goal is to prove the following two theorems:

Theorem 5.4.3. *Notation as in Setting 5.4.1. Let $(X, L)$ be a singular contact variety in dimension 3 that is projective. Then $X$ admits a crepant resolution $f\colon \mathbb{P}(TS) \to X$ for some $S$. Going the other way around, every $\mathbb{P}(TS)$ (over a base curve $B$) admits a section $\sigma\colon S \to \mathbb{P}(TS)$ and a crepant morphism onto a singular contact threefold $X$. This crepant* *morphism contracts the image of $\sigma$ onto a curve isomorphic to $B$. Moreover, if $X$ is not Fano, then $\rho(X) = 2$ and $X$ is a locally trivial bundle over $B$, whose fibers are cones $\mathcal{C}_2$. In this case, we have the following commutative diagram of contractions and sections:*

$$\begin{tikzcd} \mathbb{P}(TS) \arrow[r, "f"] \arrow[d, "\pi"] & X \arrow[d, "\pi'"] \\ S \arrow[u, bend left, "\sigma"] \arrow[r, "p"] & B. \arrow[l, bend left, "s"] \end{tikzcd}$$

Theorem 5.4.4. *There exists a unique singular projective Fano contact variety $X$ in dimension 3. It has $\rho(X) = 1$ and is resolved by $\mathbb{P}(T(\mathbb{P}^1 \times \mathbb{P}^1))$. The resolution morphism is given by the contraction associated to the linear system $|-bK_{\mathbb{P}(T(\mathbb{P}^1\times\mathbb{P}^1))}|$ for $b >> 0$. Moreover, it coincides with the variety described in Example 5.3.27 and Remark 5.3.31 for dimension 3, i.e. it can be described either as a quotient of $\mathbb{P}^3$ by $\mathbb{Z}_2$ or as a projectivization of the nilpotent cone of $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$.*

Remark 5.4.5. Note that contact manifolds of the form $\mathbb{P}(TS)$ may admit different contact forms, and in fact we have a bijection ([**KPSW00**, Prop. 2.14]):

$$H^0(S, \mathrm{End}(\Omega^1_S)) \to H^0(\mathbb{P}(TS), \Omega^1_{\mathbb{P}(TS)}(1))$$

between automorphisms of $\Omega^1_S$ and contact forms on $\mathbb{P}(TS)$. In light of this identification, we should not expect that the contact structure on $X$ is unique.

We split the proofs of both theorems into a few auxiliaries. First, we will show that our threefolds can always be resolved by a projective contact manifold.

Proposition 5.4.6. *Let $(X, L)$ be a projective singular contact threefold. Then $X$ has a crepant resolution of singularities by the projective contact manifold $(\mathbb{P}(TS), \mathcal{O}_{\mathbb{P}(TS)}(1))$ for some $S$ as in Setting 5.4.1. Moreover, a ruling on $S$ is given by the image (via $\pi$) of the rational curve contracted by the resolution morphism.*

Proof. To begin with, we show that $X$ admits a crepant resolution of singularities. Indeed, let us denote by $f\colon \widetilde{X} \to X$ the crepant terminalization of $X$, that exists by [**BCHM10**, Cor. 1.4.4] (the assumption are satisfied by Proposition 5.3.3). It is a contact variety by Theorem 5.3.13. The singular locus of a terminal variety has codimension at least 3 by [**BS95**, Lem. 1.3.1], but on the other hand by Corollary 5.3.9 its codimension must be even, therefore it is empty and $f$ already resolves all singularities and is crepant. Moreover, Corollary 5.3.9 shows that the singular locus consists of disjoint, smooth curves. As $\widetilde{X}$ is a projective contact manifold that is not Fano (it admits a crepant contraction), it must be isomorphic to $\mathbb{P}(T\Sigma)$ with the contact line bundle $\mathcal{O}_{\widetilde{X}}(1)$ for some smooth projective surface $\Sigma$ by Theorem 2.3.2 and moreover $f^*L = \mathcal{O}_{\widetilde{X}}(1)$.

Now we will show that $\Sigma$ has to be a ruled surface. To that end, recall that every fiber of $f$ is covered by rational curves by [**HM07**, Cor. 1.6]. Denote by $E$ an irreducible (divisorial) component of the exceptional locus of $f$ that gets mapped onto some curve $C_X$ in the singular locus of $X$. We have a rational curve $C_E \subset E$ such that $C_E \cdot \mathcal{O}_{\widetilde{X}}(1) = 0$, as the morphism contracting $C_E$ is crepant and $\mathcal{O}_{\widetilde{X}}(-K_{\widetilde{X}}) = \mathcal{O}_{\widetilde{X}}(2)$ by Proposition 5.3.3. Denote by $\gamma : \mathbb{P}^1 \to C_E \subset \mathbb{P}(T\Sigma)$ the normalization of $C_E$. Consider the relative Euler sequence ([**KPSW00**, Lem. 2.5]):

$$0 \to \Omega^1_{\mathbb{P}(T\Sigma)/\Sigma} \otimes \mathcal{O}_{\widetilde{X}}(1) \to \pi^* T\Sigma \to \mathcal{O}_{\widetilde{X}}(1) \to 0$$

and pull it back via $\gamma$, remembering that every vector bundle on $\mathbb{P}^1$ splits and that $\mathcal{O}_{\widetilde{X}}(1)$ is trivial on $C_E$. We obtain:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(b) \to \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \to \mathcal{O}_{\mathbb{P}^1} \to 0,$$

where we know that $a_1 \geq 2$ by [**Kol96**, II, Lemma 3.13]. Consequently, the map $\mathcal{O}_{\mathbb{P}^1}(a_1) \to \mathcal{O}_{\mathbb{P}^1}$ must be trivial, so we have $a_2 = 0$. It follows that there exists a free rational curve on $\Sigma$, so it is uniruled. Moreover, the numerical class of $C_E$ belongs to a $K_{\mathbb{P}(T\Sigma)}$-trivial extremal ray in the Mori cone of $\mathbb{P}(TS)$, so $\pi(C_E)$ belongs to a $K_\Sigma$-negative extremal ray in the Mori cone of $\Sigma$. Let $C$ be a rational curve on $\Sigma$ spanning the latter ray. By the Cone Theorem 2.2.2 we have $-K_\Sigma \cdot C \in \{1, 2, 3\}$, so via the adjunction formula we obtain $C^2 \in \{-1, 0, 1\}$. By [**KM98**, Thm 1.28] these three possible self intersections correspond to three different contractions:

- If $C^2 = -1$, then $C$ is a smooth contractible $(-1)$-curve and it particular it does not move,
- if $C^2 = 0$, then $\Sigma$ is a ruled surface $S$ and $C$ is its fiber $\ell$,
- if $C^2 = 1$, then $\Sigma \simeq \mathbb{P}^2$.

We easily exclude the last possibility, as we already know that $\mathbb{P}(T\mathbb{P}^2)$ is a homogeneous Fano manifold, so in particular it does not admit any curves intersecting trivially with the anticanonical, consequently $C_E$ could not exist in this case.

To see that the first case is also impossible, observe that if $C$ does not move, then we necessarily have $\pi(E) = C$, so $E$ is a ruled surface $\mathbb{P}(T\Sigma_{|C})$ over $C$. Now consider two maps from $E$, namely $\pi_{|E}$ onto $C$ and $f_{|E}$ onto $C_X$. Note that both have connected fibers and by the Rigidity Lemma [**KM98**, Lem. 1.6] no fiber of $\pi_{|E}$ gets contracted to a point by $f_{|E}$ and vice versa. In particular, rational curves that are fibers of $\pi_{|E}$ are mapped onto $C_X$, so we have $C_X \simeq \mathbb{P}^1$. But this means that $E$ is a trivial ruled surface over $C \simeq \mathbb{P}^1$, so $T\Sigma_{|C} \simeq \mathcal{O}_C \oplus \mathcal{O}_C$. We have reached a contradiction, as we have $-K_\Sigma \cdot C = 1 \neq 0$. We conclude that $\Sigma$ is indeed a ruled surface $S$.

The last statement is clear: the resolution morphism contracts $C_E$ and we have just argued that $\pi(C_E)$ is the curve whose numerical class is a multiple of a ruling on $S$. $\square$

Now we will establish the existence of section for $\pi$ and show some consequences of it.

PROPOSITION 5.4.7. *Notation as in Setting 5.4.1. A ruling $p\colon S \to B$ induces a section $\sigma\colon S \to \mathbb{P}(TS)$ that in particular allows us to embed $N_1(S)$ in $N_1(\mathbb{P}(TS))$. We have the following intersection table:*

| $\cdot$ | $\sigma_*[\ell]$ | $\sigma_*[B_0]$ | $[C_\pi]$ |
|---|---|---|---|
| $\pi^*\ell$ | 0 | 1 | 0 |
| $\pi^*B_0$ | 1 | $-e$ | 0 |
| $\mathcal{O}_{\mathbb{P}(TS)}(1)$ | 0 | $2-2g$ | 1 |

*Consequently, the classes of curves $\sigma_*[\ell], \sigma_*[C_0], [C_\pi]$ form a basis of the vector space $N_1(\mathbb{P}(TS))$.*

PROOF. The map $p\colon S \to B$ induces an epimorphism $TS \twoheadrightarrow p^*TB$ that corresponds to the section $\sigma$ by [**Har77**, II, Prop. 7.12] and it holds that $\pi \circ \sigma = \mathrm{id}_S$. In consequence, we have the following maps of vector spaces:

$$N_1(S) \xrightarrow{\sigma_*} N_1(\mathbb{P}(TS)) \xrightarrow{\pi_*} N_1(S)$$

$$N^1(S) \xrightarrow{\pi^*} N^1(\mathbb{P}(TS)) \xrightarrow{\sigma^*} N^1(S),$$

where both compositions are identity.

To compute two first rows of the intersection table, simply use the projection formula, remembering that on $S$ we have $\ell \cdot \ell = 0$, $\ell \cdot B_0 = 1$ and $B_0 \cdot B_0 = -e$ (the last equality is shown in [**Har77**, Ch. V, Prop. 2.9]). Moreover, $\pi_*[C_\pi] = 0$, so $C_\pi$ necessarily intersects trivially with divisors pulled back from $S$.

