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**Limits of saturated ideals of points with applications to secant
varieties**

PhD dissertation

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Abstract

The main object of study of this thesis is a class of multigraded Hilbert schemes. Given a smooth projective toric variety X with the Cox ring $S[X]$ we consider the Hilbert function of r points on X in general position, i.e. $h_{r,X} : \text{Pic}(X) \rightarrow \mathbb{N}$ given by

$$h_{r,X}([D]) = \min\{\dim_{\mathbb{C}} \Gamma(X, \mathcal{O}_X(D)), r\}.$$

The multigraded Hilbert scheme $\text{Hilb}_{S[X]}^{h_{r,X}}$ associated with $S[X]$ and $h_{r,X}$ has a distinguished irreducible component $\text{Slip}_{r,X}$ which is the closure of the locus of points corresponding to radical ideals that are saturated. The aim of this dissertation is to find necessary or sufficient conditions for a point of $\text{Hilb}_{S[X]}^{h_{r,X}}$ to belong to $\text{Slip}_{r,X}$. This problem is motivated by the border apolarity lemma established by Buczyńska and Buczyński.

Our main focus is on the case $X = \mathbb{P}^n$. We present three necessary conditions for $[I] \in \text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,\mathbb{P}^n}}$ to be in $\text{Slip}_{r,\mathbb{P}^n}$. The first of them is obtained by bounding the degrees of minimal generators of saturated ideals $J \subseteq S[\mathbb{P}^n]$ with $[J] \in \text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,\mathbb{P}^n}}$. The second criterion is based on the computation of the Hilbert polynomial of a power of a radical ideal J with $[J] \in \text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,\mathbb{P}^n}}$ and establishing the bound on the degree from which it agrees with the Hilbert function. The proof of the third necessary condition uses deformation theory and flag multigraded Hilbert schemes. We also present a sufficient condition for $[I] \in \text{Hilb}_{S[\mathbb{P}^2]}^{h_{r,\mathbb{P}^2}}$ to be in $\text{Slip}_{r,\mathbb{P}^2}$.

We consider a morphism with connected fibers $f: X \rightarrow Y$ between smooth projective toric varieties. We obtain a necessary condition for $[I] \in \text{Hilb}_{S[X]}^{h_{r,X}}$ to be in $\text{Slip}_{r,X}$. Namely, we show that there is a natural morphism $\text{Hilb}_{S[X]}^{h_{r,X}} \rightarrow \text{Hilb}_{S[Y]}^{h_{r,Y}}$ and that it maps $\text{Slip}_{r,X}$ onto $\text{Slip}_{r,Y}$. We also prove another necessary condition in the case that X is the product of $k \geq 2$ projective spaces.

We illustrate the criteria with examples. In particular, we describe all ideals which correspond to points of $\text{Slip}_{r,\mathbb{P}^2}$ for all $r \leq 6$. Furthermore, we apply our techniques to obtain some results on wild polynomials.

Keywords: multigraded Hilbert schemes, saturated ideals of points, smooth projective toric varieties, secant varieties, border rank.

AMS MSC 2020 classification: 14C05, 14M25, 14N07.

Streszczenie

Głównym obiektem badań niniejszej rozprawy jest klasa schematów Hilberta z wielogradacją. Dla gładkiej rzutowej rozmaitości torycznej X z pierścieniem Coxa $S[X]$, rozważamy funkcję Hilberta r punktów w położeniu ogólnym na X , tzn. $h_{r,X}: \text{Pic}(X) \rightarrow \mathbb{N}$ zadaną przez

$$h_{r,X}([D]) = \min\{\dim_{\mathbb{C}} \Gamma(X, \mathcal{O}_X(D)), r\}.$$

Schemat Hilberta z wielogradacją $\text{Hilb}_{S[X]}^{h_{r,X}}$ stowarzyszony z $S[X]$ i $h_{r,X}$ ma wyróżnioną składową nieprzywiedlną $\text{Slip}_{r,X}$, która jest domknięciem zbioru punktów odpowiadających ideałom radykalnym i nasyconym. Celem tej rozprawy jest znalezienie kryteriów koniecznych lub wystarczających do tego by punkt $\text{Hilb}_{S[X]}^{h_{r,X}}$ należał do $\text{Slip}_{r,X}$. Motywacja do badania tego problemu pochodzi z lematu o brzegowej abiegunowości udowodnionego przez Buczyńską i Buczyńskiego.

Główny nacisk kładziemy na przypadek $X = \mathbb{P}^n$. Prezentujemy trzy warunki konieczne do tego by punkt $[I] \in \text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,\mathbb{P}^n}}$ należał do $\text{Slip}_{r,\mathbb{P}^n}$. Pierwszy z nich jest uzyskany poprzez ograniczenie stopni minimalnych generatorów ideałów nasyconych $J \subseteq S[\mathbb{P}^n]$ takich, że $[J] \in \text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,\mathbb{P}^n}}$. Drugie kryterium bazuje na obliczeniu wielomianu Hilberta potęgi ideału radykalnego J takiego, że $[J] \in \text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,\mathbb{P}^n}}$ oraz uzyskaniu ograniczenia na stopień od którego zgadza się on z funkcją Hilberta. Dowód trzeciego kryterium wykorzystuje teorię deformacji i flagowe schematy Hilberta z wielogradacją. Prezentujemy również warunek wystarczający do tego aby $[I] \in \text{Hilb}_{S[\mathbb{P}^2]}^{h_{r,\mathbb{P}^2}}$ należał do $\text{Slip}_{r,\mathbb{P}^2}$.

Rozważamy morfizm o spójnych włóknach $f: X \rightarrow Y$ pomiędzy gładkimi rzutowymi rozmaitościami torycznymi. Uzyskujemy warunek konieczny do tego aby $[I] \in \text{Hilb}_{S[X]}^{h_{r,X}}$ należał do $\text{Slip}_{r,X}$. Mianowicie pokazujemy, że istnieje naturalny morfizm $\text{Hilb}_{S[X]}^{h_{r,X}} \rightarrow \text{Hilb}_{S[Y]}^{h_{r,Y}}$, który odwzorowuje $\text{Slip}_{r,X}$ na $\text{Slip}_{r,Y}$. Dowodzimy również innego warunku koniecznego w przypadku gdy X jest produktem $k \geq 2$ przestrzeni rzutowych.

Kryteria ilustrujemy przykładami. W szczególności, opisujemy wszystkie ideały, które odpowiadają punktom $\text{Slip}_{r,\mathbb{P}^2}$ dla wszystkich $r \leq 6$. Co więcej, wykorzystujemy nasze metody do uzyskania pewnych wyników o dzikich wielomianach.

Słowa kluczowe: schematy Hilberta z wielogradacją, nasycone ideały punktów, gładkie rzutowe rozmaitości toryczne, rozmaitości siecznych, ranga brzegowa.

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Chapter 1

Introduction

In real life, it is often the case, that certain objects naturally appear together but from human perspective one of them is more interesting than other ones. In such situations, a need arises to separate the part that we care about from the part that is redundant or less attractive from our point of view. In many cases the distinction is pretty clear. However, there are also subtle examples. One of them is the farmland. Here the crops and the weeds grow together and distinguishing between them requires both attention and certain knowledge.

The main topic of this thesis is about

identifying the "good" inside the set of "all"

in the setting that is described below.

Problems of similar nature appear commonly in mathematics. For a basic example, consider a finite-dimensional real vector space V , the set $\text{End}_{\mathbb{R}}(V)$ of all linear endomorphisms of V and its subset $\text{Aut}_{\mathbb{R}}(V)$ consisting of invertible maps. Given an element $\varphi \in \text{End}_{\mathbb{R}}(V)$ we can check whether it belongs to $\text{Aut}_{\mathbb{R}}(V)$ by computing its determinant.

Another easy to state problem is provided by univariate polynomials with real coefficients. We might be interested in understanding which of them have a real root. There is an easy sufficient condition. Namely, if the degree of the polynomial is odd, then it has a real root. However, there are also more subtle criteria like Sturm's theorem [58, §5.2] which gives the number of distinct real roots of a given polynomial in a given interval.

The main motivational example for this thesis in the realm of algebraic geometry is the Hilbert scheme of r points in the projective n -space over the complex numbers \mathbb{C} . Before explaining this example in more detail, we outline the main results about the Hilbert schemes concentrating on Hilbert schemes of points. The Hilbert scheme $\mathcal{Hilb}(\mathbb{P}^n)$ is a scheme parametrizing all closed subschemes of \mathbb{P}^n . It was constructed by Grothendieck [44]. It has a decomposition into the disjoint union $\mathcal{Hilb}(\mathbb{P}^n) = \coprod_P \mathcal{Hilb}_P(\mathbb{P}^n)$ where P is the Hilbert polynomial of a closed subscheme of \mathbb{P}^n and $\mathcal{Hilb}_P(\mathbb{P}^n)$ parametrizes all closed subschemes of \mathbb{P}^n with Hilbert polynomial P . More generally, one may consider $\mathcal{Hilb}(X)$ or $\mathcal{Hilb}_P(X)$ for a projective scheme $X \subseteq \mathbb{P}^N$ over \mathbb{C} . In 1966 Hartshorne [47] proved that $\mathcal{Hilb}_P(\mathbb{P}^n)$ is connected for every Hilbert polynomial P . If the Hilbert polynomial P is constant and equal to r for some positive integer r , we write $\mathcal{Hilb}_r(\mathbb{P}^n)$ instead of $\mathcal{Hilb}_P(\mathbb{P}^n)$. In 1968 Fogarty [36] showed that $\mathcal{Hilb}_r(X)$ is smooth and irreducible if X is an irreducible smooth surface. On the other hand, for all $n \geq 3$ the scheme $\mathcal{Hilb}_r(\mathbb{P}^n)$ is reducible for all $r \gg 0$. This was established by Iarrobino [54] in 1972. The question, whether $\mathcal{Hilb}(X)$ is reduced was also addressed. Mumford [69] showed in 1962 that $\mathcal{Hilb}_P(\mathbb{P}^3)$ is not

reduced for some polynomial P of degree 1. Jelisiejew [61] showed in 2020 that $\mathcal{Hilb}_r(\mathbb{P}^n)$ is in general non-reduced.

We return to the description of our main motivational example of identifying "good" from "all". For fixed positive integers r and n , the closure of the locus of all points of $\mathcal{Hilb}_r(\mathbb{P}^n)$ corresponding to r -tuples of points in \mathbb{P}^n is an irreducible component of $\mathcal{Hilb}_r(\mathbb{P}^n)$. We call this component the smoothable component and we denote it by $\mathcal{Hilb}_r^{sm}(\mathbb{P}^n)$. By the above-mentioned result of Iarrobino, in general, $\mathcal{Hilb}_r(\mathbb{P}^n)$ is not equal to $\mathcal{Hilb}_r^{sm}(\mathbb{P}^n)$. Thus, if we care only about the set of r -tuples of points of the projective space \mathbb{P}^n together with their limits, we need to have methods of identifying whether a given point of the Hilbert scheme $\mathcal{Hilb}_r(\mathbb{P}^n)$ belongs to $\mathcal{Hilb}_r^{sm}(\mathbb{P}^n)$. This problem was studied by Cartwright, Erman, Velasco and Viray [20]. Similar problem for Gorenstein subschemes was considered by Casnati, Jelisiejew and Notari [21] and Jelisiejew [59].

In this thesis we concentrate on an analogous problem in the setting of multigraded Hilbert schemes "of points in general position". Multigraded Hilbert schemes were introduced by Haiman and Sturmfels [46] in 2004. Let S be a polynomial ring over \mathbb{C} , graded by an abelian group \mathbf{A} . Given a numerical function $h: \mathbf{A} \rightarrow \mathbb{N}$, there is a corresponding multigraded Hilbert scheme Hilb_S^h parametrizing homogeneous ideals I of S such that S/I has Hilbert function h .

It is important to emphasize, that even when S is a standard \mathbb{Z} -graded polynomial ring and h is the Hilbert function of a closed subscheme of a projective space, this leads to an object different than the classical Hilbert scheme, which is specified by Hilbert *polynomial*. In the case of multigraded Hilbert schemes we care about the Hilbert function in all degrees, while in the case of the usual Hilbert scheme we are interested only in the Hilbert function in large degrees.

The concept of a multigraded Hilbert scheme is a common generalization of various notions of Hilbert schemes:

1. $\mathcal{Hilb}_r(\mathbb{A}^n)$ —the Hilbert scheme of r points in affine n -space;
2. $\mathcal{Hilb}_P(\mathbb{P}^n)$ —the Hilbert scheme parametrizing closed subschemes of \mathbb{P}^n with Hilbert polynomial P ;
3. the so-called toric Hilbert schemes whose special cases were studied by Peeva and Stillman [71].

We present some known results about general multigraded Hilbert schemes Hilb_S^h . Maclagan and Smith [65] showed in 2010 that if S is a polynomial ring in two variables, then Hilb_S^h is smooth and irreducible (for any grading of S in any abelian group \mathbf{A} and for any Hilbert function $h: \mathbf{A} \rightarrow \mathbb{N}$). Beside this, little is known about general multigraded Hilbert scheme. People usually study one of the three particular cases described above. It turns out that there exists a non-connected toric Hilbert scheme. In 2005 Santos [76] gave such an example for a polynomial ring in 26 variables graded by \mathbb{Z}^6 . This is in sharp contrast with the Hartshorne's result [47] concerning $\mathcal{Hilb}_P(\mathbb{P}^n)$.

It is worth comparing the above-mentioned facts about the Hilbert scheme $\mathcal{Hilb}_r(\mathbb{P}^n)$ with the results on the multigraded Hilbert schemes. The Hilbert scheme $\mathcal{Hilb}_r(\mathbb{P}^n)$ is nicely-behaved, i.e. smooth and irreducible, for $n = 2$ [36]. On the other hand, the multigraded Hilbert scheme Hilb_S^h is smooth and irreducible when the polynomial ring S has two variables which in some sense corresponds to the case of the projective line. Similarly, $\mathcal{Hilb}_r(\mathbb{P}^3)$ is in general reducible [54], while Hilb_S^h can already be reducible for a polynomial ring in three variables. In fact, the

class of multigraded Hilbert schemes discussed in this thesis provides natural examples of such behavior.

We introduce our main object of study in the case of projective space. Let n be a positive integer and $S = \mathbb{C}[\alpha_0, \dots, \alpha_n]$ be the homogeneous coordinate ring of \mathbb{P}^n . Let r be a positive integer and $h_{r,n}: \mathbb{Z} \rightarrow \mathbb{N}$ be the Hilbert function of r points in general position in \mathbb{P}^n :

$$h_{r,n}(a) = \min\{\dim_{\mathbb{C}} S_a, r\}.$$

Let $\text{Sip}_{r,n}$ denote the locus of all points of $\text{Hilb}_S^{h_{r,n}}$ corresponding to radical ideals. Let $\text{Slip}_{r,n}$ be the closure of $\text{Sip}_{r,n}$ in $\text{Hilb}_S^{h_{r,n}}$. The acronyms Sip and Slip introduced in [15] stand for the "set of ideals of points" and the "scheme of limits of ideals of points", respectively. The subset $\text{Slip}_{r,n}$ is an irreducible component [15, Prop. 3.13]. Points of $\text{Hilb}_S^{h_{r,n}}$ that belong to $\text{Slip}_{r,n}$ are the "good" points inside the set of "all" points from the scheme $\text{Hilb}_S^{h_{r,n}}$. This is a consequence of the border apolarity lemma which we discuss below.

The main goal of this thesis is to establish sufficient and necessary conditions for a closed point in $\text{Hilb}_S^{h_{r,n}}$ to be in the irreducible component $\text{Slip}_{r,n}$.

We also consider analogous problem for a smooth projective toric variety X . We study the multigraded Hilbert scheme associated with the Cox ring $S[X]$ of X and the Hilbert function $h_{r,X}: \text{Pic}(X) \rightarrow \mathbb{N}$ of r points in general position in X , i.e.

$$h_{r,X}([D]) = \min\{\dim_{\mathbb{C}} S[X]_{[D]}, r\}.$$

See Section 4.1 for relevant definitions. Again, there is a distinguished irreducible component $\text{Slip}_{r,X}$ of $\text{Hilb}_{S[X]}^{h_{r,X}}$ that is the closure of the locus of all radical ideals that are saturated with respect to the irrelevant ideal of X . We want to find criteria that identify points in $\text{Hilb}_{S[X]}^{h_{r,X}}$ that belong to $\text{Slip}_{r,X}$.

Significance of the considered problem

It is necessary to explain how does the irreducible component $\text{Slip}_{r,n}$ fit into the above philosophy of identifying "good" from "all". This is based on the border apolarity lemma introduced by Buczyńska and Buczyński [15]. This result shows that there is a connection between border rank of a homogeneous polynomial and the multigraded Hilbert scheme $\text{Hilb}_S^{h_{r,n}}$. Our discussion here is informal. The precise statement of the border apolarity, as well as the formal definitions of the border rank and secant varieties appear in Chapter 2. Suppose that F is a homogeneous polynomial of degree d in the polynomial ring $S^* = \mathbb{C}[x_0, \dots, x_n]$. Here, S^* is the graded dual of the polynomial ring $S = \mathbb{C}[\alpha_0, \dots, \alpha_n]$, see Subsection 2.4.1 for more details. We say that F has *rank* r if r is the smallest integer such that we have $F = \sum_{i=1}^r \ell_i^d$ for some linear forms $\ell_i \in S_1^*$. We also consider the *border rank* of F which is the smallest integer r such that $[F] \in \mathbb{P}S_d^*$ is in the closure of the set of all polynomials with rank at most r . Calculating the border rank of a given polynomial is a classical problem in algebraic geometry and is strongly related to studies of secant varieties of the Veronese variety. The border apolarity lemma says that F has border rank at most r for a positive integer r if and only if there exists a point $[I] \in \text{Slip}_{r,n}$ such that I is apolar to F . Thus,

**points from $\text{Slip}_{r,n}$ are the "good" points among "all" points of $\text{Hilb}_S^{h_{r,n}}$
since they serve as witnesses of small border rank.**

As a result, the more conditions (both sufficient and necessary) for a point in $\text{Hilb}_S^{h_{r,n}}$ to be in the irreducible component $\text{Slip}_{r,n}$ we have at our disposal, the greater the scope of applicability of the border apolarity lemma.

One potential application of border apolarity is in studying homogeneous polynomials, called wild polynomials, whose border rank is smaller than the smoothable rank (see Section 2.4 for definitions of these ranks). These polynomials are known to exist (see [12] and [53]). They appear naturally in the context of the border apolarity lemma. Indeed, by definition, it is precisely for these polynomials that the apolarity for smoothable rank (depending on the smoothable component $\mathcal{Hilb}_r^{sm}(\mathbb{P}^n)$ of the Hilbert scheme $\mathcal{Hilb}_r(\mathbb{P}^n)$) fails to compute the border rank. For them, it is necessary to consider the multigraded Hilbert scheme $\text{Hilb}_S^{h_{r,n}}$ and its irreducible component $\text{Slip}_{r,n}$.