Now we will consider the last row. We have $\mathcal{O}_{\mathbb{P}(TS)}(1) \cdot C_\pi = 1$ by definition. If $C$ is any curve on $S$, then to compute its intersection with $\mathcal{O}_{\mathbb{P}(TS)}(1)$ we again employ the projection formula, this time for the morphism $\sigma$:

$$\sigma_*[C] \cdot \mathcal{O}_{\mathbb{P}(TS)}(1) = [C] \cdot \sigma^*(\mathcal{O}_{\mathbb{P}(TS)}(1)) = [C] \cdot p^*TB,$$

where the last equality comes from [**Har77**, II, Prop. 7.12]. In particular, we have $[\ell] \cdot p^*TB = 0$ and $[B_0] \cdot p^*TB = 2 - 2g$. Recall from our earlier discussion in Setting 5.4.1 that $\mathcal{O}_{\mathbb{P}(TS)}(2) = \mathcal{O}(-K_{\mathbb{P}(TS)})$, so the morphism contracting $\sigma_*[\ell]$ – if exists – is crepant.

Moreover, note that if $g = 0$ then the curve $\sigma_*[B_0]$ intersect the anticanonical positively, but by [**KPSW00**, Prop. 2.13] the cone $\overline{NE(\mathbb{P}(TS))}$ admits no other $-K$-positive extremal rays other than $[C_\pi]$. Therefore, either $\sigma_*[B_0]$ lies in the interior of the Mori cone, or inside of the two-dimensional wall, bounded by $[C_\pi]$. The first possibility can be discarded as we have assumed in Setting 5.4.1 that $S$ is normalized, so in particular $[B_0]$ is a ray bounding the cone $\overline{NE(S)}$ and $\sigma_*[B_0]$ gets mapped onto this ray by $\pi_*$. We conclude that $\sigma_*[B_0]$ lies in a two-dimensional wall and we deduce that the class of $\sigma_*[B_0] - 2[C_\pi]$ must lie in $\overline{NE(\mathbb{P}(TS))}$. In fact, one can directly show the existence of a curve in this class by considering the ruled surface $\mathbb{P}(TS_{|B_0})$, but we won't need it.

For the last claim, the space $N_1(\mathbb{P}(TS))$ is of dimension 3 as a dual of $N^1(\mathbb{P}(TS))$ for which we have already picked a base (in Setting 5.4.1). The computations of the intersection table show clearly that picked representatives are linearly independent in $N_1(\mathbb{P}(TS))$, so they form a base. $\square$

We also need to determine two intersection numbers:

LEMMA 5.4.8. *Notation as in Setting 5.4.1. We have:*

$$\xi^3 = 4(1-g),$$
$$\xi^2 \cdot \pi^*\ell = 2.$$

PROOF. To begin with, recall that the pullback of cocycles $\pi^*$ gives the Chow ring $A(\mathbb{P}(TS))$ the structure of a free $A(S)$-module with a basis $1, \xi$ ([**Har77**, App. A, 2.A.11]), where $\xi$ is the class of $\mathcal{O}_{\mathbb{P}(TS)}(1)$, so that $2\xi$ is the class of $-K_{\mathbb{P}(TS)}$. Moreover, we have the following relation ([**Har77**, App. A, 3, Definition on p. 429]):

$$\xi^2 - \xi \cdot \pi^*c_1(TS) + \pi^*c_2(TS) = 0. \tag{23}$$

Our task is to compute:

$$\begin{aligned}
\xi^3 = \cdot\xi \cdot \xi^2 = \xi(\xi \cdot \pi^*c_1(TS) - \pi^*c_2(TS)) = \\
= (\pi^*c_1(TS))^2 - \pi^*c_1(TS) \cdot \pi^*c_2(TS) - \xi \cdot \pi^*c_2(TS) = \\
= \xi \cdot (\pi^*c_1(TS))^2 - \xi \cdot \pi^*c_2(TS), \\
(-K_{\mathbb{P}(TS)})^2 \cdot \pi^*\ell = \xi^2 \cdot \pi^*\ell = \xi \cdot \pi^*\ell \cdot \pi^*c_1(TS) - \pi^*c_2(TS) \cdot \pi^*\ell = \\
= \xi \cdot \pi^*\ell \cdot \pi^*c_1(TS).
\end{aligned}$$

Although $c_1(TS)$ is dependent on $\mathcal{E}$ ([**Har77**, IV, Lem. 2.10]), its square depends only on the genus $g$ of the base curve $B$ and we have $c_1(TS)^2 = 8(1-g)$ [**Har77**, V, Cor. 2.11]. Moreover, as $[\ell] \cdot [\ell] = 0$ and $[\ell] \cdot [C_0] = 1$ we obtain that:

$$\pi^*\ell \cdot \pi^*c_1(TS) = [\ell] \cdot c_1(TS) = 2.$$

Recall the Riemann-Roch formula for surfaces [**Har77**, V, Remark 1.6.1]

$$12(1 + p_a) = c_1(TS)^2 + c_2(TS),$$

where $p_a$ is the arithmetic genus that is equal to $-g$ [**Har77**, V, Cor. 2.5]. It allows us to compute that $c_2(TS) = 4(1-g)$ and consequently we easily obtain desired intersection numbers ($[x]$ denotes the class of a point):

$$\xi^3 = \xi \cdot 4(1-g)\pi^*[x] = 4(1-g)$$
$$\xi^2 \cdot \pi^*\ell = 2.$$

□

Now we will determine a nef divisor that will serve as the supporting divisor of the contraction.

LEMMA 5.4.9. *Notation as in Setting 5.4.1. For any $\mathbb{P}(TS)$, let us consider a family of divisors $D_{a,b} := a \cdot \pi^*\ell + b \cdot \xi$, where $a, b$ are positive integers and $\xi$ is the Weil divisor corresponding to $\mathcal{O}_{\mathbb{P}(TS)}(1)$, so that $2\xi = -K_{\mathbb{P}(TS)}$. For any fixed positive value of $b$, if $a \geq b \cdot \max\{e, 2g-2\}$, then $D_{a,b}$ is nef.*

PROOF. First note that if a divisor $D$ is nef, then so is $m \cdot D$ for any $m \in \mathbb{Z}_{>0}$, as nefness is a convex condition. Consequently, if we find an integer $a$ guaranteeing the nefness of $D_{a,1}$, then $D_{b \cdot a,b}$ is also nef. Therefore, it is enough to fix $b = 1$ and find an integer $a$ for this particular value.

In fact, it will be much easier to argue using the generalization of the notion of being nef to vector bundles[2] (see [**CP91**] or [**Laz04**, Part II] for a discussion). Namely, to show that $D_{a,1} = a \cdot \pi^*\ell + \xi$ is nef it is enough to prove that the vector bundle $TS \otimes \mathcal{O}_S(a \cdot \ell)$ is nef. Indeed, its pullback via $\pi$ maps epimorphically onto $\mathcal{O}_{\mathbb{P}(TS)}(1) \otimes \pi^*\mathcal{O}_S(a \cdot \ell)$ that is the line bundle corresponding to the divisor $D_{a,1}$, and the quotient of a nef vector bundle is nef by [**CP91**, Prop. 1.2.(4)].

The considered vector bundle sits in the middle of a twisted relative tangent short exact sequence:

$$0 \to T_{S/B} \otimes \mathcal{O}_S(a \cdot \ell) \to TS \otimes \mathcal{O}_S(a \cdot \ell) \to p^*TB \otimes \mathcal{O}_S(a \cdot \ell) \to 0,$$

and to show its nefness, it is enough by [**CP91**, Prop. 1.2.(5)] to prove that two other terms in the short exact sequence are nef. From the relative Euler sequence for $S = \mathbb{P}(\mathcal{E})$:

$$0 \to \mathcal{O}_S \to p^*(\mathcal{E}^*) \otimes \mathcal{O}_S(1) \to T_{S/B} \to 0$$

we obtain $T_{S/B} \simeq p^*(\det \mathcal{E}^*) \otimes \mathcal{O}_S(2)$, so numerically we have $T_{S/B} \equiv 2B_0 + e \cdot \ell$ and $p^*TB \equiv (2-2g) \cdot \ell$. Therefore, it is enough to twist both bundles by $\mathcal{O}_S(a \cdot \ell)$ for any $a \geq \max\{e, 2g-2\}$ to ensure that both are nef and the nefness of $D_{a,1}$ follows, so we are done. □

We are ready to prove the existence of the contraction:

PROPOSITION 5.4.10. *Notation as in Setting 5.4.1. For any $\mathbb{P}(TS)$ and a positive integer $a$ large enough, the linear system of some positive multiple of the divisor $D_{a,1} = a \cdot \pi^*\ell + \xi$ gives a crepant and birational morphism, contracting precisely the class $\sigma_*[\ell]$. If $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ then the linear system of some positive multiple of $-K_{\mathbb{P}(T(\mathbb{P}^1 \times \mathbb{P}^1))}$ gives another crepant and birational morphism, that contracts the 2-dimensional wall of the Mori cone of $\mathbb{P}(T(\mathbb{P}^1 \times \mathbb{P}^1)$ lying on the hyperplane $K_{\mathbb{P}(T(\mathbb{P}^1 \times \mathbb{P}^1))} = 0$.*

PROOF. To prove that the linear system of a divisor $D'$ gives a birational morphism, we need to show that $D'$ is nef, big and semiample. Such morphism contracts precisely those curves that intersect $D'$ trivially. Moreover, if the anticanonical class is trivial on any contracted curve, then the morphism is crepant. By Lemma 5.4.9, if we have a positive integer $a \geq 3 \cdot \max\{e, 2g-2\}$, then both $D_{a,1}$ and $D_{a,3} = D_{a,1} - K_{\mathbb{P}(TS)}$ are nef.

[2]The author is grateful to the anonymous referee of [**Śmi23**] for this argument, that is clearer than the one provided originally by the author.

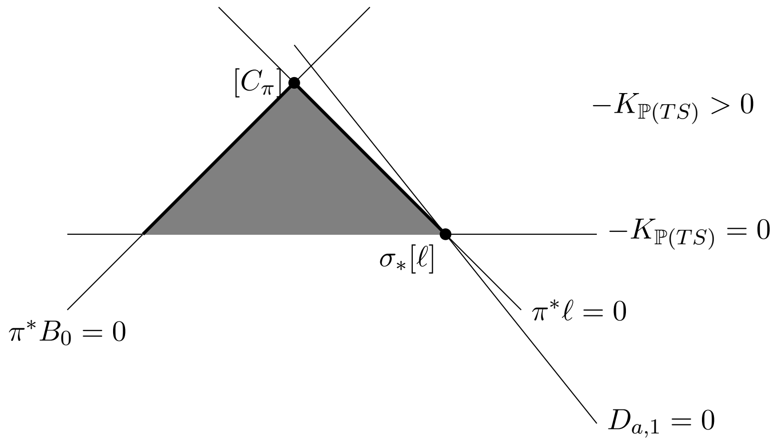


FIGURE 1. Hyperplane section of the (part of the) Mori cone of $\mathbb{P}(TS)$.

Now we need to argue that curves in the class of $\sigma_*[\ell]$ are the only ones intersecting $D_{a,1}$ trivially. The picture above is obtained by intersecting the part of the Mori cone of $\mathbb{P}(TS)$ that we have determined in Proposition 5.4.7 with any hyperplane corresponding to an ample divisor. The thick lines denote (parts of) walls lying in the $K$-negative part of the space $N_1(\mathbb{P}(TS))$ and the shaded part is always in the interior (but there might be more). The picture does not contain $\sigma_*[B_0]$ as its intersection with the anticanonical divisor can be positive (case $g = 0$), trivial (for $g = 1$) or negative (for $g \geq 2$). We have $D_{a,b} \cdot \sigma_*[\ell] = 0$ for any $a, b > 0$ by Proposition 5.4.7.

Observe that for the family of divisors $D_{a,b}$, if we increase $b$ relatively to $a$, then we rotate clockwise the hyperplane determined by $D_{a,b}$ along the axis given by $\sigma_*[\ell]$. So in particular there can be no other curve than $\sigma(\ell)$ intersecting $D_{a,1}$ trivially, as we have picked such $a$ that already $D_{a,3}$ is nef.

In the special case when $S = \mathbb{P}^1 \times \mathbb{P}^1$, we see that $-K_{\mathbb{P}(T(\mathbb{P}^1 \times \mathbb{P}^1))}$ is nef, as both classes of effective curves coming from $S$ give a ruling, so they intersect the anticanonical trivially.

To prove that a nef divisor is big, it is enough to show that the top self-intersection is positive. Observe that $(\pi^*\ell)^2 = 0$, so we have:

$$D_{a,1}^3 = \xi^3 + 3a \cdot \xi^2 \cdot \pi^*\ell \tag{24}$$

$$D_{a,3}^3 = 27\xi^3 + 27a \cdot \xi^2 \cdot \pi^*\ell. \tag{25}$$

We have calculated these intersection numbers in Lemma 5.4.8, so plug them into Equation (24) and Equation (25):

$$D_{a,1}^3 = 4(1-g) + 6a,$$

$$D_{a,3}^3 = 108(1-g) + 54a.$$

If $g \leq 1$ then we are done, as both numbers are positive, so $D_{a,3}$ and $D_{a,1}$ are big. If the base curve is of general type, then recall that we have demanded in particular that $a \geq 6g - 6$, so that both self-intersections are positive and we again reach the desired conclusion. In particular, the map defined by the linear system of some multiple of $D_{a,1}$ is birational.

To prove that the contraction is a morphism we need to show the semiampleness of its defining divisor. To that end, simply invoke the Shokurov's Theorem [**KM98**, Thm 3.3]: a divisor $D'$ on a variety $Y$ is semiample if it is nef and $c \cdot D' - K_Y$ is nef and big for some $c > 0$. We have just verified that in our case these assumptions hold (and we can even take

$c = 1$), so we have shown that the map defined by the linear system of some multiple of $D_{a,1}$ is a crepant and birational morphism, contracting precisely curves in the class $\sigma_*[\ell]$.

Using the same argument we can also show that in the special case $S = \mathbb{P}^1 \times \mathbb{P}^1$, the map given by the linear system of some multiple of the anticanonical divisor is also a morphism that is clearly crepant. But in this particular case $S$ admits two rulings (denote them by $\ell_1$ and $\ell_2$), so there are two surjections from $T(\mathbb{P}^1 \times \mathbb{P}^1)$, and $\pi$ admits two sections that we denote by $\sigma_1$ and $\sigma_2$. Again, from Proposition 5.4.7 it is clear that we have $\sigma_2(\ell_2) \cdot K_{\mathbb{P}(T(\mathbb{P}^1 \times \mathbb{P}^1))} = 0$, so the morphism defined by the linear system of the anticanonical divisor contracts both sections along the chosen rulings.

□

We are ready to finish proofs of both main theorems, starting from Theorem 5.4.4. To do it quickly, we will use the classical notion of a Campana-Peternell manifold. A projective manifold $M$ is Campana-Peternell (CP) if it has nef tangent bundle (i.e. such that $\mathcal{O}_{\mathbb{P}(TM)}(1)$ is nef). Conjecturally, in the Fano case CP manifolds are rational homogeneous spaces, a characterization extending the one proved by Mori for the projective space. Moreover, all CP surfaces were classified in [**CP91**, Thm 3.1]. In particular, for a ruled surface $S$ over $B$, $\mathcal{O}_{\mathbb{P}(TS)}(1)$ is nef if and only if $S = \mathbb{P}^1 \times \mathbb{P}^1$ or $B$ is an elliptic curve and the vector bundle $\mathcal{E}$ is semistable.

Proof of Theorem 5.4.4. Let $X$ be a contact Fano variety of dimension 3 with a nonempty singular locus. By Proposition 5.4.6 it admits a crepant resolution of singularities by $\mathbb{P}(TS)$ for some $S$. The anticanonical divisor $-K_X$ is ample by definition, so $-K_{\mathbb{P}(TS)} = f^*(-K_X)$ is nef and moreover we may take $-K_{\mathbb{P}(TS)}$ to be the supporting divisor for the contraction $f$, as it is the pullback of an ample divisor from the target. The nefness of $\mathcal{O}(-K_{\mathbb{P}(TS)}) = \mathcal{O}_{\mathbb{P}(TS)}(2)$ implies that $S$ is Campana-Peternell and in particular $B$ is either a projective line or an elliptic curve. By Lemma 5.4.8, the anticanonical linear system gives a birational map only if the genus $g(B) = 0$, so we conclude that $S = \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, as we have already observed in Proposition 5.4.10, the linear system $| - b \cdot K_{\mathbb{P}(TS)}|$ for $b >> 0$ does indeed define a birational morphism that contracts two sections of $\pi$ along their chosen rulings.

To see that $\rho(X) = 1$ it is enough to observe that the Mori cone of $\mathbb{P}(T(\mathbb{P}^1 \times \mathbb{P}^1))$ is spanned by three rays, namely $[C_\pi]$, $[\sigma_1(\ell_1)]$ and $[\sigma_2(\ell_2)]$ and the two latter are contracted by $f$.

The prove the last claim simply observe that the variety described in Example 5.3.27 is Fano for every possible dimension (it can be done either by a direct toric computation or invoking adequate result from the theory of nilpotent orbits) and has a nonempty singular locus (we have already described it). □

Proof of Theorem 5.4.3. Let $X$ be a projective singular contact variety of dimension 3. By Proposition 5.4.6 it has a crepant resolution of singularities of the form $\mathbb{P}(TS)$ and the resolution morphism contracts the class of a rational curve $C$ such that $\pi_*[C] = [\ell]$. If $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $\rho(X) = 1$, then $X$ is Fano, so we may exclude this case from our current reasoning, as it was discussed in Theorem 5.4.4. In particular, it means that $C$ distinguishes one ruling if $S$ admits two, that in turn determines the section $\sigma\colon S \to \mathbb{P}(TS)$. Then Proposition 5.4.7 shows that we can identify the ray spanned by $[C]$ with the one spanned by $\sigma_*[\ell]$, as both lie in the intersection of the hyperplanes $K_{\mathbb{P}(TS)} = 0$ and $\pi^*\ell = 0$. In Proposition 5.4.10 we have proven that the birational morphism contracting $\sigma_*[\ell]$ is given by the linear system of some positive multiple of $D_{a,1} = a \cdot \pi^*\ell + \xi$ for some integer $a > 0$.

Now we want to prove that for every $\mathbb{P}(TS)$ we can contract the section $\sigma$ and obtain a contact variety. To that end, let us denote by $\phi$ the morphism given by $|b \cdot D_{a,1}|$ for $b >> 0$ and by $Y$ the target variety. We need to show that there exists a commutative square of contractions, $Y$ is a locally trivial bundle of cones with $\rho(Y) = 2$ and that it admits a contact structure. First, let us observe that there are two distinct classes of curves on $Y$, whose representatives are $\phi(C_\pi)$ and $\phi(\sigma(B_0))$, so in particular $\rho(Y) = 2$. The morphism $\phi$ is crepant, so we have $K_Y \cdot \phi_*[C_\pi] = \phi^* K_Y \cdot [C_\pi] = K_{\mathbb{P}(TS)} \cdot [C_\pi] < 0$ and we conclude that the extremal ray $\phi_*[C_\pi]$ gives a Mori contraction. It is necessarily of fiber type, as so is $[C_\pi]$ in $\mathbb{P}(TS)$. We denote it by $\pi'$. It is clear that $p \circ \pi = \pi' \circ \phi$, as both compositions are elementary contractions that differ only in the order.

$S$ is a locally trivial $\mathbb{P}^1$-bundle over $B$, so we may consider $\mathbb{P}(TS)$ as a locally trivial bundle over $B$ with fibers $\mathcal{H}_2$. Observe that $\sigma(\ell) \subset \mathcal{H}_2$ is precisely the minimal section of $\mathcal{H}_2$. If not, then the class of the minimal section of $\mathcal{H}_2$ would be $\sigma_*[\ell] - m \cdot [C_\pi]$ for $m > 0$, but the existence of a curve in such class contradicts the nefness of $D_{a,1}$ for any $a > 0$. Consequently, the morphism $\phi$ restricted to any fiber of $\mathbb{P}(TS) \to B$ is the contraction onto a cone $\mathcal{C}_2$, so we conclude that $Y$ is a $\mathcal{C}_2$-bundle over $B$.

To verify that the resulting variety $Y$ is a contact variety, we need to check that there exists a globally defined contact line bundle $L$ and that the singularities are rational. For the first claim simply observe that $\mathcal{O}_{\mathbb{P}(TS)}(1)$ is trivial on any contracted curve $\sigma(\ell)$, as we have $\sigma(\ell) \cdot \mathcal{O}_{\mathbb{P}(TS)}(1) = 0$ by Proposition 5.4.7 and $\sigma(\ell)$ is rational. Consequently, local trivialization of the sheaf $\mathcal{O}_{\mathbb{P}(TS)}(1)$ induces a local trivialization of the sheaf $\pi_* \mathcal{O}_{\mathbb{P}(TS)}(1)$. Therefore, the latter is a line bundle that is the contact line bundle on $Y$.