A crucial motivation for studying $\text{Slip}_{r,X}$ for more general toric varieties than \mathbb{P}^n is the problem of computing the border rank of matrix multiplication tensors. This is a vitally important problem for complexity theory, however it is very complicated. For instance the border rank of matrix multiplication tensor for 3×3 matrices is unknown. See [23] for recent progress on that problem.

Structure of the thesis and main results

In Chapter 2 we present the relevant background from commutative algebra and scheme theory. In particular, we formally define the multigraded Hilbert schemes by their functors of points. We also study the flag multigraded Hilbert schemes and basic notions of deformation theory.

Chapter 3 contains the main results of the thesis in the case of projective space. We present three necessary conditions for a point $[I] \in \text{Hilb}_S^{h_{r,n}}$ to be in the irreducible component $\text{Slip}_{r,n}$. Moreover, we prove a sufficient condition for a point $[I] \in \text{Hilb}_S^{h_{r,2}}$ to be in $\text{Slip}_{r,2}$. We illustrate these criteria with simple examples. Furthermore, we end the chapter with the complete description of points from $\text{Slip}_{r,2}$ for all $r \leq 6$. The complexity of these examples is perhaps surprising, especially in view of the fact that the usual Hilbert scheme $\mathcal{Hilb}_r(\mathbb{P}^2)$ is smooth and irreducible.

We now summarize the main results presented in Chapter 3. We simplify the statements of some of the more technical theorems by considering their special cases, or omitting some parts of the conclusions. Given a polynomial ring $S = \mathbb{C}[\alpha_0, \dots, \alpha_n]$, we denote by \mathfrak{m} the ideal $(\alpha_0, \dots, \alpha_n)$.

Proposition 3.1 provides a necessary condition for $[I] \in \text{Hilb}_S^{h_{r,n}}$ to be in $\text{Slip}_{r,n}$. It is based on bounding from the above, the degree in which all saturated ideals corresponding to points of $\text{Hilb}_S^{h_{r,n}}$ are generated.

Proposition 1.1 (Proposition 3.1). *Let r and n be positive integers and $I \subseteq S = \mathbb{C}[\alpha_0, \dots, \alpha_n]$ be a homogeneous ideal such that S/I has Hilbert function $h_{r,n}$. Let $e = \min\{a \in \mathbb{Z} \mid h_{r,n}(a) = r\}$ and $d \geq e + 2$. If the inequality $\dim_{\mathbb{C}} \text{Hom}_S(I + \mathfrak{m}^d, S/(I + \mathfrak{m}^d))_0 < rn$ holds, then the point $[I]$ does not belong to $\text{Slip}_{r,n}$.*

Theorem 3.5 shows that if the point $[I]$ belongs to $\text{Slip}_{r,n}$, then the Hilbert function of S/I^k for any positive integer k is bounded from below by $r \cdot \dim_{\mathbb{C}} S_{k-1}$ for all large enough degrees (depending on k). This result is obtained by calculating the Hilbert polynomials of powers of

a radical ideal corresponding to a point of $\text{Hilb}_S^{h_{r,n}}$ and establishing a bound from which they agree with the Hilbert functions.

Theorem 1.2 (Theorem 3.5). *Let r and n be positive integers and $I \subseteq S = \mathbb{C}[\alpha_0, \dots, \alpha_n]$ be a homogeneous ideal such that S/I has Hilbert function $h_{r,n}$. Let $e = \min\{a \in \mathbb{Z} \mid h_{r,n}(a) = r\}$. If the point $[I]$ belongs to $\text{Slip}_{r,n}$, then $H_{S/I^k}(d) \geq r \cdot \dim_{\mathbb{C}} S_{k-1}$ holds for every positive integer k and for every $d \geq ke + k$.*

Theorem 3.13 presents a sufficient condition for $[I] \in \text{Hilb}_S^{h_{r,2}}$ to be in the irreducible component $\text{Slip}_{r,2}$. We show that if the Hilbert function of $S/(I : \mathfrak{m}^\infty)$ differs from $h_{r,2}$ only in one degree, then the point $[I]$ belongs to $\text{Slip}_{r,2}$. The proof is obtained by showing that $[I]$ belongs to an irreducible subset of $\text{Hilb}_S^{h_{r,2}}$ which intersects $\text{Slip}_{r,2}$ at a smooth point of $\text{Hilb}_S^{h_{r,2}}$. Furthermore, we comment why natural generalizations of this criterion for \mathbb{P}^n with $n > 2$ fail.

Theorem 1.3 (Theorem 3.13). *Let r be a positive integer and $S = \mathbb{C}[\alpha_0, \alpha_1, \alpha_2]$ be a polynomial ring. Consider a closed point $[I]$ of the multigraded Hilbert scheme $\text{Hilb}_S^{h_{r,2}}$. If $(I : \mathfrak{m}^\infty)_d \neq I_d$ holds for a unique integer d , then we have $[I] \in \text{Slip}_{r,2}$.*

Theorem 3.42 is the most technically involved result in this thesis. It is stated in a general setup in which the proof follows by a short argument using deformation theory. Then we discuss some conditions which imply the assumptions of Theorem 3.42. Finally, we present two applications of this theorem, Theorems 3.66 and 3.75. These are the versions of the theorem that we use in the rest of the thesis.

Theorem 1.4 (Theorem 3.42). *Let r and n be positive integers, $S = \mathbb{C}[\alpha_0, \dots, \alpha_n]$ be a polynomial ring and $[I] \in \text{Hilb}_S^{h_{r,n}}$ be a closed point. Assume that we have $I \neq (I : \mathfrak{m}^\infty)$ and let d be such that I_d is not equal to $(I : \mathfrak{m}^\infty)_d$. Let $J = \mathfrak{m}^d \cap (I : \mathfrak{m}^\infty)$ and $K = \mathfrak{m}^d \cap I$. When the following hold:*

1. *the natural map $\text{Hom}_S(J, S/J)_0 \rightarrow \text{Hom}_S(K, S/J)_0$ is surjective;*
2. *$[J] \in \text{Hilb}_S^h$ is a smooth point where h is the Hilbert function of S/J ;*
3. *the natural map $\text{Hom}_S(K, S/K)_0 \rightarrow \text{Hom}_S(K, S/J)_0$ is surjective,*

there is no point $[I'] \in \text{Slip}_{r,n}$ with $I'_{\geq d} = I_{\geq d}$. In particular, we have $[I] \notin \text{Slip}_{r,n}$.

As an application of Theorem 1.4 we obtain the following result.

Theorem 1.5 (Theorem 3.66). *Let $[I] \in \text{Hilb}_S^{h_{r,n}}$ be a closed point corresponding to an ideal I such that S/\bar{I} has Hilbert function $h_{r,1}$. There exists $[I'] \in \text{Slip}_{r,n}$ with $I_{\geq r-2} = I'_{\geq r-2}$ if and only if there is an inclusion $((I : \mathfrak{m}^\infty)^2)_{r-2} \subseteq I_{r-2}$.*

We end Chapter 3 with the complete set-theoretic description of $\text{Slip}_{r,2}$ for all $r \leq 6$. To give some insight into the complexity of this problem we present here a short discussion. For all $r \leq 3$, the scheme $\text{Hilb}_S^{h_{r,2}}$ is irreducible (see Propositions 3.38, 3.39 and 3.40). However, Corollary 3.79 shows that $\text{Hilb}_S^{h_{4,2}}$ is reducible. In fact, it has two irreducible components. Here, the description of $\text{Slip}_{4,2}$ follows easily from Theorem 3.66. In the next case, $r = 5$, the scheme $\text{Hilb}_S^{h_{5,2}}$ still has only two irreducible components but the description of $\text{Slip}_{5,2}$ obtained in Proposition 3.89 requires some further observations. Finally, $\text{Hilb}_S^{h_{6,2}}$ has four irreducible components. We use all

four criteria mentioned above, to obtain the description of $\text{Slip}_{6,2}$ (see Proposition 3.104). Still, the proof is of significant complexity. Since there was no prior systematic study of the component $\text{Slip}_{r,X}$, it was not clear what to expect. By analogy to Fogarty's result [36] on smoothness of $\mathcal{H}ilb_r(\mathbb{P}^2)$, we expected that $\text{Hilb}_{S[\mathbb{P}^2]}^{h_{r,\mathbb{P}^2}}$ should not be too complicated. However, it seems that the proper analogue is rather $\mathcal{H}ilb_r(\mathbb{P}^3)$, where little is known about the smoothable component.

Chapter 4 is concerned with the case of a smooth, projective toric variety X . Here again, one may consider the multigraded Hilbert scheme $\text{Hilb}_{S[X]}^{h_{r,X}}$ where $S[X]$ is the Cox ring of X and $h_{r,X}$ is the Hilbert function of r points in general position on X . Again, there is an irreducible component $\text{Slip}_{r,X}$ which is defined analogously to $\text{Slip}_{r,n}$ considered above. Theorem 4.15 describes a relation between $\text{Slip}_{r,X}$ and $\text{Slip}_{r,Y}$ where $f: X \rightarrow Y$ is a map of smooth projective toric varieties with $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. The proof is based on the possibility of lifting f to a homomorphism of Cox rings of X and Y . This is discussed in Subsection 4.1.3.

Theorem 1.6 (Theorem 4.15). *Let $f: X \rightarrow Y$ be a morphism between smooth projective toric varieties with $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Let r be a positive integer and $[I] \in \text{Hilb}_{S[X]}^{h_{r,X}}$ be a closed point. Let $\bar{f}^\# : S[Y] \rightarrow S[X]$ be a lift of f as in Definition 4.2. If the point $[I]$ belongs to $\text{Slip}_{r,X}$ then we have*

$$[(\bar{f}^\#)^{-1}(I)] \in \text{Slip}_{r,Y}.$$

Theorem 4.25 presents another necessary condition for $[I] \in \text{Hilb}_{S[X]}^{h_{r,X}}$ to be in the irreducible component $\text{Slip}_{r,X}$, when X is the product of $k \geq 2$ projective spaces. The proof of the theorem is based on simple properties of Hilbert functions of saturated ideals of points in X .

Theorem 1.7 (Theorem 4.25). *Let $k \geq 2$ and n_1, \dots, n_k be positive integers. Let $X = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ and for all $i \in \{1, \dots, k\}$ let $B(\Sigma_i) \subseteq S[X]$ be the extension of the irrelevant ideal of \mathbb{P}^{n_i} under the natural inclusion $S[\mathbb{P}^{n_i}] \rightarrow S[X]$. If the point $[I]$ belongs to $\text{Slip}_{r,X}$ for some positive integer r , then*

$$\dim_{\mathbb{C}} \text{Hom}_{S[X]} \left(I + B(\Sigma_i)^2, S[X]/(I + B(\Sigma_i)^2) \right)_0 \geq r(n_1 + \dots + n_k)$$

holds for all $i \in \{1, \dots, k\}$.

In Chapter 5 we present some applications of border apolarity lemma to secant varieties. In Section 5.1 we study polynomials whose border rank is smaller than the smoothable rank (see Section 2.4 for relevant definitions). This is in accordance with the initial motivation for developing criteria for points of $\text{Hilb}_S^{h_{r,n}}$ to belong to the irreducible component $\text{Slip}_{r,n}$.

Results from Sections 5.2, 5.3 and 5.4 are contained in [40]. They are about identifying (in special cases) points in the cactus variety that are not in the secant variety. Here, we use the border apolarity lemma without actually needing any insight into the irreducible component $\text{Slip}_{r,n}$. The problem of distinguishing the secant variety from the cactus variety is another illustration of identifying "good" inside the set of "all". Secant varieties are classical objects of study but their equations are in general unknown. Moreover, various classes of known equations have been shown to actually vanish on a larger variety—the cactus variety. See [11], [39] and [62, §10.2].

Open problems

We end this chapter with a short list of natural directions of further investigation.

Given a smooth projective toric variety X and a positive integer r , we may divide closed points of $\text{Hilb}_{S[X]}^{h_{r,X}}$ into four sets depending on whether $[I]$ is in the closure of the locus of all saturated ideals and whether the subscheme of X defined by I is smoothable. The irreducible component $\text{Slip}_{r,X}$ consists of points that are in the closure of the locus of all saturated ideals and that define smoothable subschemes. However, the following natural problem remains open.

Problem 1.8. Is there a projective toric variety X and a positive integer r such that there exists $[I] \in \text{Hilb}_{S[X]}^{h_{r,X}} \setminus \text{Slip}_{r,X}$ which satisfies conditions:

1. $[I]$ is in the closure of the locus of all saturated ideals;
2. the subscheme of X defined by I is smoothable?

If the answer to the above question is negative, this could allow us to split the problem of describing $\text{Slip}_{r,X}$ into two. One of them, which has been studied longer, is describing the smoothable component of the usual Hilbert scheme. The other problem would be studying the closure of the locus of all saturated ideals inside $\text{Hilb}_{S[X]}^{h_{r,X}}$.

Another problem is related to the geometry of $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,\mathbb{P}^n}}$.

Problem 1.9. Is $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,\mathbb{P}^n}}$ ever non-reduced?

We also want to discuss the criterion based on the flag condition for secant varieties [24, Prop. 2.3]. It seems that a natural analogue for $\text{Slip}_{r,X}$ should hold. Namely, we expect that if the point $[I]$ belongs to $\text{Slip}_{r,X}$ then there is a flag of ideals $I_r = I \subseteq I_{r-1} \subseteq \cdots \subseteq I_0 = S$ such that $[I_k]$ is in $\text{Slip}_{k,X}$ for every k .

One natural question, especially in view of Problem 1.8 is the following.

Problem 1.10. Let X be a smooth projective toric variety and r be a positive integer. Assume that $[I] \in \text{Hilb}_{S[X]}^{h_{r,X}}$ is in the closure of the locus of all saturated ideals. Is there a flag of ideals $I_r = I \subseteq I_{r-1} \subseteq \cdots \subseteq I_0 = S$ such that for all k the point $[I_k] \in \text{Hilb}_{S[X]}^{h_{r,X}}$ is in the closure of the locus of all saturated ideals?

A final general problem that is worth studying is as follows.

Problem 1.11. Is there a homogeneous ideal in three variables whose border rank is strictly smaller than the smoothable rank?

It is known that if F is such a polynomial then we have $\text{br}(F) > \deg(F) + 1$ (see [11, Prop. 2.5]). We show in Proposition 5.5 that the inequality $\text{br}(F) > \deg(F) + 2$ holds.

There are also some natural problems related more closely to our methods. They are less general and thus, not as important. However, we would like to discuss them shortly.

Criterion from Theorem 1.4 is stated in a general version and we describe two situations where its assumptions are fulfilled: Theorems 3.66 and 3.75. It seems that there might be more general setups where Theorem 1.4 could be applied. We intend to investigate this in the future.

In a similar spirit, the description of $\text{Slip}_{6,2}$ is quite lengthy and involved. It is a natural question, whether the methods used there could be abstracted to work in more general situations.

Chapter 2

Background material

In this chapter we collect some definitions and results that are used in the rest of the thesis. Section 2.1 deals with commutative algebra. Material from Subsections 2.1.1 and 2.1.2 is standard but it was hard to find a reference for some of the results discussed there. We present the proofs for the sake of completeness. Subsections 2.1.3, 2.1.4 and 2.1.5 contain some results that are used in Chapter 3. These subsections are based on [67]. In Subsection 2.2.1 we present some general results related to scheme theory. Subsections 2.2.2 and 2.2.3 are concerned with multigraded Hilbert schemes and Subsection 2.2.4 recalls the notion of a flag multigraded Hilbert scheme. Section 2.3 deals with deformation theory. We present basic definitions and results that are used in the subsequent chapters. Section 2.4 is devoted to various notions of ranks and related apolarity lemmas.

Notation

Throughout this chapter \mathbb{k} is a fixed algebraically closed field. Unless stated otherwise, all polynomial rings over \mathbb{k} that are considered have the standard \mathbb{Z} -grading. Therefore, \mathbb{Z} -graded modules over these rings are simply called graded modules.

2.1 Commutative algebra

In this section we present some results from commutative algebra that are needed for the proofs of the main results.

In Subsection 2.1.1 we study locally free modules of finite rank since these appear in the definition of the functor of points of a multigraded Hilbert scheme. Since these schemes are the main object of our investigation we feel that it is appropriate to recall the notion explicitly.

Subsection 2.1.2 deals with saturated ideals. We present a few results that are mainly used in Chapter 3.

In Subsection 2.1.3 we study the Hilbert function of a power of a radical ideal defining a zero-dimensional subscheme of a projective space. The obtained results are key in the proof of Theorem 3.5.

Subsection 2.1.4 is concerned with the computation of the dimension of the vector space $\text{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{>0}$ for a monomial ideal $\mathfrak{a} \neq T$ in the homogeneous coordinate ring T of projective line such that $\dim_{\mathbb{k}} T/\mathfrak{a}$ is finite. This is related to the tangent space to an appropriate multigraded Hilbert scheme. This observation is used in the proof of Theorem 3.13.

In Subsection 2.1.5 we study the Ext groups $\text{Ext}_S^i(\mathbb{k}, M)$ and $\text{Ext}_S^i(M, \mathbb{k})$ where S is a polynomial ring and M is a finitely generated graded S -module. These results are used in Chapter 3.

For the sake of completeness, we give detailed proofs even though some of the results might be well known. On the other hand, we only refer to the basic properties of Gröbner bases and local cohomology as we need them. We believe that the theory of Gröbner bases is already well established, and in any case there are many excellent introductions to the topic, e.g. [27] and [31]. The theory of local cohomology is only used as a tool in one of the proofs so it seems to be more natural to just cite the relevant result from [57].

Subsections 2.1.3, 2.1.4 and 2.1.5 are based on [67].

2.1.1 Locally free modules

We recall the definition of a locally free module and study some simple properties of such modules.

Definition 2.1. Given a ring R , we say that a module M is *locally free of finite rank* if, for every prime ideal \mathfrak{p} of R , there is an element $f \in R \setminus \mathfrak{p}$ such that M_f is a finite free R_f -module.

Observe that an R -module M is locally free of finite rank if and only if the corresponding quasicohherent sheaf \widetilde{M} on $\text{Spec } R$ is locally free of finite rank. We prove algebraically some properties of locally free modules of finite rank. These results are intuitively clear from the interpretation in terms of sheaves of $\mathcal{O}_{\text{Spec } R}$ -modules.