Concerning the class of singularities, consider the situation fiberwise. Over every $b \in B$ we have $\phi_{|\pi^{-1}(b)} \colon \mathcal{H}_2 \to \mathcal{C}_2$ and this map is a resolution of singularities of a surface $\mathcal{C}_2$. The exceptional divisor $E$ of $\phi_{|\pi^{-1}(b)}$ is a rational curve, so in particular $H^1(E, \mathcal{O}_E) = 0$. By **[Rei97**, Cor. 4.9] it implies that $R^1(\phi_{|\mathcal{S}})_* \mathcal{O}_{\mathcal{S}} = 0$, so the singularity is rational and we are done by the local triviality of $Y$. Consequently, $Y$ is a projective singular contact threefold and $\phi$ is its crepant resolution of singularities, so using the notation from the formulation of the theorem, we have $Y = X$ and $\phi = f$. □

We have therefore settled the situation in dimension 3. Unfortunately, to give classification in higher dimensions, one needs more refined arguments: in higher dimensions a resolution of singularities may not produce a projective contact manifold. In the particular case when $\dim(X) = 5$ the terminalization produces a contact variety whose singular locus consists of disjoint, smooth curves by Theorem 5.3.8 and resolving them destroys the contact structure as illustrated by Example 5.3.28.

Interestingly enough, in dimension 3 smooth examples are more frequent than singular ones: every smooth projective surface $\Sigma$ produces a projective contact manifold of the form $\mathbb{P}(T\Sigma)$, while as we have just seen, singular contact threefolds essentially correspond to ruled surfaces. On the other hand, just as in the smooth case, the only Fano example comes from the projectivized orbit closure. Nevertheless, the author does not dare to hypothesize that in higher dimensions such singular analogue of LeBrun-Salamon conjecture holds.

Finally, as a byproduct of our reasoning, we are able to construct a variety whose singularities are contact in the sense of **[CF02**], but that does not satisfy Definition 5.3.1:

EXAMPLE 5.4.11. [Threefold *à la* Campana-Flenner] This time let $S$ be the trivial ruled surface over an elliptic curve $E$ given by the rank 2 vector bundle $\mathcal{O}_E \oplus \mathcal{O}_E$, that in particular is semistable and has invariant $e = 0$. Any section of the Mori cone of $\mathbb{P}(TS)$ is triangular with two rays lying on the hyperplane $K_{\mathbb{P}(TS)} = 0$. As we have just discussed, the contraction of the ray $\sigma_*[\ell]$ (recall that $\sigma$ by Propostion 5.4.7 corresponds to the surjection $TS \to TE$) gives us a singular contact threefold. Instead of that, let us contract the other crepant ray, i.e. given by $\sigma_*[E_0]$, where $E_0 = E \times \{p\}$ for any $p \in \mathbb{P}^1 = \ell$. By a reasoning analogous to the one conducted in this section one can show that the divisor $D \coloneqq \pi^* E_0 + \pi^* \ell - K_{\mathbb{P}(TS)}$ is big, nef and semiample, so the linear system of some positive multiple of it gives a crepant and birational morphism that will be denoted by $h$. This morphism contracts $\sigma_*[E_0]$, as it is the only class intersected trivially by $D$.

We can construct (via the pushforward) the contact structure on the resulting variety $Y$ from the one on $\mathbb{P}(TS)$, however there is one subtlety. Namely, for a contact form to be defined around a singular point $x$, we need the contact bundle to be trivial on some neighbourhood of $x$. In our case, we demand $\mathcal{O}_{\mathbb{P}(TS)}(1)$ (the contact bundle on the resolution) to be trivial on $\sigma(E_0)$. By the projection formula and the definition of $\mathcal{O}(1)$, this is equivalent to the triviality of pullback of $TE$ to $E_0$, but it clearly holds as $E$ is elliptic. Consequently, $Y$ has contact singularities in the sense of Campana and Flenner [**CF02**].

However, this variety does not satisfy Definition 5.3.1, as the singularities are not rational. It can be checked directly: $Y$ has a resolution of singularities $h\colon \mathbb{P}(TS) \to Y$. If we restrict ourselves to $\mathcal{S} := \mathbb{P}(TS_{|E_0})$, then $h$ becomes a contraction from a ruled surface $\mathcal{S}$ over $E$ onto a cone $\mathcal{C}$ (over $E$). In particular, the exceptional divisor is isomorphic to $E$, so it has $h^1(E, \mathcal{O}_E) = 1$. The hyperplane exact sequence for the exceptional divisor in $\mathcal{S}$ gives a surjective homomorphism $R^1(h_*)_{|\mathcal{S}}\mathcal{O}_{\mathcal{S}} \to H^1(E, \mathcal{O}_E)$, as shown in [**Rei97**, Section 4.8], so in particular $R^1 h_* \mathcal{O}_{\mathbb{P}(TS)} \neq 0$ and the singularities are not rational.

Finally, let us note another peculiarity of this example. Namely, even though the singularities are not rational, the contact form clearly extends to the resolution, a property that in light of known results on extension of differential forms should not be expected.

## 5.5. Some open problems

We end Chapter 5 by listing a few open problems that can serve as a basis for the future work on contact varieties.

- **Give an example of a singular Fano contact variety that is not the closure of a projectivized nilpotent orbit.** As we have just seen, there are no such examples in dimension 3 and our quotient examples can be shown to be isomorphic to orbit closures, therefore this interesting question remains open.
- **Relation between the group of automorphisms of a contact variety $X$ and $H^0(X, L)$.** In the smooth case we have an isomorphism between the algebra of automorphisms of $X$ preserving the contact structure and the cohomology group $H^0(X, L)$, discussed in Lemma 4.4.1. We do not have yet an analogous result for singular contact varieties. Moreover, as we have seen in Section 5.3.3 the situation in the singular case is more subtle, and in many situations we need to take into account not just the action on the contact distribution, but also the contact form and the linearization at points with nontrivial stabilizers, so it is possible that the sought relation is more complicated.
- **Parametrizing different contact structures on a given variety.** As we should not expect that the contact structure is unique, there is a natural question on enumerating different contact structures on a given variety.
- **Examples with $L \cdot C > 1$.** In all of the examples that we have presented the intersection of the contact line bundle with the minimal rational curve giving a Mori contraction is equal to 1, however as we have discussed in Section 5.3.5, larger intersection numbers are allowed by the Cone Theorem 2.2.2. There is a closely related question whether $L$ can be divisible in the Picard group.
- **Application of linear systems in classification in higher dimensions.** As we have discussed in Chapter 3, we have sufficiently strong results to estimate the dimension of the fundamental (or anticanonical) linear system for rational Gorenstein-Fano varieties of dimension 5, so in particular for singular contact Fano fivefolds. The method that we used in the study of threefolds cannot be

applied as easily in higher dimensions, therefore the additional information given by estimation $h^0(X, L)$ could turn out to be helpful in the Fano case.

- **Better undestanding of a terminalization.** So far, we have only used the existence of a terminalization as a tool. In the setting of nilpotent orbit closures, recent work by Namikawa has shown that there can exist multiple nonisomorphic terminalizations and related them to chambers in the movable cone. Therefore it would be worthwhile to pursuit similar ideas in the singular contact setting.
- **Links between distinct families of contact manifolds.** By the means of Example 5.3.27 we have shown that a particular quotient of $\mathbb{P}^{2n+1}$ results in a contact variety whose (carefully chosen) crepant resolution is $\mathbb{P}(T(\mathbb{P}^1 \times ... \times \mathbb{P}^1))$. It would be interesting to provide some other links between distinct families of contact manifolds or show that this was an exceptional case.

CHAPTER 6

# The geometry of Monge-Ampère equations

## 6.1. Introduction

In this chapter we will see how contact structures and other algebro-geometric objects arise naturally in the study of partial differential equations (PDEs). The geometric theory of PDEs is a rich and multifaceted subject (see e.g. [**KLR07**] for an introduction), so our presentation is focused on showing selected statements that were obtained in a collaboration between Gutt, Manno, Moreno and the author [**GMMŚ21**]. In particular, in Section 6.2 we begin by introducing the setting of jet spaces $J^k$ and their compactifications for $k = 1, 2$ that is useful for geometric theory of PDEs and in particular allows us to consider PDEs as hypersurfaces in those spaces. Then we restrict our attention to a particular class of PDEs, that is 3-dimensional symplectic Monge-Ampère equations in complex setting. They admit a remarkably simple geometric description as hyperplane sections of the Lagrangian Grassmannian $\mathrm{LGr}(3, F)$ for a 6-dimensional symplectic vector space $F$. For that reason, in Section 6.3.1 we discuss classical results concerning the minimal projective embedding of $\mathrm{LGr}(3, F)$ along with the $\mathrm{Sp}(F)$-orbit structure of the ambient projective space. Its dual space – whose elements are projective classes of 3-forms on $F$ – parametrizing hyperplane sections of $\mathrm{LGr}(3, F)$ (and consequently Monge-Ampère equations) has an analogous structure and different orbits give non-isomorphic hyperplane sections (resp. different classes of equations). Finally, in Section 6.3.2 we are able to present aforementioned results from [**GMMŚ21**]. We discuss how one can identify the Hitchin's moment map for 3-forms [**Hit00**, Section 3] with the invariant quadratic form associated to a Monge-Ampère equation by Kushner, Lychagin and Rubtsov in [**KLR07**, Section 8.1]. We moreover show that this common invariant determines the characteristic variety of the equation that plays a crucial role in determining the existence of solutions.

As both the description in terms of jet spaces and the notion of the symplectic Monge-Ampère equation are equally valid in the real analytic and complex holomorphic setting, we initially denote the base field by $\mathbb{K}$. Later, when describing the minimal projective embedding of $\mathrm{LGr}(3, F)$ and its hyperplane sections we abandon this ambiguity and focus on the complex projective setting. Thanks to that, we are able to utilize classical descriptions of subvarieties of Lagrangian Grassmannian, the representation theory of complex Lie groups and the Borel-Weil-Bott theorem.

## 6.2. Jet spaces and Grassmannians

For an introduction to matters presented in this section see [**IL16**, Section 1.9] or [**EMMS18**].

DEFINITION 6.2.1. Let $f : U \subset \mathbb{K}^n \to \mathbb{K}$ be a smooth/holomorphic function defined on some Euclidean open domain $U$, which for simplicity we assume to be simply connected. We can define its $k$-jet at point $y$ as its Taylor expansion in $y$ up to order $k$ and denote it by $j^k_y f$. *The jet space* $J^k(U)$ is then the set of $k$-jets of smooth/holomorphic functions defined on $U$.

As we will not consider the change of domain, we will usually omit the dependence on $U$ in the notation. Moreover, we follow the convention that $J^0 = U \times \mathbb{K}$.

In our setting, the space of $k$-jets has a coordinate system, that will be denoted by $(y_1, ..., y_n, u, u_1, ..., u_n, u_{11}, u_{12}, ..., u_{nn}, ..., u_{n...n})$ ($y_i$ correspond to coordinates on $U$, $u$ to the value of the function and $u_{i_1...i_k}$ to the derivative with respect to $y_{i_1}, ..., y_{i_k}$). We have natural morphisms (projections) $\pi_{m,l}\colon J_m \to J_l$ for any $m \geq l$, that give $J_m$ the structure of an affine space bundle over $J_l$.

Now consider a scalar partial differential equation (PDE) of order $k$:

$$\phi\left(y_i, u, \frac{\partial u}{\partial y_i}, ..., \frac{\partial^k u}{\partial y_{i_1}...\partial y_{i_k}}\right) = 0. \tag{26}$$

It can be considered as a subset (hypersurface) $\mathcal{E}$ of $J^k$. From now on, we put $k = 2$, i.e. we will focus on the PDEs of second order.

The space $J^1$ admits a natural contact structure, defined by the 1-form:

$$\vartheta = du - \sum_i u_i dy_i.$$

Observe that every vector of the form $\partial_{u_i}$ or $\partial_{y_i} + u_i\partial_u$ for $i = 1, ..., n$ is killed by $\vartheta$, so the contact distibution $F$ is the span of $\{\partial_{u_1}, ..., \partial_{u_n}, \partial_{y_1} + u_1\partial_u, ..., \partial_{y_n} + u_n\partial_u\}$. Note however, that $J^1$ is not even compact (nor is any other $J^i$), but it can be embedded as an open and dense subset into a fiber bundle over $J^0$ with compact fibers. Namely, a 1-jet of a function determines a hyperplane in $TJ^0$ via the tangent space to its graph, so a point in $\mathbb{P}(TJ^0)$ (recall that we still use Grothendieck's convention, which is highly nonstandard in the setting of PDEs), which is well-known to admit a contact structure agreeing with that of $J^1$ both in the real analytic and holomorphic setting. Observe that it can also be thought of as a (trivial) Grassmannian bundle. Moreover, points that do not lie in the image of the embedding $J^1 \hookrightarrow \mathbb{P}(TJ^0)$ correspond to functions having undefined differential at the projection of this point to $U$. It is also possible to cover $\mathbb{P}(TJ^0)$ by open sets isomorphic to $J^1$ by some change of coordinates among independent and dependent variables. Therefore, from now on we will consider $\mathbb{P}(TJ^0)$ instead of $J^1$ – sometimes in literature it is taken as a definition of the first jet space.

In a similar fashion, one can describe compactifications of fibers of $J^2$ (both over $J^1$ and $\mathbb{P}(TJ^0)$):

Definition 6.2.2. For any $j^2_y f$ let $L_{j^2_y f}$ be the subspace of $TJ^0$ spanned by elements $D^{(2)}_i = \partial_{y_i} + u_i\partial_u + \sum_{i\leq j} u_{ij}\partial_{u_j}$ for $i = 1, ..., n$ (truncated total derivative operators) evaluated at $j^2_y f$. $L_{j^2_y f}$ is an isotropic subspace, as all tangent vectors of the form $D^{(2)}_i$ are annihilated by the contact form, and as it has the maximal possible dimension, it is a Lagrangian subspace. Consequently, the correspondence $j^2_y f \to L_{j^2_y f}$ gives an open embedding of the fiber of $J^2 \to J^1$ at $j^1_y$ into the Lagrangian Grassmannian $\mathrm{LGr}(F_{j^1_y})$.

The embedding that we have defined allows us to consider the bundle of Lagrangian Grassmanians over $\mathbb{P}(TJ^0)$ instead of a trivial vector bundle $J^2 \to J^1$. As in the case of $J^1$, the points not in the image of the embedding can be interpreted in terms of undefined second derivative and a change of coordinates provides a covering. Again, from now on we will focus on this particular compactification of our bundle and in particular consider $\mathcal{E}$ as its subset. Initial conditions for a PDE can also be defined in this geometric picture, using the notion of a prolongation.

Definition 6.2.3. Let us pick a point $j^1_y f \in J^1$ and let $J^2_{j^1_y f}$ be the fiber of the jet bundle $J^2 \to J^1$ over it. For any $l$ dimensional subspace $H_{j^1_y f} \subset F_{j^1_y f}$ ($l \leq n$) we can define a submanifold $H^{(1)}_{j^1_y f}$ of $J^2_{j^1_y f}$ in the following way:

$$H^{(1)}_{j^1_y f} = \{j^2_y f \in J^2_{j^1_y f} \mid L_{j^2_y f} \supset H_{j^1_y f}\}.$$

This submanifold is called *a prolongation* of $H_{j^1_y f}$.

DEFINITION 6.2.4. The *Cauchy datum* for a second order PDE $\mathcal{E} \subset J^2$ is an $(n-1)$-dimensional submanifold $M \subset J^1$ isotropic with respect to the contact distribution $F$. We call it *characteristic* at a point $j^2_y f \in J^2$ if the prolongation $(T_{j^1_y f} M)^{(1)}$ is tangent to $\mathcal{E}_{j^1_y f}$ at that point.

In the theory of PDEs characteristics play a crucial role in the study of the initial-value problems and are responsible for the existence of solutions or the non-existence thereof. In particular, a nowhere characteristic Cauchy datum allows to construct a unique solution via so-called *method of characteristics* [**IL16**, §6.1], [**EMMS18**, §7]. For a more detailed discussion of characteristic Cauchy data, also in the jet formulation, and its relation with singularities of solutions see [**Vit14**].

## 6.3. Symplectic Monge-Ampère equations

In this section we present two important classes of PDEs for which the geometric picture that we have just described is simpler.

DEFINITION 6.3.1. A second order PDE $\phi = 0$ is called *symplectic* if $\phi$ depends only on second derivatives and not on coordinates $y_i$ of $U$, nor on $u$ or its derivatives $u_i$.

For the discussion of examples of symplectic equations and their utility, see e.g. [**FHK09**] (note that authors use the term *dispersionless Hirota type* for what we call a symplectic equation).

REMARK 6.3.2. We note that the notion of symplecticity only makes sense when the Lagrangian Grassmannian bundle is trivial and this is why we have restricted ourselves to only considering simply connected domains $U \subset \mathbb{K}^n$. On the other hand, one can define jet spaces and compactify them also over more complicated subsets of $\mathbb{K}^n$ or even over smooth (holomorphic) manifolds, but at the price of losing global triviality of considered bundles.

DEFINITION 6.3.3. A second order PDE $\phi$ that can be expressed as a linear combination of minors of the Hessian with coefficients being smooth (holomorphic) functions on $J^1$ is called *a Monge-Ampère equation*. It is *symplectic* if the coefficients are constant.

From the geometric point of view, i.e. when we consider the bundle of Lagrangian Grassmanians, being symplectic means that restrictions of $\mathcal{E}$ to different fibers of this bundle are analytically isomorphic, therefore such equation is determined by a subset of a fixed Lagrangian Grassmannian. Moreover, if the equation is Monge-Ampère then this subset is a hyperplane section. This observation motivates the study of the Lagrangian Grassmannian along with its minimal projective embedding and hyperplane sections that will be conducted in the next section.

We should mention that the discussion of the topic of Monge-Ampère equations and their applications can easily reach the size of a long monograph. The definitive reference that uses the geometric approach and presents concrete applications in acoustics, thermal conductivity or fluid dynamics is [**KLR07**]. A more classical perspective along with the discussion of the utilization of Monge-Ampère equations in the problem of prescribed curvature or the optimal transport can be found in [**Fig17**]. We also have the following conjecture:

Conjecture 5 (Ferapontov's conjecture, [**DF10**, §1, Conjecture]). Every non-degenerate, integrable symplectic equation in dimension $n \geq 4$ is Monge-Ampère.

The work of Ferapontov and collaborators on this conjecture [**DF10**], [**FHK09**] has led them to the geometric approach that we present in this chapter.

**6.3.1. The minimal projective embedding of a Lagrangian Grassmannian.** From now on, we work in the complex holomorphic category.

Any Grassmann variety embeds into a projective space via the Plücker embedding (note that we are still following Grothendieck's convention for projectivizations):

$$\mathrm{Grass}(k, V) \to \mathbb{P}(\Lambda^k V^*),$$
$$W = \mathrm{span}\{v_1, ..., v_k\} \mapsto [v_1 \wedge ... \wedge v_k].$$

However, in the case of Lagrangian Grassmannians, there is a smaller subspace containing it: assume that we have a symplectic space $(F, \omega)$ of dimension $2n$ with a standard basis $e_1, ..., e_{2n}$, then $\Lambda^n F$ is a representation of $\mathrm{Sp}(2n)$, which is usually not irreducible. The vector $e_1 \wedge ... \wedge e_n$ is the highest weight vector for an irreducible representation that will be denoted by $F_{\lambda_n}$ [**FH91**, §17.2] and its projective class lies in the image of the Lagrangian Grassmannian $\mathrm{LGr}(n, F)$. We will usually denote this representation by $\Lambda^n_0 F$. If $n = 2, 3$ then this representation coincides with the kernel of the map:

$$\iota_\omega \colon \Lambda^n F \to \Lambda^{n-2} F,$$

that is a contraction of a multivector with the symplectic form. In conclusion, we have an embedding:

$$\mathrm{LGr}(n, F) \hookrightarrow \mathbb{P}(F^*_{\lambda_n}) = \mathbb{P}(\Lambda^n_0 F^*).$$

We have already discussed how we can identify an $n$-dimensional Monge-Ampère equation with a hyperplane section of $\mathrm{LGr}(n, F)$. Now, having an embedding of $\mathrm{LGr}(n, F)$ into $\mathbb{P}(\Lambda^n_0 F^*)$ we can further identify hyperplane sections with elements of $\mathbb{P}(\Lambda^n_0 F)$. Observe that the symplectic form $\omega$ on $F$ gives a $\mathrm{Sp}(2n)$-module isomorphism $\Lambda^k F \simeq \Lambda^k F^*$ for all $k > 0$. In consequence, points $\eta \in \mathbb{P}(\Lambda^n_0 F)$ lying in the same orbit of $\mathrm{Sp}(2n)$ give isomorphic hyperplane sections.