Lemma 2.2. *Let M be an R -module and let \mathfrak{p} be a prime ideal of R . If $f \in R \setminus \mathfrak{p}$ is such that M_f is a free R_f -module of rank r , then $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module of rank r . In particular, if $f' \in R \setminus \mathfrak{p}$ is such that $M_{f'}$ is a free $R_{f'}$ -module of rank r' , then we have $r = r'$.*

Proof. We may rewrite $M_{\mathfrak{p}}$ as

$$M_{\mathfrak{p}} \cong M \otimes_R R_{\mathfrak{p}} \cong M \otimes_R (R_f)_{\mathfrak{p}R_f}$$

where the latter isomorphism comes from [6, Prop. 11 §1.2]. This can be further transformed as

$$M \otimes_R (R_f)_{\mathfrak{p}R_f} \cong M \otimes_R (R_f \otimes_{R_f} (R_f)_{\mathfrak{p}R_f}) \cong M_f \otimes_{R_f} (R_f)_{\mathfrak{p}R_f} \cong (M_f)_{\mathfrak{p}R_f} \cong (R_f^r)_{\mathfrak{p}R_f} \cong R_{\mathfrak{p}}^r$$

where the last isomorphism follows from [6, Prop. 11 §1.2] since localization commutes with direct sums. \square

Definition 2.3. We say that an R -module M is *locally free of rank r* if, given any prime ideal \mathfrak{p} of R , there is an element $f \in R \setminus \mathfrak{p}$ such that M_f is a free R_f -module of rank r .

It follows from Lemma 2.2 that an R -module M can be locally free of rank r for a unique integer r . Moreover, if R is a Noetherian ring, $\text{Spec } R$ is connected (equivalently, if R has no non-trivial idempotents) and M is locally free of finite rank, then M is locally free of rank r for some integer r by [48, Ex. II.5.8].

We study how the locally free condition behaves in short exact sequences.

Lemma 2.4. *Let R be a Noetherian ring and let $M \subseteq N \subseteq R^d$ be R -submodules. Assume that R^d/N is a locally free R -module of rank a for a positive integer a . For a positive integer b , the following conditions are equivalent:*

- (i) R^d/M is a locally free R -module of rank b ;

(ii) N/M is a locally free R -module of rank $b - a$.

Proof. Localization is exact, so it is enough to show that R^d/M is a locally free R -module of finite rank if and only if N/M is a locally free R -module of finite rank. An R -module U is a locally free R -module of finite rank if and only if U is a flat R -module of finite presentation (see [6, Prop. 3 §4.4]). Moreover, since R is a Noetherian ring, it is also equivalent to U being a flat R -module of finite type.

We have an exact sequence of R -modules

$$0 \rightarrow N/M \rightarrow R^d/M \rightarrow R^d/N \rightarrow 0. \quad (2.5)$$

Since R^d/M is an R -module of finite type, it is a Noetherian module. Thus, N/M is an R -module of finite type.

The R -module R^d/N is flat by assumption. Hence, using the Tor exact sequence coming from short exact sequence (2.5), we obtain that N/M is R -flat if and only if R^d/M is R -flat. \square

2.1.2 Saturation and homogeneous saturated ideals in polynomial ring

Let $S = \mathbb{k}[\underline{\alpha}] := \mathbb{k}[\alpha_0, \alpha_1, \dots, \alpha_n]$ be a polynomial ring. By \mathfrak{m} we denote the irrelevant ideal $(\alpha_0, \alpha_1, \dots, \alpha_n)$. Given an ideal I in S , we write \bar{I} for the saturation of I with respect to \mathfrak{m} .

In the proof of Lemma 2.6 we use the notion of local cohomology. See [57, Ch. 7] for its basic properties. Recall that $H_{\mathfrak{m}}^i(-)$ are the right derived functors of the functor

$$\Gamma_{\mathfrak{m}}(-): \mathfrak{Mod}(S) \rightarrow \mathfrak{Mod}(S)$$

defined by

$$\Gamma_{\mathfrak{m}}(M) = \{m \in M \mid \mathfrak{m}^d m = 0 \text{ for some } d \in \mathbb{Z}_{>0}\}.$$

There is a close connection between the zeroth local cohomology group $H_{\mathfrak{m}}^0(S/I)$ and the saturation \bar{I} of I . Namely, \bar{I} is the kernel of the natural map $S \rightarrow (S/I)/H_{\mathfrak{m}}^0(S/I)$. Therefore, we are able to use the general results about local cohomology to prove the following lemma. Note that we are interested only in the zeroth local cohomology group.

Lemma 2.6. *Let $R \rightarrow T$ be a flat homomorphism of \mathbb{k} -algebras. Let $I \subseteq R[\underline{\alpha}] = \bigoplus_d (S_d \otimes_{\mathbb{k}} R)$ be a homogeneous ideal saturated with respect to $\mathfrak{m} \otimes_{\mathbb{k}} R$. The ideal $I \otimes_R T \subseteq S \otimes_{\mathbb{k}} T$ is saturated with respect to $\mathfrak{m} \otimes_{\mathbb{k}} T$.*

Proof. The saturation of I is the kernel of the natural map

$$S \otimes_{\mathbb{k}} R \rightarrow ((S \otimes_{\mathbb{k}} R)/I)/H_{\mathfrak{m} \otimes_{\mathbb{k}} R}^0((S \otimes_{\mathbb{k}} R)/I).$$

The ideal I is assumed to be saturated. Thus, $H_{\mathfrak{m} \otimes_{\mathbb{k}} R}^0((S \otimes_{\mathbb{k}} R)/I)$ is the zero module. Since T is a flat R -algebra, we have

$$H_{\mathfrak{m} \otimes_{\mathbb{k}} T}^0((S \otimes_{\mathbb{k}} T)/(I \otimes_R T)) \cong \left(H_{\mathfrak{m} \otimes_{\mathbb{k}} R}^0((S \otimes_{\mathbb{k}} R)/I) \right) \otimes_R T = 0$$

by [57, Prop. 7.15]. It follows that $I \otimes_R T$ is saturated with respect to $\mathfrak{m} \otimes_{\mathbb{k}} T$. \square

Lemma 2.7 states, that if an initial ideal $\text{in}_{<}(I)$ of a homogeneous ideal I is saturated, then I is saturated. This is a typical situation. The process of taking the initial ideal usually worsens

the properties of the corresponding quotient algebra. As a key example, for a homogeneous ideal $I \subseteq S$ we have an inequality of Betti numbers

$$\beta_{ij}(S/I) \leq \beta_{ij}(S/\text{in}_{<}(I))$$

for all $i, j \in \mathbb{Z}_{\geq 0}$, see [50, Cor. 3.3.3]. Therefore, if $S/\text{in}_{<}(I)$ has some nice property, then often S/I has the same property. Compare the following lemma to [50, Cor. 3.3.5].

Lemma 2.7. *Let I be an ideal in S and let $<$ be a monomial order. We have $\text{in}_{<}(\bar{I}) \subseteq \overline{\text{in}_{<}(I)}$. In particular, if I is a homogeneous ideal and $\text{in}_{<}(I)$ is a saturated ideal, then I is a saturated ideal.*

Proof. Let $f \in \bar{I}$. There is an integer l with $\alpha_i^l f \in I$ for all $i = 0, 1, \dots, n$. Therefore, $\alpha_i^l \cdot \text{in}_{<}(f)$ belongs to $\text{in}_{<}(I)$ for all $i = 0, 1, \dots, n$. Consequently, we have $\text{in}_{<}(f) \in \overline{\text{in}_{<}(I)}$.

Now assume that I is a homogeneous ideal such that $\text{in}_{<}(I)$ is a saturated ideal. There is a chain of inclusions

$$\text{in}_{<}(I) \subseteq \text{in}_{<}(\bar{I}) \subseteq \overline{\text{in}_{<}(I)} = \text{in}_{<}(I).$$

It follows that there is an equality $\text{in}_{<}(I) = \text{in}_{<}(\bar{I})$. Thus, I and \bar{I} have the same Hilbert function. As a result, we get $I = \bar{I}$. \square

We frequently use the following observation.

Lemma 2.8. *Let M be a graded S -module and let I be a homogeneous ideal of S with $I = \bar{I} \cap \mathfrak{m}^d$ for a positive integer d . Assume that $M_{<d}$ is the zero vector space and that there is a positive integer r with $\mathfrak{m}^r \cdot M = 0$. We have $\text{Hom}_S(M, S/I)_0 = 0$.*

Proof. Pick $\varphi \in \text{Hom}_S(M, S/I)_0$ and $x \in M_e$ for some $e \geq d$. We have $\mathfrak{m}^r \cdot \varphi(x) = 0$. Therefore, in the quotient algebra S/I , the element $\varphi(x)$ is represented by an element from $\bar{I}_e = I_e$. Thus, it is zero. \square

Next we study some properties of Hilbert functions of saturated ideals.

Lemma 2.9. *Let $I \neq S$ be a homogeneous saturated ideal of S .*

- (i) *There exists a linear form $f \in S_1$ that is a nonzerodivisor on S/I ;*
- (ii) *The inequality $H_{S/I}(d+1) - H_{S/I}(d) \geq 0$ holds for all integers d ;*
- (iii) *If $H_{S/I}(d) = H_{S/I}(d-1)$ holds for a positive integer d , then we have $H_{S/I}(d+1) = H_{S/I}(d)$.*

Proof.

- (i) Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the associated primes of S/I . It is enough to show that $\bigcup_{i=1}^k (\mathfrak{p}_i)_1$ is not equal to S_1 . Suppose that the equality $\bigcup_{i=1}^k (\mathfrak{p}_i)_1 = S_1$ holds. Since \mathbb{k} is infinite we have $(\mathfrak{p}_i)_1 = S_1$ for some i and therefore, \mathfrak{m} is an associated prime of S/I . This gives a contradiction with the assumption that I is saturated.
- (ii) If $f \in S_1$ is a nonzerodivisor on S/I , then the map $(S/I)_d \xrightarrow{f} (S/I)_{d+1}$ is injective for every d .

(iii) Let $f \in S_1$ be a nonzerodivisor on S/I . Suppose that $H_{S/I}(d) = H_{S/I}(d-1)$ holds. The multiplication map $(S/I)_{d-1} \xrightarrow{f} (S/I)_d$ is an isomorphism of \mathbb{k} -vector spaces. We claim that also $(S/I)_d \xrightarrow{f} (S/I)_{d+1}$ is an isomorphism. It is injective since f is a nonzerodivisor on S/I . Let $g \in (S/I)_{d+1}$. We have $g = \sum_{i=0}^n \alpha_i h_i$ for some $h_i \in (S/I)_d$. By assumptions, there are $k_0, \dots, k_n \in (S/I)_{d-1}$ with $h_i = f k_i$ for all $i = 0, 1, \dots, n$. It follows that $g = f(\sum_{i=0}^n \alpha_i k_i)$ is in the image of the map $(S/I)_d \xrightarrow{f} (S/I)_{d+1}$. \square

The following corollary of a theorem by Bayer and Stillman enables us to deform a saturated ideal to a saturated ideal with special properties—a Borel-fixed ideal. Consider the action of $\mathrm{GL}_{n+1}(\mathbb{k})$ on the set of ideals of S given by a linear change of coordinates. Given an ideal $I \subseteq S$ and a monomial order $<$, there is a non-empty open subset $U \subseteq \mathrm{GL}_{n+1}(\mathbb{k})$ such that the ideal $\mathrm{in}_{<}(g \cdot I)$ is independent of the choice of $g \in U$ [31, Thm. 15.18]. We call this ideal the *generic initial ideal* of I with respect to $<$.

Corollary 2.10. *Suppose that $I \neq S$ is a homogeneous saturated ideal of S . The generic initial ideal I' of I with respect to the grevlex order with $\alpha_0 > \dots > \alpha_n$ is a saturated ideal.*

Proof. Since I is a saturated ideal we have $\mathrm{depth}(S/I) \geq 1$ by Lemma 2.9(i). Thus, the inequality $\mathrm{depth} S/I' \geq 1$ is a consequence of [50, Cor. 4.3.18]. It follows that I' is saturated. \square

The following lemma gives a useful property of saturated Borel-fixed ideals.

Lemma 2.11. *Let $T = \mathbb{k}[\alpha_0, \dots, \alpha_{n-1}]$. If $I \subseteq S$ is a saturated Borel-fixed ideal then there exists an ideal $\mathfrak{a} \subseteq T$ with $\mathfrak{a} \cdot S = I$.*

Proof. Let G be the set of all minimal monomial generators of I . It is enough to show that there is no element $\prod_{i=0}^n \alpha_i^{a_i} \in G$ with $a_n > 0$. Assume that we have $\prod_{i=0}^n \alpha_i^{a_i} \in G$ with $a_n > 0$.

We claim that the element

$$\left(\prod_{i=0}^{j-1} \alpha_i^{a_i} \right) \cdot \alpha_j^{\sum_{k=j}^n a_k}$$

belongs to I for every $0 \leq j \leq n$. This follows from [68, Prop. 2.3] if $\mathrm{char} \mathbb{k} = 0$ and from [51, Prop. 1.2] if $\mathrm{char} \mathbb{k} > 0$. Therefore, $\prod_{i=0}^{n-1} \alpha_i^{a_i}$ belongs to $\bar{I} = I$ since we have

$$\alpha_j^{\sum_{k=j}^n a_k} \cdot \left(\prod_{i=0}^{n-1} \alpha_i^{a_i} \right) \in I$$

for all $j \in \{0, \dots, n\}$. This shows that g is not a minimal monomial generator and gives a contradiction. \square

The following observation is a special case of Macaulay's theorem [10, Thm. 4.2.10].

Lemma 2.12. *Let $T = \mathbb{k}[\alpha_0, \alpha_1]$ and $\mathfrak{a} \subseteq T$ be a homogeneous ideal. The inequality*

$$H_{T/\mathfrak{a}}(d) - H_{T/\mathfrak{a}}(d+1) \geq 0$$

holds for every d with $\mathfrak{a}_d \neq 0$.

Proof. If \mathfrak{a}_d is non-zero, then we have $H_{T/\mathfrak{a}}(d) \leq \dim_{\mathbb{k}} T_{d-1} = d$. It follows from [10, Thm. 4.2.10] that there is an inequality $H_{T/\mathfrak{a}}(d+1) \leq H_{T/\mathfrak{a}}(d)$. \square

As a consequence of the above observation we obtain some bounds on the number of minimal homogeneous generators of a homogeneous saturated ideal $\bar{I} \subseteq S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$.

Lemma 2.13. *Let $\bar{I} \subseteq S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$ be a homogeneous saturated ideal. Let m be a positive integer with $\bar{I}_m \neq 0$ and let f be the Hilbert function of S/\bar{I} . For every $d \geq m$ we have*

$$\beta_{1,d+1}(S/\bar{I}) \leq 2f(d) - f(d+1) - f(d-1).$$

Proof. Let I' be the generic initial ideal (see [31, §15.9]) of \bar{I} with respect to the grevlex order with $\alpha_0 > \alpha_1 > \alpha_2$. By Corollary 2.10 the ideal I' is saturated. Moreover, $\beta_{1,a}(S/I') \geq \beta_{1,a}(S/\bar{I})$ holds for every a by [50, Cor. 3.3.3]. Therefore, it is enough to prove the lemma for a saturated Borel-fixed ideal I' . Let $\mathfrak{a} = I' \cap \mathbb{k}[\alpha_0, \alpha_1]$ and let g be the Hilbert function of $\mathbb{k}[\alpha_0, \alpha_1]/\mathfrak{a}$. We have $I' = \mathfrak{a} \cdot S$ by Lemma 2.11, so we get $f(a) - f(a-1) = g(a)$ for every $a \in \mathbb{Z}$.

Let $d \geq m$ be such that there is an inequality $\beta_{1,d+1}(S/I') = \beta_{1,d+1}(\mathbb{k}[\alpha_0, \alpha_1]/\mathfrak{a}) > 2f(d) - f(d+1) - f(d-1)$. Since \mathfrak{a}_d is non-zero, it follows from Lemma 2.12 that we have $2f(d) - f(d+1) - f(d-1) = g(d) - g(d+1) \geq 0$. Let G be the set of all minimal monomial generators of \mathfrak{a} and let G' be obtained from G by deleting $s = g(d) - g(d+1) + 1$ minimal monomial generators of degree $d+1$. Let \mathfrak{a}' be the ideal of $\mathbb{k}[\alpha_0, \alpha_1]$ generated by monomials from G' and let $R = \mathbb{k}[\alpha_0, \alpha_1]/\mathfrak{a}'$. We have $\mathfrak{a}'_d = \mathfrak{a}_d \neq 0$, but

$$H_R(d+1) - H_R(d) = g(d+1) + s - g(d) = 1.$$

This contradicts Lemma 2.12. □

2.1.3 Hilbert function of a power of an ideal of points

We keep the notation of Subsection 2.1.2. In particular, $S = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ is a polynomial ring. The main result of this subsection is Proposition 2.19. Let I be a radical homogeneous ideal in S such that S/I has constant Hilbert polynomial. In the proposition we compute the Hilbert polynomial of S/I^k (for a positive integer k) and bound the degree from which it agrees with the Hilbert function of S/I^k . This result is crucial in the proof of Theorem 3.5.

We begin with studying the following condition on homogeneous ideals J of S :

$$\text{there exists a positive integer } d \text{ with } S_d \subseteq J. \tag{*}$$

Condition (*) is equivalent to the condition that J contains a power of the irrelevant ideal $\mathfrak{m} = (\alpha_0, \alpha_1, \dots, \alpha_n)$. It can be restated geometrically. Namely, a homogeneous ideal J in the polynomial ring S satisfies condition (*) if and only if the corresponding closed subset of the projective space $\text{Proj } S$ is the empty set. See [77, Lem. 1.1] for a proof of this equivalence.

We collect some useful properties of condition (*) in the following lemma. They are probably all well-known. Nevertheless, we could not find a reference for all of them so we present a simple proof.

We stress that in the following lemma, the lower index of I_s does not indicate the degree s part of the homogeneous ideal I , as it usually does in the rest of the thesis.

Lemma 2.14. *Let $m \geq 2$ be an integer and J, K, I_1, \dots, I_m be homogeneous ideals of S .*

(i) *We have $\overline{J \cap K} = \overline{J} \cap \overline{K}$;*

- (ii) $J + K$ satisfies condition (*) if and only if $\sqrt{J} + \sqrt{K}$ satisfies condition (*);
- (iii) If $I_i + I_m$ satisfies condition (*) for all $i = 1, 2, \dots, m-1$, then $I_1 I_2 \cdots I_{m-1} + I_m$ and $I_1 \cap I_2 \cap \cdots \cap I_{m-1} + I_m$ satisfy condition (*);
- (iv) If I_1, I_2, \dots, I_m are homogeneous ideals such that $I_i + I_j$ satisfies condition (*) for all $1 \leq i < j \leq m$, then the equality $\overline{I_1 I_2 \cdots I_m} = \overline{I_1 \cap I_2 \cap \cdots \cap I_m}$ holds.

Proof.

- (i) Let $f \in \overline{J} \cap \overline{K}$. By definition of saturation, there are integers k_1 and k_2 with $\alpha_i^{k_1} f \in J$ and $\alpha_i^{k_2} f \in K$ for all $i = 0, 1, \dots, n$. Therefore, we get $\alpha_i^{\max\{k_1, k_2\}} f \in J \cap K$ for all $i = 0, 1, \dots, n$. Since f was arbitrary, we obtain $\overline{J} \cap \overline{K} \subseteq \overline{J \cap K}$. On the other hand, we have $\overline{J \cap K} \subseteq \overline{J} \cap \overline{K}$ since $J \cap K$ is contained in both J and K .
- (ii) An ideal of S satisfies condition (*) if and only if its radical satisfies condition (*). Therefore, it is enough to observe that we have

$$\sqrt{J + K} = \sqrt{\sqrt{J} + \sqrt{K}}.$$

This follows from the definition of a radical of an ideal, and is well-known [2, Ex. 1.13 v)].

- (iii) By [2, Ex. 1.13 iii)] and induction we have

$$\sqrt{I_1 I_2 \cdots I_{m-1}} = \sqrt{I_1 \cap I_2 \cap \cdots \cap I_{m-1}} = \sqrt{I_1} \cap \sqrt{I_2} \cap \cdots \cap \sqrt{I_{m-1}}.$$

Thus, by part (ii) it is enough to show that $\sqrt{I_1} \cap \sqrt{I_2} \cap \cdots \cap \sqrt{I_{m-1}} + \sqrt{I_m}$ satisfies condition (*). By assumptions there is an integer d with $\alpha_i^d \in I_j + I_m$ for all $i = 0, 1, \dots, n$ and for all $j = 1, 2, \dots, m-1$. It follows that there are elements $s_{ij} \in \sqrt{I_j}$ and $t_{ij} \in \sqrt{I_m}$ satisfying $\alpha_i^d = s_{ij} + t_{ij}$ for all $i = 0, 1, \dots, n$ and for all $j = 1, 2, \dots, m-1$. Multiplying these identities for all $j \in \{1, 2, \dots, m-1\}$ and fixed i , we obtain

$$\alpha_i^{d(m-1)} = \prod_{j=1}^{m-1} (s_{ij} + t_{ij}) = \prod_{j=1}^{m-1} s_{ij} + \left(\prod_{j=1}^{m-1} (s_{ij} + t_{ij}) - \prod_{j=1}^{m-1} s_{ij} \right). \quad (2.15)$$

We have

$$\prod_{j=1}^{m-1} s_{ij} \in \sqrt{I_1} \sqrt{I_2} \cdots \sqrt{I_{m-1}} \subseteq \sqrt{I_1} \cap \sqrt{I_2} \cap \cdots \cap \sqrt{I_{m-1}}$$

and $\prod_{j=1}^{m-1} (s_{ij} + t_{ij}) - \prod_{j=1}^{m-1} s_{ij} \in \sqrt{I_m}$. Hence, by Equation (2.15) α_i belongs to

$$\sqrt{\sqrt{I_1} \cap \sqrt{I_2} \cap \cdots \cap \sqrt{I_{m-1}} + \sqrt{I_m}}$$

for all $i = 0, 1, \dots, n$. It follows that $\sqrt{I_1} \cap \sqrt{I_2} \cap \cdots \cap \sqrt{I_{m-1}} + \sqrt{I_m}$ satisfies condition (*).

- (iv) We prove it by induction on m starting with $m = 2$. The inclusion $\overline{I_1 \cdot I_2} \subseteq \overline{I_1 \cap I_2}$ is a consequence of $I_1 \cdot I_2 \subseteq I_1 \cap I_2$. In order to establish the opposite inclusion, observe that for some positive integer d we have

$$S_d \cdot (I_1 \cap I_2) \subseteq (I_1 + I_2)(I_1 \cap I_2) \subseteq I_1 \cdot I_2$$

by the assumption that $I_1 + I_2$ satisfies condition (*). The inclusion $I_1 \cap I_2 \subseteq \overline{I_1 \cdot I_2}$ follows. Thus, $\overline{I_1 \cap I_2}$ is a subset of $\overline{I_1 \cdot I_2}$.

Let $k \geq 3$ and assume that part (iv) holds for all integers m smaller than k . From part (i) we get $\overline{I_1 \cap I_2 \cap \cdots \cap I_k} = \overline{I_1 \cap I_2 \cap \cdots \cap I_{k-1} \cap I_k}$.

Applying the inductive hypothesis for $m = k - 1$ we conclude that we have

$$\overline{I_1 \cap I_2 \cap \cdots \cap I_k} = \overline{I_1 \cap I_2 \cap \cdots \cap I_{k-1} \cap I_k} = \overline{I_1 I_2 \cdots I_{k-1} \cap I_k}.$$

The ideal $I_1 I_2 \cdots I_{k-1} + I_k$ satisfies condition (*) by part (iii). Therefore, from part (i) and inductive hypothesis for $m = 2$, we get

$$\overline{I_1 \cap I_2 \cap \cdots \cap I_k} = \overline{I_1 I_2 \cdots I_{k-1} \cap I_k} = \overline{(I_1 I_2 \cdots I_{k-1}) \cap I_k} = \overline{I_1 I_2 \cdots I_k},$$

as claimed. \square

The following lemma shows that if two ideals have the same saturation, then their k -th powers for any positive integer k also have the same saturation.

Lemma 2.16. *Let I and J be homogeneous ideals of S and k be a positive integer.*

- (i) *There is an integer d_0 such that for all integers $d_1, \dots, d_k \geq d_0$ the map $\bigotimes_{i=1}^k I_{d_i} \rightarrow I_{d_1+\dots+d_k}^k$ induced by multiplication is surjective;*
- (ii) *If the equality $\overline{I} = \overline{J}$ holds, then we have $\overline{I^k} = \overline{J^k}$.*

Proof.

- (i) Consider a minimal set of homogeneous generators of I . We can take d_0 to be the maximum of degrees of elements of this set. This can be expressed in terms of Betti numbers as $d_0 = \max\{j \mid \beta_{1,j}(S/I) \neq 0\}$.
- (ii) Let $d_0 = \max\{j \mid \beta_{1,j}(S/I) \neq 0\}$ and $e_0 = \max\{j \mid \beta_{1,j}(S/J) \neq 0\}$. Let r_0 be an integer with $I_{\geq r_0} = \overline{I}_{\geq r_0}$ and $J_{\geq r_0} = \overline{J}_{\geq r_0}$. Let $s_0 = \max\{d_0, e_0, r_0\}$. For all $d_1, \dots, d_k \geq s_0$ we have

$$I_{d_1+\dots+d_k}^k = \overline{I}_{d_1+\dots+d_k}^k = \overline{J}_{d_1+\dots+d_k}^k = J_{d_1+\dots+d_k}^k$$

where the first and last equality follow from part (i). We get $\overline{I^k} = \overline{J^k}$. \square

Using the above algebraic results we compute the Hilbert polynomial of a power of a homogeneous radical ideal which defines a closed, zero-dimensional subscheme of projective space.

Lemma 2.17. *Let $I \subseteq S$ be a homogeneous radical ideal such that the Hilbert polynomial of the quotient algebra S/I is constant, equal to r for some positive integer r . For a positive integer k , the Hilbert polynomial of S/I^k is constant equal to $r \cdot \dim_{\mathbb{k}} S_{k-1}$.*

Proof. Let P_1, \dots, P_r be the (distinct) points of the support of $\text{Proj } S/I \subseteq \mathbb{P}^n$. Define \mathfrak{p}_i to be the homogeneous prime ideal of S defining P_i . We have $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ and $\mathfrak{p}_i + \mathfrak{p}_j = \mathfrak{m}$ for all $1 \leq i < j \leq r$. Therefore, by Lemma 2.14(iv), we get $\overline{I} = \overline{J}$, where J is the product $\mathfrak{p}_1 \cdots \mathfrak{p}_r$. Hence $\overline{I^k} = \overline{J^k}$ holds by Lemma 2.16(ii). As a result, it suffices to show that the Hilbert polynomial of S/J^k is $r \cdot \dim_{\mathbb{k}} S_{k-1}$. Let $K = \mathfrak{p}_1^k \cap \cdots \cap \mathfrak{p}_r^k$. Observe that $\mathfrak{p}_i^k + \mathfrak{p}_j^k$ satisfies condition

(*) for every $1 \leq i < j \leq m$ by Lemma 2.14(ii). Therefore, we have $\overline{K} = \overline{J^k}$ by Lemma 2.14(iv). Thus, it is enough to consider the Hilbert polynomial of S/K . As a set, $\text{Proj } S/K$ is the disjoint union of r points P_1, \dots, P_r . Consequently, it is enough to show that the degree of $\text{Proj } S/\mathfrak{p}_i^k$ is $\dim_{\mathbb{k}} S_{k-1}$ for every $i = 1, \dots, r$. This is clear, since up to a linear change of variables \mathfrak{p}_i is the ideal $(\alpha_1, \dots, \alpha_n)$. \square

The following example shows that the assumption in Lemma 2.17 that I is reduced cannot be weakened to the assumption that I is saturated.

Example 2.18. Let $I = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2) \subseteq S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$. It is a saturated ideal and the corresponding subscheme $\text{Proj } S/I \subseteq \mathbb{P}^2$ is zero-dimensional of degree 3. However, $\text{Proj } S/I^2$ has degree 10.

In Lemma 2.17 we have calculated the Hilbert polynomial of S/I^k for a homogeneous radical ideal I defining a zero-dimensional closed subscheme of a projective space and a positive integer k . Now we provide an upper bound on the least degree, from which the Hilbert function of S/I^k agrees with the Hilbert polynomial of S/I^k . The proof uses the notion of regularity. We recall its definition in terms of Betti numbers. For a finitely generated graded S -module M , its regularity $\text{reg } M$ is defined to be $\text{reg } M = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}$.

Proposition 2.19. *Let r and k be positive integers and $I \subseteq S$ be a homogeneous radical ideal with the Hilbert polynomial of the quotient algebra S/I equal to r . Define $e = \min\{a \in \mathbb{Z} \mid H_{S/I}(a) = r\}$. We have $H_{S/I^k}(d) = r \cdot \dim_{\mathbb{k}} S_{k-1}$ for every $d \geq ke + k$.*

Proof. The Hilbert polynomial of S/I^k is $r \cdot \dim_{\mathbb{k}} S_{k-1}$ by Lemma 2.17. Therefore, we are left with establishing the bound on the degree from which the Hilbert function agrees with the Hilbert polynomial. This is related to the regularity. By [32, Thm. 4.2], it is enough to show that the inequality

$$ke + k - 1 \geq \text{reg } S/I^k \tag{2.20}$$

holds. We have $\text{reg } S/I = e$ by [32, Thm. 4.2]. Hence, from the definition of regularity in terms of Betti numbers, we get $\text{reg } I = e + 1$. Thus, $\text{reg } I^k \leq ke + k$ is a consequence of [22, Thm. 6]. Inequality (2.20) follows. \square

Unlike in Lemma 2.17, to obtain the bound on the degree from which the Hilbert function agrees with the Hilbert polynomial, the assumption in Proposition 2.19 that I is radical could be replaced by the weaker assumption that I is saturated. However, we need to control both the value of the Hilbert polynomial and the degree from which the Hilbert function has this value. Therefore, we need to restrict our attention to radical ideals.

2.1.4 Tangent space at extended ideal

In this subsection we consider polynomial ring $T = \mathbb{k}[\alpha_0, \alpha_1]$. The main result is Proposition 2.22 which computes

$$\dim_{\mathbb{k}} \text{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{>0}$$

for a monomial ideal $\mathfrak{a} \neq T$ of T such that $\dim_{\mathbb{k}} T/\mathfrak{a}$ is finite. This is later used to compute the dimension of the tangent space to the multigraded Hilbert scheme at the point corresponding to the extended ideal $\mathfrak{a}^{ex} \subseteq \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$.

If M and N are graded T -modules and M is finitely generated then the Ext groups $\text{Ext}_T^i(M, N)$ are graded T -modules in a natural way (see [10, §1.5]). For a graded T -module M and an integer d , by $M(d)$ we denote the graded T -module given by $M(d)_e = M_{e+d}$ for all $e \in \mathbb{Z}$.

Let \mathfrak{a} be a monomial ideal in T with $\dim_{\mathbb{k}} T/\mathfrak{a} = r$ for a positive integer r . We can consider the associated staircase diagram (see [68, §3.1]). We recall its construction. For each pair of non-negative integers (s, t) with $\alpha_0^s \alpha_1^t \notin \mathfrak{a}$ put a 1×1 box with sides parallel to coordinate axis and (s, t) as lower left corner of the box. The diagram corresponding to \mathfrak{a} is denoted by $\mathcal{D}_{\mathfrak{a}}$. The set of boxes of the diagram $\mathcal{D}_{\mathfrak{a}}$ (or, the set of monomials outside \mathfrak{a}) is denoted by $\Lambda_{\mathfrak{a}}$. There is a canonical minimal free resolution of T/\mathfrak{a} (see [68, Prop. 3.1]). The set of minimal monomial generators of \mathfrak{a} is denoted by $M_{\mathfrak{a}}$ and the generating set of relations (or more precisely the set of their degrees when T is considered with the natural \mathbb{Z}^2 -grading) used in that resolution is denoted by $R_{\mathfrak{a}}$.

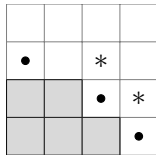


Figure 2.1: Staircase diagram of the ideal $\mathfrak{a} = (\alpha_0^3, \alpha_0^2 \alpha_1, \alpha_1^2)$.

Example 2.21. Figure 2.1 presents the staircase diagram of $\mathfrak{a} = (\alpha_0^3, \alpha_0^2 \alpha_1, \alpha_1^2)$. Filled boxes correspond to monomials outside \mathfrak{a} (i.e. elements of $\Lambda_{\mathfrak{a}}$), dots correspond to elements of $M_{\mathfrak{a}}$ (i.e. minimal monomial generators of \mathfrak{a}) and asterisks correspond to elements of $R_{\mathfrak{a}}$ (i.e. minimal relations between those generators).

We have $\#\Lambda_{\mathfrak{a}} = \dim_{\mathbb{k}} T/\mathfrak{a} = r$ and $\#R_{\mathfrak{a}} = \#M_{\mathfrak{a}} - 1$. We identify monomials of T with lattice points in \mathbb{Z}^2 . Given a point $\mathbf{u} = (s, t)$ in \mathbb{Z}^2 we write $|\mathbf{u}|$ for $s + t$. We define three functions from integers to integers:

$$\begin{aligned} \lambda_{\mathfrak{a}}(a) &= H_{T/\mathfrak{a}}(a) = \#\{\mathbf{u} \in \Lambda_{\mathfrak{a}} \mid |\mathbf{u}| = a\}, \\ \mu_{\mathfrak{a}}(a) &= \beta_{1,a}(T/\mathfrak{a}) = \#\{\mathbf{u} \in M_{\mathfrak{a}} \mid |\mathbf{u}| = a\}, \\ \rho_{\mathfrak{a}}(a) &= \beta_{2,a}(T/\mathfrak{a}) = \#\{\mathbf{u} \in R_{\mathfrak{a}} \mid |\mathbf{u}| = a\}. \end{aligned}$$

The goal of this subsection is the proof of the following proposition.

Proposition 2.22. *Let r be a positive integer. Given a monomial ideal \mathfrak{a} in T with $\dim_{\mathbb{k}} T/\mathfrak{a} = r$ we have*

$$\dim_{\mathbb{k}} \text{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{>0} = \sum_{\mathbf{u} \in M_{\mathfrak{a}}} \sum_{a > |\mathbf{u}|} \lambda_{\mathfrak{a}}(a) - \sum_{\mathbf{u} \in R_{\mathfrak{a}}} \sum_{a > |\mathbf{u}|} \lambda_{\mathfrak{a}}(a). \quad (2.23)$$

Observe that [34, Lem. 3.2] presents a more general formula for $\dim_{\mathbb{k}} \text{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{\mathbf{u}}$ where $\mathbf{u} \in \mathbb{Z}^2$ and we consider T with the natural \mathbb{Z}^2 -grading.

The proof of Proposition 2.22 is based on the following observation.

Lemma 2.24. *Let \mathfrak{a} be a monomial ideal in T such that T/\mathfrak{a} is a finite \mathbb{k} -vector space.*

(i) *The natural map $T \rightarrow \text{Hom}_T(\mathfrak{a}, T)$ given by $f \mapsto (g \mapsto fg)$ is an isomorphism of graded T -modules.*

(ii) *We have $\text{Ext}_T^1(\mathfrak{a}, T/\mathfrak{a})_{>0} = 0$.*

Proof. Since $\dim_{\mathbb{k}} T/\mathfrak{a}$ is finite, \mathfrak{a} is equal to $(\alpha_0^{a_0}, \alpha_0^{a_1} \alpha_1^{b_1}, \dots, \alpha_0^{a_{s-1}} \alpha_1^{b_{s-1}}, \alpha_1^{b_s})$ for some positive integers $a_0 > a_1 > \dots > a_{s-1}$ and $b_1 < b_2 < \dots < b_s$. Set $a_s = b_0 = 0$.

- (i) Let $\varphi: \mathfrak{a} \rightarrow T$ be a homomorphism of T -modules. It is enough to show that there exists an element $f \in T$ with $\varphi(g) = fg$ for every $g \in \mathfrak{a}$. Define $f_i = \varphi(\alpha_0^{a_i} \alpha_1^{b_i})$ for all $i = 0, \dots, s$. For each $i \in \{1, \dots, s\}$ we have relations of the form

$$\alpha_1^{b_i - b_{i-1}} f_{i-1} = \varphi(\alpha_0^{a_{i-1}} \alpha_1^{b_i}) = \alpha_0^{a_{i-1} - a_i} f_i. \quad (2.25)$$

From Equation (2.25) for $i = s$ we deduce that $\alpha_0^{a_{s-1}}$ divides f_{s-1} . It follows by induction that $\alpha_0^{a_i}$ divides f_i for all i . Thus, we have $f_0 = \alpha_0^{a_0} f$ for some $f \in T$. From Equation (2.25) we conclude that f_i is equal to $\alpha_0^{a_i} \alpha_1^{b_i} f$ for each i .