Now we will present a more detailed description of the situation for $n = 3$. First, recall Example 4.2.1 and the notation therein. We may moreover introduce symbols $e_{ijk} = e_i \wedge e_j \wedge e_k$ for $1 \leq i < j < k \leq 6$ forming a basis of $\Lambda^3 F$. The dual basis consists of $x_{ijk} = x_i \wedge x_j \wedge x_k$. Now pick a decomposition of $F$ into a pair of Lagrangian subspaces: $F = V_0 \oplus V_1$, $V_0 = \mathrm{span}(e_1, e_2, e_3)$, $V_1 = \mathrm{span}(e_4, e_5, e_6)$, so that $F^* = V_0^\perp \oplus V_1^\perp$, where $V_0^\perp = \mathrm{span}(x_4, x_5, x_6)$ and $V_1^\perp = \mathrm{span}(x_1, x_2, x_3)$. Consequently, we obtain a decomposition of $\Lambda^3 F^*$:

$$\Lambda^3 F^* = \Lambda V_1^\perp \oplus (\Lambda^2 V_1^\perp \otimes V_0^\perp) \oplus (V_1^\perp \otimes \Lambda^2 V_0^\perp) \oplus \Lambda^3 V_0^\perp. \tag{27}$$

Now, interpreting the basis of $\Lambda^3 F$ as coordinates on $\Lambda^3 F^*$, we can write a 3-form $\eta \in \Lambda^3 F^*$ using the matrices:

$$u^* = e_{123}, \quad X^* = \begin{pmatrix} e_{234} & -e_{134} & e_{124} \\ e_{235} & -e_{135} & e_{125} \\ e_{236} & -e_{136} & e_{126} \end{pmatrix}, \quad Y^* = \begin{pmatrix} e_{156} & -e_{146} & e_{145} \\ e_{256} & -e_{246} & e_{245} \\ e_{356} & -e_{346} & e_{345} \end{pmatrix}, \quad z^* = e_{456}. \tag{28}$$

By a direct computation we can verify that $\eta \in \Lambda^3_0 F^*$ if and only if $X^*$ and $Y^*$ are symmetric. Note that $\Lambda^3 F$ admits a dual description, for which we will use the notation $w = (u, X, Y, z)$.

Theorem 6.3.4 ([**LM01**, Section 5.3], [**IR05**, Thm 2.3.2]). *For a 6 dimensional symplectic vector space $F$, let $\Sigma = \mathrm{LGr}(3, F)$ be the Lagrangian Grassmannian embedded in $\mathbb{P}(\Lambda^3_0 F^*) = \mathbb{P}^{13}$. Then the* $\mathrm{Sp}(6)$ *action on* $\mathbb{P}(\Lambda^3_0 F^*)$ *has four orbits:*

(1) *the open orbit* $\mathcal{O} = \mathbb{P}^{13} \setminus H$*, where* $H$ *is a hypersurface of degree 4 which is isomorphic to the projective dual of* $\Sigma$*,*
(2) $H \setminus \Omega$*, where* $\Omega$ *is the singular locus of* $H$ *that has dimension* 9 *and degree* 21*,*
(3) $\Omega \setminus \Sigma$*, where* $\Sigma$ *is the Lagrangian Grassmannian,*
(4) $\Sigma$*, which is the only closed orbit.*

*Moreover,* $H$ *is described by the polynomial equation*

$$f(w) = (uz - \operatorname{tr}(XY))^2 + 4u \det Y + 4z \det X - 4\sum_{i,j} \det(X_{ij}) \det(Y_{ij}), \tag{29}$$

*where* $X_{ij}$ *(resp.* $Y_{ij}$*) is the matrix* $X$ *(resp.* $Y$*) with crossed out* $i$*-th row and* $j$*-th column. Consequently,* $\Omega$ *is defined by the Jacobian ideal of* $f$*.*

The dual projective space $\mathbb{P}(\Lambda^3_0 F)$ admits an analogous description coming from the $\mathrm{Sp}(F)$-equivariant isomorphism. We will use the notation $H^\vee, \Omega^\vee, \Sigma^\vee$ for the corresponding closures of orbits.

THEOREM 6.3.5 ([**LM01**, Prop. 8.2], [**IR05**, Prop. 2.5.1]). *Let* $\eta \in \mathbb{P}(\Lambda^3_0 F)$ *and* $\mathcal{E}_\eta = \ker \eta \cap \mathrm{LGr}(3, F)$ *be the corresponding hyperplane section of the Lagrangian Grassmannian.*

(1) *If* $\eta$ *belongs to the open orbit, then* $\mathcal{E}_\eta$ *is nonsingular.*
(2) *If* $\eta \in H^\vee \setminus \Omega^\vee$ *then* $\mathcal{E}_\eta$ *has a unique isolated quadratic singularity.*
(3) *If* $\eta \in \Omega^\vee \setminus \Sigma^\vee$ *then the singular locus of an associated hyperplane section is a smooth quadric surface, isomorphic to a Schubert cycle in* $\mathrm{LGr}(2, 4)$ *and the singularities are quadratic.*
(4) *If* $\eta \in \Sigma^\vee$ *then* $\mathcal{E}_\eta$ *is singular along a projective cone over a Veronese surface.*

REMARK 6.3.6. The 4 orbit closures of $\mathbb{P}(\Lambda^3_0 F)$ correspond to 4 types of Monge-Ampère equations [**GMMŚ21**, §8.2.1], [**FHK09**, Rmk 3]:

| Orbit | equation type |
|---|---|
| open | general |
| $H^\vee$ | linearizable |
| $\Omega^\vee$ | Goursat |
| $\Sigma^\vee$ | parabolic |

In particular, representing a Monge-Ampère equation as a hyperplane section determined by an element $\eta \in \mathbb{P}(\Lambda^3_0 F)$ provides a simple criterion for the linearizability: it is enough to check whether $\eta$ satisfies the polynomial equation defining $H^\vee$.

DEFINITION 6.3.7. Let $\eta \in \Lambda^3_0 F^*$ be a 3-form whose class is associated to some symplectic Monge-Ampère equation in dimension 3. If we denote by $\Theta \in \Lambda^2 F$ the bivector corresponding to the symplectic form $\omega$, then we can define the following $\mathrm{Sp}(F)$-equivariant quadratic form $q_\eta \in S^2 F^*$:

$$q_\eta(v) = -\frac{1}{4}\iota^2_\Theta(\iota_v \eta)^2,$$

which is called the Kushner-Lychagin-Rubtsov (KLR) invariant.

DEFINITION 6.3.8. The zero locus of the homogeneous quadratic polynomial $q_\eta$ in $\mathbb{P}(F^*)$ is known as *the cocharacteristic variety* of the corresponding Monge-Ampère equation in dimension 3.

In the next sections we will see how the objects that we have just introduced can be defined using more geometric language.

**6.3.2. The Hitchin moment map and the KLR invariant.** As before, let $F$ be a 6 dimensional symplectic vector space with a form $\omega$. The algebra $\mathfrak{gl}(F)$ can be regarded as the linear part of the algebra $\mathfrak{X}(F)$ of polynomial vector fields on $F$. In this way, we obtain an embedding:

$$j\colon \mathfrak{sp}(F) \hookrightarrow \mathfrak{X}(F). \tag{30}$$

Then, there exists a unique, $\mathrm{Sp}(F)$-equivariant *moment map* $\mu\colon F \to \mathfrak{sp}(F)^*$ such that $d\langle \mu, X\rangle = \iota_{j(X)}(\omega)$ for every $X \in \mathfrak{sp}(F)$. Moreover, there exists a commutative diagram:

$$\begin{tikzcd} S^2(F) \arrow[r, "\phi"] & \mathfrak{sp}(F)^* \\ F \arrow[u, "v_2"] \arrow[ur, "\mu"] & \end{tikzcd}$$

where the map $\phi$ is an isomorphism of representations of $\mathrm{Sp}(F)$ and $v_2$ is the natural (Veronese) embedding. Observe that it allows us to compute $\mu$ as the composition $\phi \circ v_2$. Similarly, we can construct the following diagram

$$\begin{tikzcd} S^2(\Lambda^3_0(F^*)) \arrow[r, "pr"] & S^2(F) \\ \Lambda^3_0(F^*). \arrow[u, "v_2"] \arrow[ur, "\mu"] & \end{tikzcd}$$

This time, the horizontal arrow is no longer an isomorphism, but a $\mathrm{Sp}(F)$-equivariant projection $S^2(\Lambda^3_0(F^*)) = W_{(2,0,0)} \oplus W_{(0,0,2)} \to W_{(2,0,0)} = S^2(F)$ (recall that $S^2F \simeq S^2F^*$ as representations of $\mathrm{Sp}(F)$) and the map denoted (by an abuse of notation) $\mu$ is known as the Hitchin moment map [**Hit00**, Section 3]. Now, we can take projectivizations of considered vector spaces to obtain a rational map:

$$\widetilde{\mu}\colon \mathbb{P}(\Lambda^3_0(F)) \dashrightarrow \mathbb{P}(\mathfrak{sp}(F)). \tag{31}$$

Our goal is to determine the image of this map and to this end observe that it is $\mathrm{Sp}(F)$-equivariant, as $\mu$ is a composition of maps with this property. Consequently, it is enough to pick a representative for each orbit of $\mathbb{P}(\Lambda^3_0F)$ and compute its image via $\widetilde{\mu}$.

PROPOSITION 6.3.9 ([**GMMŚ21**, Prop. 8.1-8.4]). *Let $\widetilde{\mu}$ be the map defined above.*

(1) *The locus of indeterminacy of the map $\widetilde{\mu}$ is precisely $\Sigma^\vee$.*
(2) *The image of $\Omega^\vee$ is $\mathbb{P}(\mathcal{O}_{[2,1^4]})$, the projectivization of the minimal nilpotent orbit of $\mathfrak{sp}(F)$. The fiber over it is equal to $\mathbb{P}^4 \setminus \mathrm{LGr}(2,V)$ for a particular $V \subset F^*$, a symplectic subspace of dimension 4. Moreover, this embedding induces an embedding of Lagrangian Grassmannians: $\mathrm{LGr}(2,V) \subset \Omega^\vee$.*
(3) *The image of $H^\vee$ is $\mathbb{P}(\mathcal{O}_{[2^3]})$, the projectivization of the 12 dimensional nilpotent orbit of $\mathfrak{sp}(F)$ and the fiber over it is equal to $\mathbb{C}$ that can be compactified to $\mathbb{P}^1$ by a point lying in $\Sigma^\vee$.*
(4) *The image of the open orbit consists of a 1 parameter family of semisimple orbits containing $\mathrm{diag}(a,a,a,-a,-a,-a)$ for each $a \in \mathbb{C}\setminus\{0\}$. The fiber over it is $\mathbb{C}^*$ that can be compactified to $\mathbb{P}^1$ by two points from $\Sigma^\vee$.*

PROOF. First observe that for any $[y] = \widetilde{\mu}([x])$ the fiber over $[y]$ is acted upon in a transitive way by $\mathrm{Stab}([y]) \subset \mathrm{Sp}(6)$. To determine images of chosen representatives, we will employ computations of weights. In particular, $x_{123}$ corresponds to (the dual of) the Lagrangian plane, so for its projective class we have $[x_{123}] \in \Sigma^\vee$. Its symmetric square, which we will denote by $x_{123}^2$ for brevity, has weight $2h_1 + 2h_2 + 2h_3$, so it is the highest weight vector for $W_{(0,0,2)}$. Consequently, it gets mapped to 0 by the affine projection $S^2(\Lambda^3_0(F^*)) \to W_{(2,0,0)}$, and the map $\widetilde{\mu}$ is not defined on this orbit.

For the second statement we can pick $[x_{163}+x_{125}]$ as a representative of $\Omega^\vee$. Its symmetric square has weight $2h_1$ and the weight space $S^2(\Lambda_0^3(F^*))_{2h_1}$ has dimension 3, spanned by the square of the chosen representative along with $x_{123}\odot x_{156}$ and $x_{153}\odot x_{126}$. On the one hand, the highest weight vector of $W_{(2,0,0)}$ is annihilated by all primitive positive root vectors. On the other, images of elements with weights $2h_1+2h_3$ and $2h_1+h_2-h_3$ via negative root vectors belong to $W_{(0,0,2)}$. Consequently, we obtain:

$$
\begin{aligned}
(W_{(2,0,0)})_{2h_1} &= \mathrm{span}((x_{163}+x_{125})^2 - 4x_{123}\odot x_{156} + 4x_{153}\odot x_{126})\\
(W_{(0,0,2)})_{2h_1} &= \mathrm{span}(x_{123}\odot x_{156} + x_{153}\odot x_{126}, (x_{163}+x_{125})^2 + 2x_{126}\odot x_{153}).
\end{aligned}
$$

By computing the coordinates of the square of the chosen representative we deduce that it projects nontrivially onto $(W_{(2,0,0)})_{2h_1}$. It is spanned by $E_{1,4}$, which clearly is a nilpotent matrix having a unique nonzero Jordan block of size 2, so $\widetilde{\mu}(\Omega^\vee\setminus\Sigma^\vee) = \mathbb{P}(\mathcal{O}_{[2,1^4]})$. To determine the fiber, observe that $\mathrm{Stab}([E_{1,4}]) = \mathrm{Sp}(V)$ for $V = \mathrm{span}(x_2, x_3, x_5, x_6)$, therefore it is enough to determine the $\mathrm{Sp}(V)$-orbit of $[x_{63}+x_{25}]$ in $\mathbb{P}(\Lambda_0^2(V)) \simeq \mathbb{P}^4$. There are two such orbits: the closed one which is the Lagrangian Grassmannian realised as a hypersurface and its complement. As $x_{63}+x_{25}$ is not a decomposable 2-form, we have $[x_{63}+x_{25}] \notin \mathrm{LGr}(2,V)$. Moreover, the wedge product of a 2-form representing Lagrangian subspace in $V$ with $x_1$ maps it to a Lagrangian subspace of $F$.

For the third claim, consider a line $[x_{423}+x_{126}+x_{153}+k\cdot x_{123}]$ and observe that for every $k\in\mathbb{C}$ it belongs to $H^\vee\setminus\Omega^\vee$, however the point in infinity lies in $\Sigma^\vee$. Symmetric squares of elements with $k\in\mathbb{C}$ have components with weights $\pm 2h_1 \pm 2h_2 \pm 2h_3$ (pure squares), $2h_i+2h_j$ (mixed squares with $k\cdot x_{123}$) and those with $2h_i$ (other mixed squares). First two types get killed by the projection and for the mixed squares without $k$ we can check as before that they project nontrivially onto $(W_{(2,0,0)})_{2h_1}$. Consequently, the whole affine line $[x_{423}+x_{126}+x_{153}+k\cdot x_{123}]$ gets mapped onto $[E_{1,4}+E_{2,5}+E_{3,6}]$, which is an element of $\mathbb{P}(\mathcal{O}_{[2^3]})$. To see that the fiber does not contain any other component besides the described line, consider the stabilizer subgroup:

$$
\mathrm{Stab}([E_{1,4}+E_{2,5}+E_{3,6}]) = \left\{ \begin{pmatrix} A & B \\ 0 & a\cdot A \end{pmatrix} | a\in\mathbb{C}^*, a\cdot A\cdot A^T = I_3, A^T\cdot B = B^T\cdot A \right\}.
$$

The presence of the parameter $a$ allows the determinant of $A$ to take any nonzero complex value, so the stabilizer group is connected and consequently the fiber over $[E_{1,4}+E_{2,5}+E_{3,6}]$ does not contain other components.

Finally, for the open orbit we can choose a family of representatives given by $[x_{123} + k\cdot x_{456}]$ for $k\in\mathbb{C}^*$. Observe that both $[x_{123}]$ and $[x_{456}]$ belong to $\Sigma^\vee$. For the symmetric square of any representative, only the mixed term component $2k\cdot x_{123}\odot x_{456}$ can project nontrivially onto $W_{(2,0,0)}$ and then it lives in the Cartan algebra, i.e. it gets mapped to some element of the form $ah_1+bh_2+ch_3$. To see that the projection is indeed nonzero, compute how weight vectors $X_{h_i}$ act upon $2k\cdot x_{123}\odot x_{456}$: in all cases $X_{h_i}\cdot(2k\cdot x_{123}\odot x_{456})$ has a component in $W_{(2,0,0)}$. Moreover, we have $X_{h_i-h_j}\cdot X_{2h_j}2k\cdot x_{123}\odot x_{456} = X_{h_j-h_i}\cdot X_{2h_i}2k\cdot x_{123}\odot x_{456}$, so $a=b=c$ and we obtain that $\widetilde{\mu}([x_{123}+k\cdot x_{456}]) = [\mathrm{diag}(1,1,1,-1,-1,-1)]$. To determine whether the fiber contains any component other than $\mathbb{C}^*$, again we consider the stabilizer:

$$
\mathrm{Stab}([h_1+h_2+h_3]) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(6) \mid \begin{pmatrix} A & -B \\ C & -D \end{pmatrix} = \begin{pmatrix} aA & aB \\ -aC & -aD \end{pmatrix}, a\in\mathbb{C}^* \right\}.
$$

This time it has two connected components, so we cannot argue as before, however we can immediately see that for any $g$ in the stabilizer we have $g\cdot[x_{123}+x_{456}] = [(\det A + \det B)x_{123} + (\det C + \det D)x_{456}]$, so there are no other components in the fiber. $\square$

Proposition 6.3.10 ([**GMMŚ21**, Thm 8.1]). *The map* $\widetilde{\mu}\colon \mathbb{P}(\Lambda^3_0(F^*)) \dashrightarrow \mathbb{P}(\mathfrak{sp}(F))$ *is equal to the map coming from the projectivization of the KLR invariant* $\mathbb{P}(\Lambda^3_0(F^*)) \dashrightarrow \mathbb{P}(S^2F)$ *composed with the isomorphism* $\mathbb{P}(\mathfrak{sp}(F)) \simeq \mathbb{P}(S^2F)$.

Proof. In light of $\mathrm{Sp}(F)$-equivariance of both maps it follows from direct computation of KLR invariant on chosen representatives of 4 orbits. For details, see cited reference. □

**6.3.3. The cocharacteristic variety.** As before, we let $\Sigma$ be the Lagrangian Grassmannian $\mathrm{LGr}(3,F)$ embedded via the Plücker map in $\mathbb{P}(\Lambda^3_0F^*)$, i.e. if $V$ is a Lagrangian subspace spanned by $v_1, v_2, v_3$ then its image is $[v_1 \wedge v_2 \wedge v_3]$. Now, for an element $h \in S^2(V^*) \subset \mathrm{Hom}(V,F)$ we can define a curve:

$$\gamma_h(t) = [(v_1 + t\cdot h(v_1)) \wedge (v_2 + t\cdot h(v_2)) \wedge (v_3 + t\cdot h(v_3))], t \in \mathbb{C}.$$

We have $\gamma_h(0) = V$ and the fact that $h \in S^2(V^*)$ implies that $\gamma_h(t) \in \mathrm{LGr}(3,F)$. By varying $h$ we obtain a family of curves $\gamma_h(t)$ that can be used to define a map:

$$\begin{aligned} \tau\colon S^2(V^*) \to& T_V\Sigma \\ \tau(h) \mapsto& \dot{\gamma_h}(0). \end{aligned}$$

Observe that the contraction

$$\begin{aligned} S^2(V^*) \to& \Lambda^3 F \\ h \mapsto& \iota_h(v_1\wedge v_2\wedge v_3) = h(v_1)\wedge v_2\wedge v_3 + v_1\wedge h(v_2)\wedge v_3 + v_1\wedge v_2\wedge h(v_3) \end{aligned}$$

is a monomorphism, so consequently $\tau$ is injective. As both vector spaces have dimension 6, $\tau$ is an isomorphism. Now let $V$ be a smooth point of a Monge-Ampère equation understood as a hyperplane section $\mathcal{E}_\eta$ of $\mathrm{LGr}(3,F)$ determined by $\eta \in \Lambda^3_0F^*$. The curve $\gamma_h$ is tangent to $\mathcal{E}_\eta$ if and only if $\eta(\iota_h(v_1\wedge v_2\wedge v_3)) = 0$. We can express it in terms of a linear map:

$$\begin{aligned} \overline{\eta}_V\colon T_V\Sigma = S^2V^* &\to \Lambda^3V^* \\ h &\mapsto \eta(\iota_h(\cdot)). \end{aligned}$$

Namely, we have $T_V\mathcal{E}_\eta = \ker(\overline{\eta}_V)$.
If we denote the tautological bundle over $\mathrm{LGr}(3,F)$ by $\mathcal{V}$, then this pointwise reasoning results in a global isomorphism $T\Sigma \simeq S^2(\mathcal{V}^*)$. Moreover, we can define a section $\overline{\eta} \in H^0((\mathcal{E}_\eta)_{sm}, S^2(\mathcal{V}\otimes\Lambda^3(\mathcal{V}^*)))$ such that its value at $V$ is $\overline{\eta}_V$. We are ready to define the variety parametrizing characteristics of the equation, that were introduced in Definition 6.2.4.