- (ii) We start with showing that $\text{Ext}_T^1(\mathfrak{a}, T)_{>0}$ is zero. Consider the canonical minimal graded free resolution

$$\bigoplus_{i=1}^s T(-a_{i-1} - b_i) \rightarrow \bigoplus_{i=0}^s T(-a_i - b_i) \rightarrow \mathfrak{a} \rightarrow 0$$

of \mathfrak{a} (see [68, Prop. 3.1]). Applying the functor $\text{Hom}_T(-, T)$ to the above resolution, we obtain for every integer c a \mathbb{k} -linear map

$$\psi_c: \bigoplus_{i=0}^s T(a_i + b_i)_c \rightarrow \bigoplus_{i=1}^s T(a_{i-1} + b_i)_c.$$

We claim that ψ_c is surjective for every $c > 0$. Observe that we have $\ker \psi_c \cong \text{Hom}_T(\mathfrak{a}, T)_c \cong T_c$ by part (i). Therefore, the claim is a consequence of the calculation

$$\dim_{\mathbb{k}} \bigoplus_{i=0}^s T_{a_i + b_i + c} = (s+1)(c+1) + \sum_{i=1}^s (a_{i-1} + b_i) = \dim_{\mathbb{k}} \bigoplus_{i=1}^s T_{a_{i-1} + b_i + c} + \dim_{\mathbb{k}} T_c.$$

Since ψ_c is surjective for every positive integer c , it follows that $\text{Ext}_T^1(\mathfrak{a}, T)_{>0}$ is zero.

Now we prove that we have $\text{Ext}_T^1(\mathfrak{a}, T/\mathfrak{a})_{>0} = 0$. Consider the following part of the long exact sequence of Ext groups obtained from the short exact sequence $0 \rightarrow \mathfrak{a} \rightarrow T \rightarrow T/\mathfrak{a} \rightarrow 0$ by applying the functor $\text{Hom}_T(\mathfrak{a}, -)$:

$$\dots \rightarrow \text{Ext}_T^1(\mathfrak{a}, T)_{>0} \rightarrow \text{Ext}_T^1(\mathfrak{a}, T/\mathfrak{a})_{>0} \rightarrow \text{Ext}_T^2(\mathfrak{a}, \mathfrak{a})_{>0} \rightarrow \dots$$

We have shown the equality $\text{Ext}_T^1(\mathfrak{a}, T)_{>0} = 0$. Moreover, $\text{Ext}_T^2(\mathfrak{a}, \mathfrak{a})_{>0}$ is trivial since \mathfrak{a} has projective dimension 1. The equality $\text{Ext}_T^1(\mathfrak{a}, T/\mathfrak{a})_{>0} = 0$ follows. \square

Proof of Proposition 2.22. Consider the canonical minimal free resolution of \mathfrak{a}

$$0 \rightarrow \bigoplus_{a \in \mathbb{Z}} T(-a)^{\rho_{\mathfrak{a}}(a)} \rightarrow \bigoplus_{b \in \mathbb{Z}} T(-b)^{\mu_{\mathfrak{a}}(b)} \rightarrow \mathfrak{a} \rightarrow 0.$$

Applying the functor $\text{Hom}_T(-, T/\mathfrak{a})_{>0}$ and using Lemma 2.24(ii) we get an exact sequence

$$0 \rightarrow \text{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{>0} \rightarrow \bigoplus_{b \in \mathbb{Z}} \text{Hom}_T(T(-b)^{\mu_{\mathfrak{a}}(b)}, T/\mathfrak{a})_{>0} \rightarrow \bigoplus_{a \in \mathbb{Z}} \text{Hom}_T(T(-a)^{\rho_{\mathfrak{a}}(a)}, T/\mathfrak{a})_{>0} \rightarrow 0.$$

This can be rewritten as

$$0 \rightarrow \text{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{>0} \rightarrow \bigoplus_{b \in \mathbb{Z}} (T/\mathfrak{a})_{>b}^{\mu \mathfrak{a}(b)} \rightarrow \bigoplus_{a \in \mathbb{Z}} (T/\mathfrak{a})_{>a}^{\rho \mathfrak{a}(a)} \rightarrow 0.$$

Thus, we have

$$\dim_{\mathbb{k}} \text{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{>0} = \sum_{\mathbf{u} \in M_{\mathfrak{a}}} \sum_{c > |\mathbf{u}|} \dim_{\mathbb{k}} (T/\mathfrak{a})_c - \sum_{\mathbf{u} \in R_{\mathfrak{a}}} \sum_{c > |\mathbf{u}|} \dim_{\mathbb{k}} (T/\mathfrak{a})_c.$$

This is equivalent to Equation (2.23). □

We end this subsection with an example.

Example 2.26. Let $\mathfrak{a} = (\alpha_0^3, \alpha_0^2 \alpha_1, \alpha_1^6)$. Its staircase diagram is presented in Figure 2.2. For this ideal Equation (2.23) takes the form

$$\dim_{\mathbb{k}} \text{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{>0} = (2 \cdot \sum_{a>3} \lambda_{\mathfrak{a}}(a) + \sum_{a>6} \lambda_{\mathfrak{a}}(a)) - (\sum_{a>4} \lambda_{\mathfrak{a}}(a) + \sum_{a>8} \lambda_{\mathfrak{a}}(a)) = (2 \cdot 5 + 0) - (3 + 0) = 7.$$

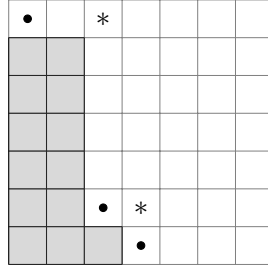


Figure 2.2: Staircase diagram of the ideal $\mathfrak{a} = (\alpha_0^3, \alpha_0^2 \alpha_1, \alpha_1^6)$.

2.1.5 Dimensions of Ext groups

In Lemmas 2.27 and 2.28 we present general results about finitely generated modules over polynomial rings. They are used in Chapter 3.

Lemma 2.27. *Let n be a positive integer and $S = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ be a polynomial ring. Let M be a finitely generated graded S -module. We have*

$$\sum_{i=0}^{n+1} (-1)^i \dim_{\mathbb{k}} \text{Ext}_S^i(\mathbb{k}, M)_e = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} \dim_{\mathbb{k}} M_{e+i}$$

for every $e \in \mathbb{Z}$.

Proof. Let P_{\bullet} be the Koszul resolution of \mathbb{k} . Applying the functor $\text{Hom}_S(-, M)_e$ we obtain a complex

$$\text{Hom}_S(P_{\bullet}, M)_e,$$

whose cohomology groups are $\text{Ext}_S^i(\mathbb{k}, M)_e$ for all $i = 0, \dots, n+1$. Therefore, by a standard

argument (see [81, Ex. 1.6.B]) by splitting the above complex into short exact sequences we get

$$\sum_{i=0}^{n+1} (-1)^i \dim_{\mathbb{k}} \text{Ext}_S^i(\mathbb{k}, M)_e = \sum_{i=0}^{n+1} (-1)^i \dim_{\mathbb{k}} \text{Hom}_S(P_i, M)_e.$$

Since there is an isomorphism $P_i \cong S(-i)^{\binom{n+1}{i}}$, we obtain the following equality

$$\dim_{\mathbb{k}} \text{Hom}_S(P_i, M)_e = \binom{n+1}{i} \dim_{\mathbb{k}} M_{e+i}. \quad \square$$

Lemma 2.28. *Let n be a positive integer and $S = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ be a polynomial ring. Given a finitely generated graded S -module M and an integer $e \in \mathbb{Z}$ we have $\dim_{\mathbb{k}} \text{Ext}_S^i(M, \mathbb{k})_e = \beta_{i, -e}(M)$.*

Proof. Apply the functor $\text{Hom}_S(-, \mathbb{k})$ to a minimal graded free resolution P_{\bullet} of M . The Ext groups $\text{Ext}_S^i(M, \mathbb{k})$ can be computed as cohomology groups of the obtained complex. Since the i -th differential in P_{\bullet} maps P_i into $\mathfrak{m}P_{i-1}$, the differentials in the complex $\text{Hom}_S(P_{\bullet}, \mathbb{k})$ are zero. Therefore, we have $\dim_{\mathbb{k}} \text{Ext}_S^i(M, \mathbb{k})_e = \dim_{\mathbb{k}} \text{Hom}_S(P_i, \mathbb{k})_e = \beta_{i, -e}(M)$. \square

2.2 Scheme theory and multigraded Hilbert schemes

In this section we give the definition of multigraded Hilbert schemes and study the basic properties of these schemes. In Subsection 2.2.1 we present general results from scheme theory that are used in the proofs of the main results. Next subsection contains a formal definition of a multigraded Hilbert scheme and its basic properties. In Subsection 2.2.3 we introduce multigraded Hilbert schemes "of points in general position". This is the main object of investigation in this thesis. The final subsection concerns the flag multigraded Hilbert scheme. This scheme is defined by its functor of points and we prove its existence using existence of multigraded Hilbert schemes.

2.2.1 Scheme theory

The following lemma gives some conditions under which a morphism of \mathbb{k} -schemes that is bijective on \mathbb{k} -valued points is a homeomorphism.

Lemma 2.29. *Let $f: X \rightarrow Y$ be a closed morphism of schemes locally of finite type over \mathbb{k} . If f induces a bijection of \mathbb{k} -valued points $X(\mathbb{k}) \rightarrow Y(\mathbb{k})$, then f is a homeomorphism.*

Proof. In both X and Y , closed points are very dense by [41, Prop. 3.35]. Since f induces a bijection of closed points, it is dominant and hence surjective. Moreover, if we have $f(p) = f(q)$, then there is a chain of equalities $f(\overline{\{p\}}) = \overline{\{f(p)\}} = \overline{\{f(q)\}} = f(\overline{\{q\}})$. As a result, the sets of closed points of X that are contained in $\overline{\{p\}}$ and $\overline{\{q\}}$ are equal. It follows that the closures of $\{p\}$ and $\{q\}$ coincide and therefore, we get $p = q$. This shows that $f: X \rightarrow Y$ is a bijective, closed, continuous map and thus, a homeomorphism. \square

In describing the intersection of irreducible components of some multigraded Hilbert schemes we use the following lemma.

Lemma 2.30. *Let X be a scheme locally of finite type over \mathbb{k} . Let Z_1 and Z_2 be irreducible closed subsets of X of dimensions d_1 and d_2 , respectively. Let $P \in Z_1 \cap Z_2$ be a closed point of the intersection and let $d = \dim_{\mathbb{k}} \mathbf{T}_P X$. Every irreducible component W of $Z_1 \cap Z_2$ such that $P \in W$ satisfies $\dim W \geq d_1 + d_2 - d$.*

Proof. By [80, Tag 0C2G], there exists an open neighborhood U of P in X and a closed immersion $i: U \rightarrow Y$ where Y is a smooth d -dimensional variety over \mathbb{k} . Let $W_1 = i(|Z_1| \cap |U|)$ and $W_2 = i(|Z_2| \cap |U|)$, where $|\cdot|$ denotes the underlying topological space. These are d_1 and d_2 -dimensional irreducible closed subsets of Y , respectively. Therefore, every irreducible component of $W_1 \cap W_2$ has dimension at least $d_1 + d_2 - d$ (see [37, §8.2]). \square

2.2.2 Multigraded Hilbert schemes

We denote the categories of sets, \mathbb{k} -algebras and \mathbb{k} -schemes by \mathfrak{Set} , $\mathbb{k}\text{-Alg}$ and $\mathfrak{Sch}_{\mathbb{k}}$, respectively. In this subsection we introduce multigraded Hilbert schemes following [46]. We give the definition in terms of the functor of points $\underline{\text{Hilb}}_S^h: \mathbb{k}\text{-Alg} \rightarrow \mathfrak{Set}$ and then verify that the scheme represents the natural extension to the functor $\underline{\text{Hilb}}_S^h: \mathfrak{Sch}_{\mathbb{k}}^{\text{op}} \rightarrow \mathfrak{Set}$.

Let n be a positive integer and let $S = \mathbb{k}[\underline{\alpha}] := \mathbb{k}[\alpha_0, \dots, \alpha_n]$ be the polynomial ring over \mathbb{k} . We identify monomials of S with \mathbb{N}^{n+1} . Let \mathbf{A} be an abelian group and let $\deg: \mathbb{N}^{n+1} \rightarrow \mathbf{A}$ be a homomorphism of semigroups. We assume that \mathbf{A} is generated by $\deg(\alpha_i)$ for all $i = 0, \dots, n$. We consider S with the \mathbf{A} -grading induced by \deg

$$S = \bigoplus_{a \in \mathbf{A}} S_a \text{ satisfying } S_a \cdot S_b \subseteq S_{a+b},$$

where S_a is the \mathbb{k} -vector space spanned by monomials $\alpha^{\mathbf{u}}$ with $\deg(\mathbf{u}) = a$. Given a \mathbb{k} -algebra R , we write $R[\underline{\alpha}]$ for $S \otimes_{\mathbb{k}} R$ together with the \mathbf{A} -grading given by $R[\underline{\alpha}]_a = S_a \otimes_{\mathbb{k}} R$.

Definition 2.31. Given a \mathbb{k} -algebra R and a function $h: \mathbf{A} \rightarrow \mathbb{N}$, we say that a homogeneous ideal $I \subseteq R[\underline{\alpha}]$ is *admissible for Hilbert function h* if $R[\underline{\alpha}]_a/I_a$ is a locally free R -module of rank $h(a)$ for every $a \in \mathbf{A}$.

We define the functor $\underline{\text{Hilb}}_S^h: \mathbb{k}\text{-Alg} \rightarrow \mathfrak{Set}$ by

$$R \mapsto \{I \subseteq R[\underline{\alpha}] \mid I \text{ is an admissible ideal for Hilbert function } h\}$$

and given $\varphi: R \rightarrow R'$ we define

$$\underline{\text{Hilb}}_S^h(\varphi): I \mapsto I \otimes_R R'.$$

The following lemma confirms that the above data define a functor.

Lemma 2.32. *Let R be a \mathbb{k} -algebra and $I \subseteq R[\underline{\alpha}]$ be an admissible ideal for Hilbert function h . If $\varphi: R \rightarrow R'$ is a homomorphism of \mathbb{k} -algebras, then $I \otimes_R R'$ is an ideal of $R'[\underline{\alpha}]$, admissible for Hilbert function h .*

Proof. Consider the exact sequence of R -modules

$$0 \rightarrow I \rightarrow R[\underline{\alpha}] \rightarrow R[\underline{\alpha}]/I \rightarrow 0.$$

Since $R[\underline{\alpha}]_a/I_a$ is a locally free R -module of finite rank it is flat (see [6, Prop. 3 §4.4]). Therefore, by tensoring the above sequence with R' over R we obtain an exact sequence of R' -modules

$$0 \rightarrow I \otimes_R R' \rightarrow R'[\underline{\alpha}] \rightarrow R'[\underline{\alpha}]/(I \otimes_R R') \rightarrow 0.$$

Since $I \otimes_R R'$ is an R' -submodule of $R'[\underline{\alpha}]$ stable under multiplication by any monomial, it is an ideal. We show that $R'[\underline{\alpha}]_a/(I_a \otimes_R R')$ is a locally free R' -module of rank $h(a)$ for every $a \in \mathbf{A}$. Let $\mathfrak{q} \in \text{Spec } R'$ and let $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Given $a \in \mathbf{A}$, there is an element $f \in R \setminus \mathfrak{p}$ with $(R[\underline{\alpha}]_a/I_a)_f \cong R_f^{h(a)}$. Let $g = \varphi(f)$. We have $g \in R' \setminus \mathfrak{q}$ and we show that there is an isomorphism $(R'[\underline{\alpha}]_a/(I_a \otimes_R R'))_g \cong (R'_g)^{h(a)}$. Indeed, this is a consequence of the following chain of isomorphisms

$$\begin{aligned} (R'[\underline{\alpha}]_a/(I_a \otimes_R R'))_g &\cong ((R[\underline{\alpha}]_a/I_a) \otimes_R R')_g \cong (R[\underline{\alpha}]_a/I_a) \otimes_R R'_g \cong \\ &\cong (R[\underline{\alpha}]_a/I_a)_f \otimes_{R_f} R'_g \cong R_f^{h(a)} \otimes_{R_f} R'_g \cong (R'_g)^{h(a)}. \end{aligned} \quad \square$$

The following existence statement is the foundation of the theory of multigraded Hilbert schemes.

Theorem 2.33 ([46, Thm. 1.1]). *Let $S = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ be a polynomial ring graded by an abelian group \mathbf{A} . Let $h: \mathbf{A} \rightarrow \mathbb{N}$ be a numerical function. There exists a quasiprojective \mathbb{k} -scheme Hilb_S^h representing the functor $\underline{\text{Hilb}}_S^h: \mathbb{k}\text{-Alg} \rightarrow \mathfrak{S}et$.*

Moreover, under additional assumptions on the grading $\text{deg}: \mathbb{N}^{n+1} \rightarrow \mathbf{A}$, the scheme Hilb_S^h is projective.

Definition 2.34. Let $S = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ be a polynomial ring and \mathbf{A} be an abelian group. The grading of S given by a semigroup homomorphism $\text{deg}: \mathbb{N}^{n+1} \rightarrow \mathbf{A}$ is called *positive* if $S_0 = \mathbb{k}$.

In the cases that are studied in this thesis, the grading is positive.

Example 2.35.

- (a) The homogeneous coordinate ring $S[\mathbb{P}^n] = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ of projective space \mathbb{P}^n is \mathbb{Z} -graded by $\text{deg}(\alpha_i) = 1$. This grading is positive.
- (b) More generally, let X be a smooth projective toric variety over the field of complex numbers. Its Cox ring $S[X]$ is a $\text{Pic}(X)$ -graded polynomial ring and $S[X]_0 = \Gamma(X, \mathcal{O}_X) = \mathbb{C}$. See Chapter 4 for more about smooth projective toric varieties and their Cox rings.

The main consequence of the assumption that S is positively graded, is the following result.

Theorem 2.36 ([46, Cor. 1.2.]). *Let $S = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ be a polynomial ring graded by an abelian group \mathbf{A} . If the grading is positive then for every function $h: \mathbf{A} \rightarrow \mathbb{N}$, the multigraded Hilbert scheme Hilb_S^h is projective over \mathbb{k} .*

Moreover, there is also an algebraic consequence of this restriction on the grading.

Theorem 2.37 ([68, Thm. 8.6]). *Let $S = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ be a polynomial ring graded by an abelian group \mathbf{A} . The following conditions are equivalent:*

1. $S_0 = \mathbb{k}$;

2. S_a is a finite-dimensional \mathbb{k} -vector space for every $a \in \mathbf{A}$.

Now we discuss the extension of the functor $\widetilde{\text{Hilb}}_S^h$ to the category $\mathfrak{Sch}_{\mathbb{k}}^{op}$. The definition comes from [68, §18.5]. Then we check that the extended functor is indeed the functor of points of Hilb_S^h .

Definition 2.38. Let X be a \mathbb{k} -scheme and $h: \mathbf{A} \rightarrow \mathbb{N}$ be a function. We say that a closed subscheme $Z \subseteq \mathbb{A}_X^{n+1}$ is an *admissible family* for the function h if for every affine open subscheme $\text{Spec } R \subseteq X$, the pullback of the ideal sheaf of Z to \mathbb{A}_R^{n+1} corresponds to an ideal of $R[\underline{\alpha}]$ that is admissible for the function h .