Definition 6.3.11. The characteristic variety of a Monge-Ampère equation $\mathcal{E}_\eta$ is the subvariety $\sigma \subset \mathbb{P}(\mathcal{V}_{|(\mathcal{E}_\eta)_{sm}})$ defined by the equation $\overline{\eta} = 0$, where by an abuse of notation we regard $\overline{\eta}$ as a map from $\mathbb{P}(\mathcal{V}_{|(\mathcal{E}_\eta)_{sm}})$ to $\mathbb{P}(((\mathcal{V}\otimes\Lambda^3(\mathcal{V}^*))_{|\mathcal{E}_\eta})^*)$. $\sigma^\vee$, the projective dual of $\sigma$, is called *the cocharacteristic variety.* It is a subvariety of $\mathbb{P}(\mathcal{V}^*)$ and the natural embedding of any Lagrangian subspace $V$ into $F$ allows us to consider $\sigma^\vee$ as a subvariety in $\mathbb{P}(F^*)$.

One can show that $\Lambda^2(\overline{\eta})$ is in fact a section of $S^2(\mathcal{V}^*)$, and when considered as a map $\mathcal{V}\to\mathcal{V}^*$ its projectivized zero locus is precisely $\sigma^\vee \subset \mathbb{P}(\mathcal{V}^*)$. With such perspective, it is possible to see that the cocharacteristic variety is actually defined by the moment map. Namely, we have:

Theorem 6.3.12 ([**GMMŚ21**, Section 9.5, Prop. 11.1]). *There exists a commutative,* $\mathrm{Sp}(F)$*-equivariant diagram:*

$$\begin{tikzcd}
\mathbb{P}(\Lambda^3_0 F) \arrow[d, dashed, "\widetilde{\mu}"] \arrow[dr, dashed, "\Lambda^2(\cdot)"] & \\
\mathbb{P}(S^2(F)) \arrow[r, "s"] & \mathbb{P}(\Gamma((\mathcal{E}_\eta)_{sm}, S^2(\mathcal{V}))).
\end{tikzcd}$$

*Moreover, the map $s$ is an isomorphism induced by the morphism of sheaves $S^2(\mathcal{O} \otimes F^*)_{|\mathcal{E}_\eta} \xrightarrow{p} S^2(\mathcal{V}^*)_{|\mathcal{E}_\eta}$ on $\mathcal{E}_\eta$. The spaces of sections of both sheaves are isomorphic, irreducible* $\mathrm{Sp}(F)$ *representations with the fundamental weight* $(2, 0, 0)$*, i.e.* $S^2(F^*) \simeq S^2(F)$.

Before presenting the proof, we will recall the crucial tool – the Borel-Weil-Bott theorem.

THEOREM 6.3.13 ([**OR06**, Thm 4.1]). *Assume that $Y = G/P$ is a rational homogeneous variety for a complex semisimple group $G$ and a parabolic subgroup $P$ determined by a simple root $\alpha_k$, i.e. $P = P(\alpha_k)$. Let $D = \{\sum a_i\lambda_i \mid a_i \geq 0, \lambda_i$ is a fundamental weight$\}$ be the fundamental Weyl chamber and put $\delta = \sum \lambda_i$. Denote by $D_k$ the fundamental Weyl chamber of the reductive part of $P$, $D_k = \{\sum a_i\lambda_i \mid a_i \geq 0$ for $i \neq k\}$. For the Weyl group $W$ of $G$ define a subset $W^k = \{w \in W \mid wD \subset D\}$. Recall that for any $w \in W$ there exists an integer $l(w)$ equal to the length of the minimal presentation in terms of reflections generating $W$. Finally, let $E_\lambda$ be a $G$-homogeneous vector bundle on $Y$ associated to an irreducible representation $W_\lambda$ of $P$ with the highest weight $\lambda$. Then:*

(1) *If $\lambda \in D_k$ then there exists a unique $w \in W$ such that $w^{-1} \in W^k$ and $w(\lambda+\delta) \in D$.*
(2) *If $w(\lambda+\delta)$ in the interior of $D$ then for $\nu = w(\lambda+\delta)-\delta$ one has $H^{l(w)}(E_\lambda) = W_\nu$ and other cohomology groups vanish. In the particular case when $\lambda \in D$ one has $H^0(E_\lambda) = W_\lambda$ and $H^i(E_\lambda) = 0$ for $i > 0$.*
(3) *If $w(\lambda + \delta) \in \partial D$ then $H^i(E_\lambda) = 0$ for all $i$ (we say that $E_\lambda$ is cohomologically trivial or immaculate).*

PROOF OF THEOREM 6.3.12. The commutativity of the diagram follows from direct computation on orbit representatives, as every arrow is Sp(6)-invariant. For details see [**GMMŚ21**, Section 9.5]. For the claim on $s$, we will construct a diagram of sheaves with exact rows and columns. We start with the tautological exact sequence for the Lagrangian Grassmannian:

$$0 \to \mathcal{V} \to \mathcal{O} \otimes F \to Q \to 0. \tag{32}$$

Dualize it and take the second symmetric power to obtain:

$$0 \to Q^* \cdot F^* \to S^2(\mathcal{O} \otimes F^*) \to S^2(\mathcal{V}^*) \to 0. \tag{33}$$

The leftmost nontrivial sheaf is usually defined as the cokernel of the inclusion $\Lambda^2 Q^* \hookrightarrow Q^* \otimes F^*$. Combine this with the hyperplane short exact sequence on $\Sigma$ to obtain the following diagram of coherent sheaves on $\Sigma$ with exact rows and columns:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Q^* \cdot F^*(-1) \arrow[r] \arrow[d] & Q^* \cdot F^* \arrow[r] \arrow[d] & (Q^* \cdot F^*)_{|\mathcal{E}_\eta} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^2(\mathcal{O} \otimes F^*)(-1) \arrow[r] \arrow[d] & S^2(\mathcal{O} \otimes F^*) \arrow[r] \arrow[d] & S^2(\mathcal{O} \otimes F^*)_{|\mathcal{E}_\eta} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^2(\mathcal{V}^*)(-1) \arrow[r] \arrow[d] & S^2(\mathcal{V}^*) \arrow[r] \arrow[d] & S^2(\mathcal{V}^*)_{|\mathcal{E}_\eta} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}
$$

Now we will compute the associated long exact sequences in cohomology for each row and column of this diagram. Observe that for $\mathrm{LGr}(3, F)$ we have $\mathcal{V} \simeq Q^*$ and this bundle is $\mathrm{Sp}(F)$-homogeneous, as is $\mathcal{O}_\Sigma(-1)$. Therefore, to compute the cohomology, one can determine weights and employ the Borel-Weil-Bott theorem. In the case of $\mathcal{V}$, the weights are $-h_1, -h_2, -h_3$ (the minus sign comes from the fact that one acts on the fiber of the associated bundle by the inverse). The highest among them is $-h_3$. For the $\mathcal{O}_\Sigma(-1)$ the highest weight is $-h_1 - h_2 - h_3$. Then we can write down highest weights associated with some products, namely for $S^2(\mathcal{V}^*)$ we have $2h_1$, for $S^2(\mathcal{V}^*)(-1)$ we obtain $h_1 - h_2 - h_3$, for $\Lambda^2(Q^*)$ we get $-h_2 - h_3$ and finally $\Lambda^2(Q^*)(-1)$ is associated to $-h_1 - 2h_2 - 2h_3$.

Recall from Example 4.2.1 that the fundamental weights of $\mathrm{Sp}(6)$ are $h_1+h_2+h_3$, $h_1+h_2$ and $h_1$, so in the setting of Theorem 6.3.13 we have the auxiliary weight $\delta = 3h_1+2h_2+h_3$. The fundamental Weyl chamber $D$ is a cone spanned by the fundamental weights and one easily sees that the highest weights for $\mathcal{V}$, $\Lambda^2(Q^*)$, $S^2(\mathcal{V}^*)(-1)$, $S^2(\mathcal{O} \otimes F^*)(-1)$, $Q^*(-1)$ and $\Lambda^2(Q^*)(-1)$ belong to the boundary $\partial D$, therefore these bundles are immaculate by Theorem 6.3.13. From the long exact sequence associated to the short exact sequence 33 twisted by $\mathcal{O}(-1)$ it follows that $(Q^* \cdot F^*)(-1)$ is immaculate. Now consider

$$0 \to \Lambda^2 Q^* \to Q^* \otimes F^* \to Q^* \cdot F^* \to 0,$$

i.e. the sequence defining $Q^* \cdot F^*$. We have already observed that the leftmost term is immaculate, and so is the middle, as a tensor product of an immaculate line bundle with a trivial one, so for the rightmost sheaf we also have $H^i = 0$ for all $i$. Finally, we obtain that $S^2(\mathcal{O} \otimes F^*)(-1)$ and $(Q^* \cdot F^*)_{|\mathcal{E}_\eta}$ are cohomologically trivial by an analogous argument.

Consequently, from the diagram above it follows that for all $i$ we have $H^i(\Sigma, S^2(\mathcal{V}^*)) = H^i(\Sigma, S^2(\mathcal{O}\otimes F^*)) = H^i(\mathcal{E}_\eta, S^2(\mathcal{O}\otimes F^*)) = H^i(\mathcal{E}_\eta, S^2(\mathcal{V}^*))$. To finish the proof it is enough to observe that $S^2(\mathcal{O}\otimes F^*)$ is a trivial vector bundle, so its only nonvanishing cohomology group is $H^0$, and we have $H^0(\Sigma, S^2(\mathcal{O} \otimes F^*)) = S^2(F^*) = W_{(2,0,0)}$. □

The theorem above allows us to determine the (co)characteristic varieties for all 4 classes of complex symplectic Monge-Ampère equations in 3 dimensions via the moment map. In particular, we have the following:

Corollary 6.3.14 ([**GMMŚ21**, Cor. 9.1]). *Let $\eta \in \mathbb{P}(\Lambda^3_0 F)$ be a Monge-Ampère equation. Then:*

(1) *if $\eta$ belongs to the open orbit then $\sigma^\vee$ is an irreducible, nondegenerate, nonsingular quadric,*
(2) *if $\eta \in H^\vee \setminus \Omega^\vee$ then $\sigma^\vee$ is an irreducible and degenerate (rank 3) quadric,*
(3) *if $\eta \in \Omega^\vee \setminus \Sigma^\vee$ then $\sigma^\vee$ is a reducible and degenerate (rank 1) quadric,*
(4) *if $\eta \in \Sigma^\vee$ then $\sigma^\vee = \mathbb{P}(F^*)$, i.e. $\sigma^\vee$ is trivial.*

PROOF. In light of the isomorphism $\mathfrak{sp}(F) \simeq S^2F^*$ and Theorem 6.3.12 it is enough to determine what are the images of chosen representatives from Proposition 6.3.9 in $S^2F^*$. For details see cited reference. $\square$

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