As is often the case, this condition can be checked on any affine open covering.

Lemma 2.39. Let $h: \mathbf{A} \rightarrow \mathbb{N}$ be a function. Let X be a \mathbb{k} -scheme with an affine open covering $X = \bigcup_{j \in J} U_j$ with $U_j = \text{Spec } R_j$. Let $Z \subseteq \mathbb{A}_X^{n+1}$ be a closed subscheme such that the pullback of its ideal sheaf to $\mathbb{A}_{R_j}^{n+1}$ is an admissible ideal of $R_j[\underline{\alpha}]$ for the function h for every $j \in J$. The subscheme Z is an admissible family over X for the function h .

Proof. Let $V = \text{Spec } R$ be an affine open subscheme of X and let I be the ideal in $R[\underline{\alpha}]$ corresponding to the pullback of the ideal sheaf of Z . The ideal I is an admissible ideal for the function h if and only if for each a in \mathbf{A} , the degree a part of $\widetilde{R[\underline{\alpha}]/I}$ is a locally free sheaf of rank $h(a)$. This can be checked on any affine open covering of \mathbb{A}_R^{n+1} . Given a point $P \in \mathbb{A}_R^{n+1}$ there is an affine open subscheme $U = \text{Spec } T \subseteq V \cap U_j$ for some $j \in J$ that is a distinguished open subscheme of U_j and such that P is in \mathbb{A}_T^{n+1} . It is enough to show that the pullback of the ideal sheaf of Z to \mathbb{A}_T^{n+1} corresponds to an admissible ideal. Since this is true for the pullback to $\mathbb{A}_{R_j}^{n+1}$, the claim follows from Lemma 2.32. \square

We define the natural extension of the multigraded Hilbert scheme functor to the category $\mathfrak{Sch}_{\mathbb{k}}^{op}$, opposite of the category of \mathbb{k} -schemes. For now we denote this functor by $\widetilde{\text{Hilb}}_S^h$.

Given a \mathbb{k} -scheme X , let

$$\widetilde{\text{Hilb}}_S^h(X) = \{ \text{admissible families } Z \subseteq \mathbb{A}_X^{n+1} \text{ for the function } h \}$$

and for a morphism $f: X \rightarrow Y$ of \mathbb{k} -schemes let

$$\widetilde{\text{Hilb}}_S^h(f): Z \mapsto (f \times \text{id}_{\mathbb{A}^{n+1}})^{-1}(Z).$$

The following lemma checks that this functor is well-defined.

Lemma 2.40. Let $f: X \rightarrow Y$ be a morphism of \mathbb{k} -schemes. If $Z \subseteq \mathbb{A}_Y^{n+1}$ is an admissible family over Y , then the scheme theoretic inverse image

$$(f \times \text{id}_{\mathbb{A}^{n+1}})^{-1}(Z) \subseteq \mathbb{A}_X^{n+1}$$

is an admissible family over X .

Proof. To simplify notation, let $Z' = (f \times \text{id}_{\mathbb{A}^{n+1}})^{-1}(Z)$. We may choose affine open subschemes $\text{Spec } T \subseteq X$ and $\text{Spec } R \subseteq Y$ with $f(\text{Spec } T) \subseteq \text{Spec } R$. By Lemma 2.39, it is enough to show that the pullback of Z' to \mathbb{A}_T^{n+1} , denoted by Z'_T , corresponds to an admissible ideal of $T[\underline{\alpha}]$. Let Z_R denote the pullback of Z to \mathbb{A}_R^{n+1} . We have the following diagrams with all inner squares pullback diagrams:

$$\begin{array}{ccccccc}
\mathbb{A}_T^{n+1} \times_{\mathbb{A}_R^{n+1}} Z_R & \longrightarrow & Z_R & \longrightarrow & Z & & Z'_T \longrightarrow Z' \longrightarrow Z \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{A}_T^{n+1} & \longrightarrow & \mathbb{A}_R^{n+1} & \longrightarrow & \mathbb{A}_Y^{n+1} & & \mathbb{A}_T^{n+1} & \longrightarrow & \mathbb{A}_X^{n+1} & \longrightarrow & \mathbb{A}_Y^{n+1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Spec } T & \longrightarrow & \text{Spec } R & \longrightarrow & Y & & \text{Spec } T & \longrightarrow & X & \longrightarrow & Y.
\end{array}$$

Since Z is an admissible ideal over Y , Z_R corresponds to an admissible ideal of $R[\underline{\alpha}]$. Thus, $\mathbb{A}_T^{n+1} \times_{\mathbb{A}_R^{n+1}} Z_R$ corresponds to an admissible ideal of $T[\underline{\alpha}]$ by Lemma 2.32. Since the bottom arrows from $\text{Spec } T$ to Y in both diagrams are the same, so are the middle arrows $\mathbb{A}_T^{n+1} \rightarrow \mathbb{A}_Y^{n+1}$. Therefore, we have $Z'_T = \mathbb{A}_T^{n+1} \times_{\mathbb{A}_R^{n+1}} Z_R$ as a closed subscheme of \mathbb{A}_T^{n+1} which finishes the proof. \square

As expected, the multigraded Hilbert scheme represents this extended functor.

Lemma 2.41. *Let $S = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ be a polynomial ring graded by an abelian group \mathbf{A} . Let $h: \mathbf{A} \rightarrow \mathbb{N}$ be a function. The functor $\widetilde{\text{Hilb}}_S^h: \mathfrak{Sch}_{\mathbb{k}}^{\text{op}} \rightarrow \mathfrak{Set}$ defined above is the functor of points of the multigraded Hilbert scheme Hilb_S^h .*

Proof. Let $h_{\text{Hilb}_S^h}$ be the functor of points of Hilb_S^h . Let Y be a \mathbb{k} -scheme and cover it by affine open subschemes $\{V_{i_s}\}_{i_s \in I}$. Moreover, cover each intersection $V_{i_s} \cap V_{i_t}$ by affine open subschemes $V_{i_s i_t k}$ for $k \in I_{i_s i_t}$. Since data of an admissible family over Y is affine local on Y , we have an equalizer diagram of sets

$$\widetilde{\text{Hilb}}_S^h(Y) \rightarrow \prod_{i_s \in I} \widetilde{\text{Hilb}}_S^h(V_{i_s}) \rightrightarrows \prod_{i_s \in I} \prod_{i_t \in I} \prod_{k \in I_{i_s i_t}} \widetilde{\text{Hilb}}_S^h(V_{i_s i_t k}).$$

See [75, p. 225] for the definition and universal property of equalizer. Since $h_{\text{Hilb}_S^h}$ is representable, we also have an equalizer diagram of sets

$$h_{\text{Hilb}_S^h}(Y) \rightarrow \prod_{i_s \in I} h_{\text{Hilb}_S^h}(V_{i_s}) \rightrightarrows \prod_{i_s \in I} \prod_{i_t \in I} \prod_{k \in I_{i_s i_t}} h_{\text{Hilb}_S^h}(V_{i_s i_t k}).$$

Since $\widetilde{\text{Hilb}}_S^h$ and $h_{\text{Hilb}_S^h}$ are isomorphic when restricted to the category of affine \mathbb{k} -schemes, the middle and right terms of the above sequences are naturally isomorphic. Thus, we have an isomorphism $\widetilde{\text{Hilb}}_S^h(Y) \rightarrow h_{\text{Hilb}_S^h}(Y)$ by the universal property of equalizer.

We claim that this isomorphism is natural with respect to $f: X \rightarrow Y$ so that it defines a natural isomorphism of functors between $\widetilde{\text{Hilb}}_S^h$ and $h_{\text{Hilb}_S^h}$. Indeed, we may choose an affine open covering $\{U_{j_a}\}_{j_a \in J}$ of X refining the open covering of X by preimages of V_{i_s} 's for $i_s \in I$. Let $\gamma: J \rightarrow I$ be the corresponding map of indexing sets, such that U_{j_a} is contained in $f^{-1}(V_{\gamma(j_a)})$ for every $j_a \in J$. Furthermore, we can cover each $U_{j_a} \cap U_{j_b}$ with $j_a, j_b \in J$ by affine open subsets $U_{j_a j_b l}$ for $l \in J_{j_a j_b}$ refining the open covering of $U_{j_a} \cap U_{j_b}$ by preimages of $V_{\gamma(j_a)\gamma(j_b)k}$ for $k \in I_{\gamma(j_a)\gamma(j_b)}$. We have a commutative diagram

$$\begin{array}{ccc}
\widetilde{\text{Hilb}}_S^h(Y) & \longrightarrow & \prod_{i_s \in I} \widetilde{\text{Hilb}}_S^h(V_{i_s}) \rightrightarrows \prod_{i_s \in I} \prod_{i_t \in I} \prod_{k \in I_{i_s i_t}} \widetilde{\text{Hilb}}_S^h(V_{i_s i_t k}) \\
\downarrow \widetilde{\text{Hilb}}_S^h(f) & & \downarrow \\
\widetilde{\text{Hilb}}_S^h(X) & \longrightarrow & \prod_{j_a \in J} \widetilde{\text{Hilb}}_S^h(U_{j_a}) \rightrightarrows \prod_{j_a \in J} \prod_{j_b \in J} \prod_{l \in J_{j_a j_b}} \widetilde{\text{Hilb}}_S^h(U_{j_a j_b l})
\end{array}$$

and a similar one for $h_{\text{Hilb}_S^h}$. Since the isomorphisms

$$\alpha: \widetilde{\text{Hilb}_S^h}(Y) \rightarrow h_{\text{Hilb}_S^h}(Y) \text{ and } \beta: \widetilde{\text{Hilb}_S^h}(X) \rightarrow h_{\text{Hilb}_S^h}(X)$$

are induced from the universal property of equalizer and isomorphism of restricted functors, the equality $\beta \circ \widetilde{\text{Hilb}_S^h}(f) = h_{\text{Hilb}_S^h}(f) \circ \alpha$ follows from the universal property of equalizer. \square

We end this subsection with two technical results. The first of them is concerned with the fact that smoothness of $[I] \in \text{Hilb}_S^h$ "does not depend" on I_1 . The following lemma makes it precise.

Lemma 2.42. *Let $m \leq n$ be positive integers and let $I \subseteq S[\mathbb{P}^m] = \mathbb{k}[\alpha_0, \dots, \alpha_m]$ be a homogeneous ideal. Denote the Hilbert function of $S[\mathbb{P}^m]/I$ by h . Let $I' = I + (\alpha_{m+1}, \dots, \alpha_n) \subseteq S[\mathbb{P}^n] = \mathbb{k}[\alpha_0, \dots, \alpha_m, \dots, \alpha_n]$. The point $[I] \in \text{Hilb}_{S[\mathbb{P}^m]}^h$ is a smooth point if and only if $[I'] \in \text{Hilb}_{S[\mathbb{P}^n]}^h$ is a smooth point.*

Proof. Let $d = h(1)$. It follows from the inequality $d \leq m + 1$ that we can consider $S[\mathbb{P}^{d-1}] = \mathbb{k}[\alpha_0, \dots, \alpha_{d-1}]$ as a subring of $S[\mathbb{P}^m]$. Up to a linear change of variables in $S[\mathbb{P}^m]$ we may assume that we have $I = I'' + (\alpha_d, \dots, \alpha_m)$ for an ideal $I'' \subseteq S[\mathbb{P}^{d-1}]$ such that $S[\mathbb{P}^{d-1}]/I''$ has Hilbert function h .

The scheme $\text{Hilb}_{S[\mathbb{P}^n]}^h$ is a $\text{Hilb}_{S[\mathbb{P}^{d-1}]}^h$ -bundle over $\text{Gr}(n + 1 - d, S[\mathbb{P}^n]_1)$ and $\text{Hilb}_{S[\mathbb{P}^m]}^h$ is a $\text{Hilb}_{S[\mathbb{P}^{d-1}]}^h$ -bundle over $\text{Gr}(m + 1 - d, S[\mathbb{P}^m]_1)$ by [20, Prop. 3.1]. Therefore,

$$[I] \text{ is a smooth point} \Leftrightarrow [I''] \text{ is a smooth point} \Leftrightarrow [I'] \text{ is a smooth point.} \quad \square$$

We compute the fiber of a natural map of multigraded Hilbert schemes, associated with the homogeneous coordinate ring of a projective space, given by restricting ideals to high degree.

Lemma 2.43. *Let n be a positive integer and I be a homogeneous ideal of $S = S[\mathbb{P}^n]$. Let $m = \min\{a \in \mathbb{Z} \mid I_a \neq 0\}$ and g be the Hilbert function of S/I . Let $d > m$ be a positive integer and define $h: \mathbb{Z} \rightarrow \mathbb{Z}$ by*

$$h(a) = \begin{cases} \dim_{\mathbb{k}} S_a & \text{for } a < d \\ g(a) & \text{for } a \geq d. \end{cases}$$

There is a natural map $\pi: \text{Hilb}_S^g \rightarrow \text{Hilb}_S^h$ given on closed points by $[J] \mapsto [J \cap \mathfrak{m}^d]$. Let $[K]$ be a closed point of Hilb_S^h with $H_{S/\overline{K}}(a) = g(a)$ for every $a \geq m + 1$. The fiber of π over $[K]$ is the Grassmannian

$$\text{Gr}(\dim_{\mathbb{k}} S_m - g(m), \overline{K}_m).$$

Proof. The point $[K] \in \text{Hilb}_S^h$ gives a natural morphism

$$\text{Spec } \mathbb{k} = \text{Spec } k([K]) \rightarrow \text{Hilb}_S^h.$$

Its functor of points $\mathbb{k}\text{-Alg} \rightarrow \mathfrak{S}et$ is given by

$$R \mapsto \{K \otimes_{\mathbb{k}} R\}.$$

The scheme theoretic fiber over $[K]$ is the fiber product

$$\text{Hilb}_S^g \times_{\text{Hilb}_S^h} \text{Spec } k([K]).$$

Therefore, its functor of points $\mathbb{k}\text{-Alg} \rightarrow \mathbf{Set}$ is the fiber product of the corresponding functors, i.e. it is given by

$$R \mapsto \{J \in \underline{\text{Hilb}}_S^g(R) \mid J \cap (\mathfrak{m} \otimes_{\mathbb{k}} R)^d = K \otimes_{\mathbb{k}} R\}.$$

Since all schemes considered in this proof are of finite type over $\text{Spec } \mathbb{k}$, they can be recovered from their functors of points restricted to the subcategory of finitely generated \mathbb{k} -algebras. In what follows we restrict to this subcategory.

The ideal $\overline{K} \otimes_{\mathbb{k}} R$ is saturated with respect to $\mathfrak{m} \otimes_{\mathbb{k}} R$ by Lemma 2.6. Therefore, by definition of m and the assumption that $H_{S/\overline{K}}(a) = g(a)$ holds for every $a \geq m+1$, the functor of points of the fiber is naturally isomorphic to the functor from the category of finitely generated \mathbb{k} -algebras to the category of sets defined by

$$R \mapsto \{J_m \subseteq \overline{K}_m \otimes_{\mathbb{k}} R \subseteq S_m \otimes_{\mathbb{k}} R \mid J_m \text{ is an } R\text{-submodule of } \overline{K}_m \otimes_{\mathbb{k}} R \text{ and} \\ (S_m \otimes_{\mathbb{k}} R)/J_m \text{ is a locally free } R\text{-module of rank } g(m)\}.$$

By Lemma 2.4, this functor coincides with the following functor:

$$R \mapsto \{J_m \subseteq \overline{K}_m \otimes_{\mathbb{k}} R \mid J_m \text{ is an } R\text{-submodule of } \overline{K}_m \otimes_{\mathbb{k}} R \text{ and } (\overline{K}_m \otimes_{\mathbb{k}} R)/J_m \text{ is a locally} \\ \text{free } R\text{-module of rank } g(m) + \dim_{\mathbb{k}} \overline{K}_m - \dim_{\mathbb{k}} S_m\}.$$

This is the functor of points of $\text{Gr}(\dim_{\mathbb{k}} S_m - g(m), \overline{K}_m)$. □

2.2.3 Multigraded Hilbert schemes of points in general position and Slip

In this subsection we introduce the main object of study of this thesis in the case of projective space. We define the multigraded Hilbert scheme $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,n}}$ of r points in general position on projective n -space. The more general definition for a smooth projective toric variety appears in Subsection 4.1.5. For positive integers r and n , the scheme $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,n}}$ has a distinguished irreducible component called $\text{Slip}_{r,n}$ which plays a key role in the border apolarity lemma—Proposition 2.92.

Fix a positive integer n and let $S = S[\mathbb{P}^n] := \mathbb{k}[\underline{\alpha}] = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ be the homogeneous coordinate ring of \mathbb{P}^n with its standard \mathbb{Z} -grading given by $\deg(\alpha_i) = 1$ for all $i = 0, \dots, n$. Let r be a positive integer and let $h_{r,n} = h_{r,\mathbb{P}^n} : \mathbb{Z} \rightarrow \mathbb{N}$ be given by

$$h_{r,n}(a) = \min\{r, \dim_{\mathbb{k}} S_a\}.$$

This is the Hilbert function of r points in \mathbb{P}^n in general position. We study the multigraded Hilbert scheme $\text{Hilb}_S^{h_{r,n}}$.

This scheme has a natural morphism into the Hilbert scheme $\mathcal{Hilb}_r(\mathbb{P}^n)$ parametrizing zero-dimensional, length r subschemes of the projective n -space. We describe this in some details. We start with the following observation.

Lemma 2.44 ([46, Lem. 4.1]). *Let $f_{r,n} : \mathbb{Z} \rightarrow \mathbb{N}$ be defined by*

$$f_{r,n}(a) = \begin{cases} \dim_{\mathbb{k}} S_a = \dim_{\mathbb{k}} \mathbb{k}[\alpha_0, \dots, \alpha_n]_a & \text{for } a < r, \\ r & \text{for } a \geq r. \end{cases}$$

The Hilbert scheme $\mathcal{Hilb}_r(\mathbb{P}^n)$ is isomorphic to $\text{Hilb}_S^{f_{r,n}}$ where on closed points, the isomorphism

identifies $[I] \in \text{Hilb}_S^{f_{r,n}}$ with $[\text{Proj}(S/I)] \in \mathcal{H}ilb_r(\mathbb{P}^n)$.

Consider a morphism of functors $\underline{\text{Hilb}}_S^{h_{r,n}} \rightarrow \underline{\text{Hilb}}_S^{f_{r,n}}$ defined for a \mathbb{k} -algebra R by

$$S \otimes_{\mathbb{k}} R \supseteq I \mapsto I \cap (\mathfrak{m} \otimes_{\mathbb{k}} R)^r,$$

where $\mathfrak{m} = (\alpha_0, \dots, \alpha_n)$ is the irrelevant ideal of S . Using the identification from Lemma 2.44 this gives a morphism of schemes $\varphi_{r,n}: \text{Hilb}_S^{h_{r,n}} \rightarrow \mathcal{H}ilb_r(\mathbb{P}^n)$ that on closed points sends an ideal to the subscheme defined by this ideal. Unless stated otherwise, we identify $\mathcal{H}ilb_r(\mathbb{P}^n)$ with $\text{Hilb}_S^{f_{r,n}}$.

Following [15] we define $\text{Sip}_{r,n}$ to be the set of all closed points of $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,n}}$ corresponding to saturated ideals of r distinct points. The next result is fundamental. Its proof for $\mathbb{k} = \mathbb{C}$ appears in [15, Prop. 3.13] for an arbitrary smooth projective toric variety.

Proposition 2.45. *For any two positive integers r and n , the closure of $\text{Sip}_{r,n}$ in $\text{Hilb}_S^{h_{r,n}}$ is an irreducible component.*

Proof. Let $\mathcal{U} \subseteq \mathcal{H}ilb_r(\mathbb{P}^n) \times \mathbb{P}^n$ be the universal family over $\mathcal{H}ilb_r(\mathbb{P}^n)$ and let $\pi: \mathcal{U} \rightarrow \mathcal{H}ilb_r(\mathbb{P}^n)$ denote the natural morphism.

Let U be the locus of all points in $\mathcal{H}ilb_r(\mathbb{P}^n)$ corresponding to smooth subschemes of \mathbb{P}^n and let V be the locus of all points in $\mathcal{H}ilb_r(\mathbb{P}^n)$ corresponding to subschemes with Hilbert function $h_{r,n}$. We claim that U and V are open. For a non-negative integer d , let $V_d \subseteq \mathcal{H}ilb_r(\mathbb{P}^n)$ be the locus of all points corresponding to subschemes $R \subseteq \mathbb{P}^n$ such that $\dim_{\mathbb{k}} H^0(\mathbb{P}^n, \mathcal{I}_R(d)) \leq \dim_{\mathbb{k}} S_d - h_{r,n}(d)$ where \mathcal{I}_R is the ideal sheaf of R . The subset V_d is open by [48, Thm. III.12.8]. Furthermore, V_d equals $\mathcal{H}ilb_r(\mathbb{P}^n)$ for all $d \geq r$ by Gotzmann's regularity theorem [10, Thm. 4.3.2] and [32, Thm. 4.2]. Therefore, $V = \bigcap_{d=0}^{r-1} V_d$ is open.

Let $W \subseteq \mathcal{U}$ be the locus of all points x such that the fiber of π over $\pi(x)$ is smooth. This is an open subset of \mathcal{U} by [45, Thm. 12.1.6]. Therefore, its image U under π is open since π is flat and locally of finite presentation and thus, open by [41, Thm. 14.33].

It follows that $\text{Sip}_{r,n} = \varphi_{r,n}^{-1}(U \cap V)$ is open. Furthermore, it is homeomorphic to $U \cap V$ by Lemma 2.29. In particular, it is irreducible since $U \cap V$ is a non-empty open subset of the smoothable component of $\mathcal{H}ilb_r(\mathbb{P}^n)$. It follows that the closure of $\text{Sip}_{r,n}$ is an irreducible component of $\text{Hilb}_S^{h_{r,n}}$. \square

Definition 2.46. For any two positive integers r and n , the irreducible component $\overline{\text{Sip}_{r,n}}$ of $\text{Hilb}_S^{h_{r,n}}$ is denoted by $\text{Slip}_{r,n}$.

We end this subsection with a remark about a relation between the irreducible component $\text{Slip}_{r,n}$ and the smoothable component $\mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$ of the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^n)$.

Remark 2.47. Let $\varphi_{r,\mathbb{P}^n} = \varphi_{r,n}: \text{Hilb}_S^{h_{r,n}} \rightarrow \mathcal{H}ilb_r(\mathbb{P}^n)$ be defined on closed points by

$$[I] \mapsto [\text{Proj } S/I].$$

Since both schemes are projective, it is a closed map. It sends $\text{Sip}_{r,n}$ onto an open subset of the locus of all reduced subschemes. It follows that we have $\varphi_{r,n}(\text{Slip}_{r,n}) = \mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$ set-theoretically. In particular, if $\mathcal{H}ilb_r(\mathbb{P}^n)$ is irreducible then for every closed point $[I] \in \text{Hilb}_S^{h_{r,n}}$ there is a point $[I'] \in \text{Slip}_{r,n}$ with $\bar{I} = \bar{I}'$. As a special case, if I is saturated, $\mathcal{H}ilb_r(\mathbb{P}^n)$ is irreducible and $[I]$ is a point of $\text{Hilb}_S^{h_{r,n}}$ then $[I]$ belongs to $\text{Slip}_{r,n}$.

2.2.4 Flag multigraded Hilbert schemes

Let $S = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ be a polynomial ring. Assume that S is graded by an abelian group \mathbf{A} . Let $f_1, f_2: \mathbf{A} \rightarrow \mathbb{N}$ be numerical functions. There are multigraded Hilbert schemes $\text{Hilb}_S^{f_1}$ and $\text{Hilb}_S^{f_2}$. The goal of this subsection is the construction of the scheme $\text{Hilb}_S^{f_1, f_2}$ parametrizing pairs (K, J) of homogeneous ideals such that we have $K \subseteq J$ and the algebras S/K and S/J have Hilbert functions f_1 and f_2 , respectively. The idea is to show that the condition that K is contained in J defines a closed subscheme of the product $\text{Hilb}_S^{f_1} \times \text{Hilb}_S^{f_2}$. The rest of this subsection makes this intuition precise.

Let $\underline{\text{Hilb}}_S^{f_1}$ and $\underline{\text{Hilb}}_S^{f_2}$ be the functors of points of $\text{Hilb}_S^{f_1}$ and $\text{Hilb}_S^{f_2}$, respectively. We start with defining the functor $\underline{\text{Hilb}}_S^{f_1, f_2}: \mathbb{k}\text{-Alg} \rightarrow \mathfrak{Set}$ that $\text{Hilb}_S^{f_1, f_2}$ should represent. It is a subfunctor of the product functor $\underline{\text{Hilb}}_S^{f_1} \times \underline{\text{Hilb}}_S^{f_2}$. Given $R \in \mathbb{k}\text{-Alg}$, let

$$\underline{\text{Hilb}}_S^{f_1, f_2}(R) = \{(K, J) \in \underline{\text{Hilb}}_S^{f_1}(R) \times \underline{\text{Hilb}}_S^{f_2}(R) \mid K \subseteq J\}.$$

We can now state the main result of this subsection.

Proposition 2.48. *Let $S = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ be a polynomial ring graded by an abelian group \mathbf{A} . Let $f_1, f_2: \mathbf{A} \rightarrow \mathbb{N}$ be numerical functions. The functor $\underline{\text{Hilb}}_S^{f_1, f_2}: \mathbb{k}\text{-Alg} \rightarrow \mathfrak{Set}$ is represented by a closed subscheme of $\text{Hilb}_S^{f_1} \times \text{Hilb}_S^{f_2}$. In particular, it is a projective scheme if the grading of S is positive.*

Before proving Proposition 2.48, we recall some results related to representable functors. See [33, Cha. VI] or [41, Cha. 8] for good introductions to this topic.

Definition 2.49. A functor $F: \mathbb{k}\text{-Alg} \rightarrow \mathfrak{Set}$ is a *sheaf in the Zariski topology* if for every \mathbb{k} -algebra R and for every open covering $\text{Spec } R = \bigcup U_i$ by distinguished open subschemes $U_i = \text{Spec } R_{f_i}$ we have an equalizer sequence

$$F(R) \rightarrow \prod_i F(R_{f_i}) \rightrightarrows \prod_{i,j} F(R_{f_i f_j}).$$

If a functor $F: \mathbb{k}\text{-Alg} \rightarrow \mathfrak{Set}$ is represented by a \mathbb{k} -scheme, then F is a sheaf in the Zariski topology [33, Thm. VI-14].

Following [46] we introduce the following notion.

Definition 2.50. Let R be a \mathbb{k} -algebra and let \mathcal{C} be a condition on R -algebras. We say that the condition \mathcal{C} is *closed* if there exists an ideal $\mathfrak{a} \subseteq R$ such that an R -algebra $\phi: R \rightarrow T$ satisfies condition \mathcal{C} if and only if $\phi(\mathfrak{a})$ is zero.

Let $F: \mathbb{k}\text{-Alg} \rightarrow \mathfrak{Set}$ be a sheaf in the Zariski topology and assume that it is a subfunctor of the functor of points h_X of a \mathbb{k} -scheme X . Given a \mathbb{k} -algebra R and an element $\lambda \in h_X(R)$ we say that an R -algebra $\phi: R \rightarrow T$ satisfies condition $V_{R, \lambda}$ if $h_X(\phi)(\lambda) \in h_X(T)$ belongs to $F(T)$.

In the following remark we show that the ideal \mathfrak{a} from the definition of a closed condition is uniquely determined by the condition.

Remark 2.51. Let R be a \mathbb{k} -algebra and let \mathcal{C} be a closed condition on R -algebras. The ideal $\mathfrak{a} \subseteq R$ such that R -algebra $\phi: R \rightarrow T$ satisfies condition \mathcal{C} if and only if $\phi(\mathfrak{a})$ is zero is uniquely determined by \mathcal{C} . Indeed, if \mathfrak{b} has analogous property then considering R -algebras $R \rightarrow R/\mathfrak{a}$ and $R \rightarrow R/\mathfrak{b}$ we conclude that we have $\mathfrak{a} = \mathfrak{b}$.

In order to show that $\underline{\text{Hilb}}_S^{f_1, f_2}$ is represented by a closed subscheme of $\text{Hilb}_S^{f_1} \times \text{Hilb}_S^{f_2}$ we use the following result.

Proposition 2.52 ([46, Prop. 2.9]). *Let $F: \mathbb{k}\text{-Alg} \rightarrow \mathfrak{Set}$ be a sheaf in the Zariski topology and assume that it is a subfunctor of the functor of points h_X of a \mathbb{k} -scheme X . Assume that for every \mathbb{k} -algebra R and for every $\lambda \in h_X(R)$ the condition $V_{R, \lambda}$ on R -algebras is closed. The functor F is represented by a closed subscheme of X .*

Being a closed condition can be checked affine locally. Lemma 2.53 makes this precise.

Lemma 2.53. *Let $F: \mathbb{k}\text{-Alg} \rightarrow \mathfrak{Set}$ be a sheaf in the Zariski topology and assume that it is a subfunctor of the functor of points h_X of a \mathbb{k} -scheme X .*

Let R be a \mathbb{k} -algebra and λ be an element of $h_X(R)$. Suppose that there is a covering $\text{Spec } R = \bigcup_{i \in I} \text{Spec } R_{g_i}$ of $\text{Spec } R$ by distinguished affine open subsets. Let $\lambda_i = h_X(\tau_i)(\lambda)$ where $\tau_i: R \rightarrow R_{g_i}$ is the localization map. If the condition $V_{R_{g_i}, \lambda_i}$ on R_{g_i} -algebras is closed for every $i \in I$, then the condition $V_{R, \lambda}$ on R -algebras is closed.

Proof. Let $\tau_{ii'}: R_{g_i} \rightarrow R_{g_i g_{i'}}$ be the localization map for every $i, i' \in I$. For all $i \in I$, let $\mathfrak{a}_i \subseteq R_{g_i}$ be the ideal such that an R_{g_i} -algebra $\psi: R_{g_i} \rightarrow U$ satisfies condition $V_{R_{g_i}, \lambda_i}$ if and only if $\psi(\mathfrak{a}_i)$ is zero. By Remark 2.51 we have $\mathfrak{a}_i R_{g_i g_{i'}} = \mathfrak{a}_{i'} R_{g_i g_{i'}}$ since both ideals show that the condition $V_{R_{g_i g_{i'}}, h_X(\tau_{ii'})(\lambda_i)}$ is closed.

It follows that there is an ideal $\mathfrak{a} \subseteq R$ with $\mathfrak{a} R_{g_i} = \mathfrak{a}_i$ for all $i \in I$. We claim that an R -algebra $\phi: R \rightarrow T$ satisfies condition $V_{R, \lambda}$ if and only if $\phi(\mathfrak{a}) = 0$ holds.

Let $h_i = \phi(g_i)$ for all $i \in I$ and $\sigma_i: T \rightarrow T_{h_i}$, $\sigma_{ii'}: T_{h_i} \rightarrow T_{h_i h_{i'}}$ be the localization maps. Since F is a sheaf in the Zariski topology, $h_X(\phi)(\lambda)$ belongs to $F(T)$ if and only if we have

$$h_X(\sigma_i \circ \phi)(\lambda) \in F(T_{h_i}) \tag{2.54}$$

for every $i \in I$ and

$$F(\sigma_{ii'})\left(h_X(\sigma_i \circ \phi)(\lambda)\right) = F(\sigma_{i'i})\left(h_X(\sigma_{i'} \circ \phi)(\lambda)\right) \tag{2.55}$$

for every $i, i' \in I$.

First we show that Equation (2.55) always holds. Indeed, F is a subfunctor of h_X . As a result, we can replace $F(\sigma_{ii'})$ by $h_X(\sigma_{ii'})$ and $F(\sigma_{i'i})$ by $h_X(\sigma_{i'i})$. Since we have $\sigma_{ii'} \circ \sigma_i \circ \phi = \sigma_{i'i} \circ \sigma_{i'} \circ \phi$ the claimed equality follows from the fact that h_X is a functor.

Therefore, we need to show that Equation (2.54) is satisfied if and only if $\phi(\mathfrak{a})$ is zero. For every $i \in I$, there is an induced map $\phi_i: R_{g_i} \rightarrow T_{h_i}$ with $\phi_i \circ \tau_i = \sigma_i \circ \phi$. Thus, there is a chain of equalities $h_X(\sigma_i \circ \phi)(\lambda) = h_X(\phi_i) \circ h_X(\tau_i)(\lambda) = h_X(\phi_i)(\lambda_i)$. It follows that Equation (2.54) is satisfied if and only if $\phi_i(\mathfrak{a}_i) = 0$ holds for every $i \in I$.

We have

$$\phi(\mathfrak{a}) = 0 \Leftrightarrow \sigma_i \circ \phi(\mathfrak{a}) = 0 \text{ for all } i \in I \Leftrightarrow \phi_i \circ \tau_i(\mathfrak{a}) = 0 \text{ for all } i \in I \Leftrightarrow \phi_i(\mathfrak{a}_i) = 0 \text{ for all } i \in I.$$

This implies that Equation (2.54) is satisfied if and only if $\phi(\mathfrak{a})$ is zero and thus, finishes the proof. \square

We start with checking that the functor $\underline{\text{Hilb}}_S^{f_1, f_2}$ is a sheaf in the Zariski topology. In fact, it is convenient to prove this in slightly greater generality. Let $D \subseteq \mathbf{A}$ and define the subfunctor

$\underline{\text{Hilb}}_{S,D}^{f_1,f_2}$ of the product functor $\underline{\text{Hilb}}_S^{f_1} \times \underline{\text{Hilb}}_S^{f_2}$ by

$$\underline{\text{Hilb}}_{S,D}^{f_1,f_2}(R) = \{(K, J) \in \underline{\text{Hilb}}_S^{f_1}(R) \times \underline{\text{Hilb}}_S^{f_2}(R) \mid K_a \subseteq J_a \text{ for every } a \in D\}.$$

Lemma 2.56. *The functor $\underline{\text{Hilb}}_{S,D}^{f_1,f_2}$ is a sheaf in the Zariski topology. In particular, $\underline{\text{Hilb}}_S^{f_1,f_2}$ is a sheaf in the Zariski topology.*

Proof. Let R be a \mathbb{k} -algebra. Consider a covering of $\text{Spec } R$ with distinguished affine open subschemes $\{\text{Spec } R_{g_i}\}_{i \in I}$. We need to show that we have an equalizer sequence

$$\underline{\text{Hilb}}_{S,D}^{f_1,f_2}(R) \rightarrow \prod_{i \in I} \underline{\text{Hilb}}_{S,D}^{f_1,f_2}(R_{g_i}) \rightrightarrows \prod_{i,i' \in I} \underline{\text{Hilb}}_{S,D}^{f_1,f_2}(R_{g_i g_{i'}}).$$

For all $i, i' \in I$, let $\tau_{ii'}: R_{g_i} \rightarrow R_{g_i g_{i'}}$ and $\tau_i: R \rightarrow R_{g_i}$ be the localization maps. Let $(K_i, J_i) \in \underline{\text{Hilb}}_{S,D}^{f_1,f_2}(R_{g_i})$ for all $i \in I$ be such that

$$\underline{\text{Hilb}}_{S,D}^{f_1,f_2}(\tau_{ii'})(K_i, J_i) = \underline{\text{Hilb}}_{S,D}^{f_1,f_2}(\tau_{i'i})(K_{i'}, J_{i'})$$

holds for all $i, i' \in I$. We need to show that there exists a unique element $(K, J) \in \underline{\text{Hilb}}_{S,D}^{f_1,f_2}(R)$ with $\underline{\text{Hilb}}_{S,D}^{f_1,f_2}(\tau_i)(K, J) = (K_i, J_i)$ for every $i \in I$.

Since $\underline{\text{Hilb}}_{S,D}^{f_1,f_2}$ is a subfunctor of the representable functor $\underline{\text{Hilb}}_S^{f_1} \times \underline{\text{Hilb}}_S^{f_2}$, it follows that there is a unique element $(K, J) \in \underline{\text{Hilb}}_S^{f_1}(R) \times \underline{\text{Hilb}}_S^{f_2}(R)$ with $\underline{\text{Hilb}}_S^{f_1}(\tau_i)(K) = K_i$ and $\underline{\text{Hilb}}_S^{f_2}(\tau_i)(J) = J_i$ for all $i \in I$. We are left with showing that K_a is contained in J_a for every $a \in D$.

Let π be the natural map $\bigoplus_{a \in D} K_a \rightarrow \bigoplus_{a \in D} (R[\underline{\alpha}]/J)_a$ and denote $\ker \pi \rightarrow \bigoplus_{a \in D} K_a$ by θ . We claim that θ is an isomorphism. Indeed, θ_{g_i} is an isomorphism for all $i \in I$ since we have $(K_i)_a \subseteq (J_i)_a$ for every $a \in D$. Subschemes $\text{Spec } R_{g_i}$ cover $\text{Spec } R$. It follows that $\theta_{\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p} \in \text{Spec } R$ and thus, θ is an isomorphism. \square

Now we can give a proof of the existence of flag multigraded Hilbert schemes.

Proof of Proposition 2.48. The functor $\underline{\text{Hilb}}_S^{f_1,f_2}$ is a sheaf in the Zariski topology by Lemma 2.56. Therefore, by Proposition 2.52, it is enough to show that the following holds. Let R be a \mathbb{k} -algebra and $(K, J) \in \underline{\text{Hilb}}_S^{f_1}(R) \times \underline{\text{Hilb}}_S^{f_2}(R)$. There exists an ideal $\mathfrak{a} \subseteq R$ such that $K \otimes_R T$ is contained in $J \otimes_R T$ for an R -algebra $\phi: R \rightarrow T$ if and only if $\phi(\mathfrak{a})$ is zero.

We start with the following reduction. Given $a \in \mathbf{A}$ consider the functor $\underline{\text{Hilb}}_{S,\{a\}}^{f_1,f_2}$. We show that there exists an ideal $\mathfrak{b}_a \subseteq R$ such that $K_a \otimes_R T$ is contained in $J_a \otimes_R T$ for an R -algebra $\phi: R \rightarrow T$ if and only if $\phi(\mathfrak{b}_a)$ is zero. Then we take $\mathfrak{a} = \sum_{a \in \mathbf{A}} \mathfrak{b}_a$.

Moreover, by Lemma 2.53 by replacing R by its localization, we may assume that $(R[\underline{\alpha}]/J)_a$ is a free R -module. Let $(R[\underline{\alpha}]/J)_a = \bigoplus_{k=1}^{f_2(a)} R \cdot e_{a,k}$. Let $\pi: K_a \rightarrow (R[\underline{\alpha}]/J)_a$ be the natural map. Let \mathcal{C} be the condition on R -algebras such that $\phi: R \rightarrow T$ satisfies condition \mathcal{C} if and only if $\pi \otimes_R \text{id}_T$ is the zero map. We need to show that condition \mathcal{C} is closed. Let $(b_{a,i})_{i \in I_a}$ be a set of generators of K_a as an R -module. Let $\pi(b_{a,i}) = \sum_{k=1}^{f_2(a)} c_{a,i,k} e_{a,k}$. The map $\pi \otimes_R \text{id}_T$ is the zero map if and only if $\phi(c_{a,i,k}) = 0$ holds for every $i \in I_a$ and $k \in \{1, \dots, f_2(a)\}$. The ideal $\mathfrak{b}_a = (c_{a,i,k})_{i \in I_a, k \in \{1, \dots, f_2(a)\}}$ shows that condition \mathcal{C} is closed. \square

2.3 Deformation theory

In this section we recall some definitions and results from deformation theory. Subsection 2.3.1 introduces a small amount of general theory that we need. Our main reference is [35].

In Subsection 2.3.2 we study the tangent-obstruction theory of multigraded Hilbert schemes and flag multigraded Hilbert schemes.

2.3.1 Definitions and basic results

Let $\mathfrak{Art}/\mathbb{k}$ be the category of local Artin \mathbb{k} -algebras with residue field \mathbb{k} . Observe that every morphism of \mathbb{k} -algebras from $\mathfrak{Art}/\mathbb{k}$ is a local morphism of local rings.

The main objects of study of deformation theory are deformation functors. We consider the class of deformation functors coming from \mathbb{k} -points of \mathbb{k} -schemes.

Definition 2.57. Let X be a \mathbb{k} -scheme and $h_X: \mathfrak{Sch}_{\mathbb{k}}^{op} \rightarrow \mathfrak{Set}$ be its functor of points. Let $x \in h_X(\text{Spec } \mathbb{k})$. We have the corresponding *deformation functor*

$$D_{X,x}: \mathfrak{Art}/\mathbb{k} \rightarrow \mathfrak{Set}$$

defined by $D_{X,x}(A) = \{\lambda \in h_X(\text{Spec } A) \mid h_X(\pi_A^\#)(\lambda) = x\}$, where $\pi_A: A \rightarrow A/\mathfrak{m}_A$ is the natural map to the residue field of A and $\pi_A^\#$ is the corresponding map of affine schemes. A *morphism of deformation functors* $D_{X,x} \rightarrow D_{Y,y}$ is a natural transformation of functors.

A morphism of \mathbb{k} -schemes determines morphisms of deformation functors.

Example 2.58. Let $f: X \rightarrow Y$ be a morphism of \mathbb{k} -schemes and let $x \in X$ and $y = f(x)$ be \mathbb{k} -points. The natural transformation $h_X \rightarrow h_Y$ of functors of points corresponding to f , induces a morphism of deformation functors $D_{X,x} \rightarrow D_{Y,y}$.

Suppose we have a \mathbb{k} -scheme X and a \mathbb{k} -point $x \in X$. A key question is whether given a surjection $\pi: B \rightarrow A$ of algebras in $\mathfrak{Art}/\mathbb{k}$ and an element of $\lambda \in D_{X,x}(A)$ we can lift it to an element of $D_{X,x}(B)$. The simplest situation is when the kernel of π is killed by the maximal ideal of B .

Definition 2.59. A *small extension* is a short exact sequence

$$0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$$

where A and B are in $\mathfrak{Art}/\mathbb{k}$ and we have $\mathfrak{m}_B \cdot M = 0$. Here \mathfrak{m}_B is the maximal ideal of B .

Definition 2.60. A deformation functor $D_{X,x}$ has a *tangent-obstruction theory* if there are finite-dimensional \mathbb{k} -vector spaces $\mathbf{T}_{X,x}$ (called the *tangent space*) and $\text{Ob}_{X,x}$ (called the *obstruction space*) such that:

1. For all small extensions $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ there exists an exact sequence of sets

$$\mathbf{T}_{X,x} \otimes_{\mathbb{k}} M \rightarrow D_{X,x}(B) \rightarrow D_{X,x}(A) \xrightarrow{\text{ob}_{B \rightarrow A}} \text{Ob}_{X,x} \otimes_{\mathbb{k}} M;$$

(Exactness at $D_{X,x}(A)$ means that an element of $D_{X,x}(A)$ lifts to $D_{X,x}(B)$ if and only if its image in $\text{Ob}_{X,x} \otimes_{\mathbb{k}} M$ is zero. Exactness at $D_{X,x}(B)$ means that there is a transitive action of $\mathbf{T}_{X,x} \otimes_{\mathbb{k}} M$ on each fiber of $D_{X,x}(B) \rightarrow D_{X,x}(A)$).

2. If $A = \mathbb{k}$ then the sequence becomes

$$0 \rightarrow \mathbf{T}_{X,x} \otimes_{\mathbb{k}} M \rightarrow D_{X,x}(B) \rightarrow D_{X,x}(\mathbb{k}) \xrightarrow{ob_{B \rightarrow \mathbb{k}}} \text{Ob}_{X,x} \otimes_{\mathbb{k}} M;$$

3. Suppose we have a commutative diagram whose rows are small extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow \varphi_M & & \downarrow \varphi_B & & \downarrow \varphi_A \\ 0 & \longrightarrow & M' & \longrightarrow & B' & \longrightarrow & A' \longrightarrow 0. \end{array}$$

There is a commutative diagram

$$\begin{array}{ccccccc} \mathbf{T}_{X,x} \otimes_{\mathbb{k}} M & \longrightarrow & D_{X,x}(B) & \longrightarrow & D_{X,x}(A) & \xrightarrow{ob_{B \rightarrow A}} & \text{Ob}_{X,x} \otimes_{\mathbb{k}} M \\ \downarrow \text{id}_{\mathbf{T}_{X,x}} \otimes_{\mathbb{k}} \varphi_M & & \downarrow D_{X,x}(\varphi_B) & & \downarrow D_{X,x}(\varphi_A) & & \downarrow \text{id}_{\text{Ob}_{X,x}} \otimes_{\mathbb{k}} \varphi_M \\ \mathbf{T}_{X,x} \otimes_{\mathbb{k}} M' & \longrightarrow & D_{X,x}(B') & \longrightarrow & D_{X,x}(A') & \xrightarrow{ob_{B' \rightarrow A'}} & \text{Ob}_{X,x} \otimes_{\mathbb{k}} M'. \end{array}$$

The tangent space $\mathbf{T}_{X,x}$ is uniquely determined and agrees with the usual definition of tangent space to scheme X at point x .

Proposition 2.61 ([35, Prop. 6.1.23]). *If $D_{X,x}$ is a deformation functor with tangent obstruction theory $(\mathbf{T}_{X,x}, \text{Ob}_{X,x})$, then $\mathbf{T}_{X,x} = D_{X,x}(\mathbb{k}[\varepsilon]/(\varepsilon^2))$ is the tangent space to X at x .*

On the other hand, the obstruction space $\text{Ob}_{X,x}$ is not uniquely determined by the deformation functor $D_{X,x}$. In fact, given a tangent-obstruction theory $(\mathbf{T}_{X,x}, \text{Ob}_{X,x})$ and an injective \mathbb{k} -linear map $\iota: \text{Ob}_{X,x} \rightarrow \text{Ob}'_{X,x}$ to a \mathbb{k} -vector space $\text{Ob}'_{X,x}$, there is a tangent-obstruction theory for $D_{X,x}$ with obstruction space $\text{Ob}'_{X,x}$ and obstruction map $ob'_{B \rightarrow A} = (\iota \otimes_{\mathbb{k}} \text{id}_M) \circ ob_{B \rightarrow A}$ for a small extension $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$.

We need the following notion of a map of tangent-obstruction theories.

Definition 2.62. Let $\eta: D_{X,x} \rightarrow D_{Y,y}$ be a morphism of deformation functors with tangent-obstruction theories $(\mathbf{T}_{X,x}, \text{Ob}_{X,x})$ and $(\mathbf{T}_{Y,y}, \text{Ob}_{Y,y})$, respectively. A map of tangent-obstruction theories is a pair of linear maps $\mathbf{T}_\eta: \mathbf{T}_{X,x} \rightarrow \mathbf{T}_{Y,y}$ and $\text{Ob}_\eta: \text{Ob}_{X,x} \rightarrow \text{Ob}_{Y,y}$ such that for every small extension $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ there is a commutative diagram

$$\begin{array}{ccccccc} \mathbf{T}_{X,x} \otimes_{\mathbb{k}} M & \longrightarrow & D_{X,x}(B) & \longrightarrow & D_{X,x}(A) & \xrightarrow{ob_{B \rightarrow A}} & \text{Ob}_{X,x} \otimes_{\mathbb{k}} M \\ \downarrow \mathbf{T}_\eta \otimes_{\mathbb{k}} \text{id}_M & & \downarrow \eta(B) & & \downarrow \eta(A) & & \downarrow \text{Ob}_\eta \otimes_{\mathbb{k}} \text{id}_M \\ \mathbf{T}_{Y,y} \otimes_{\mathbb{k}} M & \longrightarrow & D_{Y,y}(B) & \longrightarrow & D_{Y,y}(A) & \xrightarrow{ob_{B \rightarrow A}} & \text{Ob}_{Y,y} \otimes_{\mathbb{k}} M. \end{array} \quad (2.63)$$

The commutativity of the left square means that $\eta(B)$ is equivariant with respect to the natural actions of $\mathbf{T}_{X,x} \otimes M$ and $\mathbf{T}_{Y,y} \otimes M$.

Proposition 2.61 showed that the tangent space of a deformation functor is uniquely defined. Similarly, for a natural transformation of deformation functors $\eta: D_{X,x} \rightarrow D_{Y,y}$, if we set $\mathbf{T}_\eta = \eta(\mathbb{k}[\varepsilon]/(\varepsilon^2))$, then the left square of diagram (2.63) commutes.

The main result from deformation theory that we use is the smoothness criterion (see [35, Rmk. 6.3.2]). In order to state it, we recall the definition of a smooth morphism of deformation functors.

Definition 2.64. Let $f: X \rightarrow Y$ be a morphism of \mathbb{k} -schemes locally of finite type. Let x and $y = f(x)$ be \mathbb{k} -points. A morphism of deformation functors $D_{X,x} \rightarrow D_{Y,y}$ is called *smooth* if for every small extension $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$, the natural map $D_{X,x}(B) \rightarrow D_{Y,y}(B) \times_{D_{Y,y}(A)} D_{X,x}(A)$ is surjective.

Theorem 2.65 ([35, Rmk. 6.3.2]). *Let $f: X \rightarrow Y$ be a morphism of \mathbb{k} -schemes locally of finite type. Let x and $y = f(x)$ be \mathbb{k} -points. The morphism f is smooth at x if and only if the morphism of deformation functors $D_{X,x} \rightarrow D_{Y,y}$ is smooth.*

We use the following special cases of Theorem 2.65.

Corollary 2.66. *Let $f: X \rightarrow Y$ be a morphism of \mathbb{k} -schemes locally of finite type. Let x and $y = f(x)$ be \mathbb{k} -points.*

- (i) *The point x is a smooth point of X if and only if $D_{X,x}(B) \rightarrow D_{X,x}(A)$ is surjective for every small extension $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$.*
- (ii) *Assume that deformation functors $D_{X,x}$ and $D_{Y,y}$ have tangent-obstruction theories. Let $\eta: D_{X,x} \rightarrow D_{Y,y}$ be the morphism of deformation functors defined by f . If there is a map of tangent-obstruction theories $(\mathbf{T}_\eta, \text{Ob}_\eta)$ which is surjective on tangent spaces and injective on obstruction spaces, then the morphism f is smooth at x .*

Proof.

- (i) This follows from the fact that if we have $Y = \text{Spec } \mathbb{k}$ then $D_{Y,y}(A)$ is a singleton for every $A \in \mathfrak{Art}/\mathbb{k}$.
- (ii) This follows from chasing the diagram (2.63). □

We end this subsection with two lemmas from deformation theory that are used in Chapter 3.

Lemma 2.67. *Assume that $f: X \rightarrow Y$ is a morphism of \mathbb{k} -schemes locally of finite type. Let x and $y = f(x)$ be closed points. Assume that functors $D_{X,x}$ and $D_{Y,y}$ have tangent-obstruction theories with obstruction spaces $\text{Ob}_{X,x}$ and $\text{Ob}_{Y,y}$, respectively. If y is a smooth point and f induces a map of obstruction theories which is injective on obstruction spaces then x is a smooth point.*

Proof. By Corollary 2.66(i) it is enough to show that for every small extension $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$, the map $D_{X,x}(B) \rightarrow D_{X,x}(A)$ is surjective. Consider the commutative diagram of sets

$$\begin{array}{ccccc} D_{X,x}(B) & \longrightarrow & D_{X,x}(A) & \longrightarrow & \text{Ob}_{X,x} \otimes M \\ \downarrow & & \downarrow & & \downarrow \\ D_{Y,y}(B) & \longrightarrow & D_{Y,y}(A) & \longrightarrow & \text{Ob}_{Y,y} \otimes M \end{array}$$

whose rows are the exact sequences from Definition 2.60. Since y is a smooth point, the lower left map is surjective and thus, the lower right map takes every element of $D_{Y,y}(A)$ to 0. By assumption the map on obstruction spaces is injective. It follows that every element of $D_{X,x}(A)$ is mapped to 0 by the upper right map. Hence the upper left map is surjective, as claimed. □

Before presenting the final lemma of this subsection, we introduce some notation. Let $g': X \rightarrow X'$ and $g'': X \rightarrow X''$ be morphisms of \mathbb{k} -schemes locally of finite type. Let $x \in X$, $x' = g'(x)$ and $x'' = g''(x)$ be \mathbb{k} -points.

Assume that there are tangent-obstruction theories for $D_{X',x'}$ with obstruction space $\text{Ob}_{X',x'}$ and for $D_{X'',x''}$ with obstruction space $\text{Ob}_{X'',x''}$. Assume that there is a \mathbb{k} -vector space L and \mathbb{k} -linear maps $\alpha': \text{Ob}_{X',x'} \rightarrow L$ and $\alpha'': \text{Ob}_{X'',x''} \rightarrow L$ such that there is a tangent-obstruction theory for $D_{X,x}$ with obstruction space $\text{Ob}_{X,x}$ given by the pullback

$$\begin{array}{ccc} \text{Ob}_{X,x} & \xrightarrow{\beta''} & \text{Ob}_{X'',x''} \\ \downarrow \beta' & & \downarrow \alpha'' \\ \text{Ob}_{X',x'} & \xrightarrow{\alpha'} & L. \end{array}$$

Moreover, assume that β' and β'' determine maps of tangent-obstruction theories.

Lemma 2.68. *Assume that x' is a smooth point of X' . There is a tangent-obstruction theory for $D_{X,x}$ with obstruction space $\ker \alpha''$. Moreover, the canonical injection*

$$\iota: \ker \alpha'' \rightarrow \text{Ob}_{X'',x''}$$

induces a map of tangent-obstruction theories.

Proof. By the universal property of fiber product, there is a map $\gamma: \ker \alpha'' \rightarrow \text{Ob}_{X,x}$ with

$$\beta'' \circ \gamma = \iota \tag{2.69}$$

and

$$\beta' \circ \gamma = 0. \tag{2.70}$$

Let $\eta': D_{X,x} \rightarrow D_{X',x'}$ and $\eta'': D_{X,x} \rightarrow D_{X'',x''}$ be the natural transformations induced by g' and g'' , respectively. Given a small extension $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$, we denote the obstruction map $D_{X,x}(A) \rightarrow \text{Ob}_{X,x} \otimes M$ by $ob_{X,x;B \rightarrow A}$. We do similarly with $D_{X',x'}$ and $D_{X'',x''}$.

Step 1: Definition of obstruction maps

Fix a small extension $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$. We construct a map $D_{X,x}(A) \rightarrow \ker \alpha'' \otimes M$. Consider the commutative diagram:

$$\begin{array}{ccccc} D_{X,x}(A) & \xrightarrow{ob_{X,x;B \rightarrow A}} & \text{Ob}_{X,x} \otimes M & \xrightarrow{\beta' \otimes \text{id}_M} & \text{Ob}_{X',x'} \otimes M \\ & & \downarrow \beta'' \otimes \text{id}_M & & \downarrow \alpha' \otimes \text{id}_M \\ \ker \alpha'' \otimes M & \xrightarrow{\iota \otimes \text{id}_M} & \text{Ob}_{X'',x''} \otimes M & \xrightarrow{\alpha'' \otimes \text{id}_M} & L \otimes M. \end{array}$$

We have

$$(\beta' \otimes \text{id}_M) \circ ob_{X,x;B \rightarrow A} = ob_{X',x';B \rightarrow A} \circ \eta'(A) = 0, \tag{2.71}$$

where the first equality follows from the fact that β' defines a map of tangent-obstruction theories and the second is a consequence of smoothness of x' (see Corollary 2.66(i)).

It follows that the image of $(\beta'' \otimes \text{id}_M) \circ ob_{X,x;B \rightarrow A}$ is contained in $\ker \alpha'' \otimes M$. Thus, there is a map $ob_{B \rightarrow A}: D_{X,x}(A) \rightarrow \ker \alpha'' \otimes M$ with

$$(\iota \otimes \text{id}_M) \circ ob_{B \rightarrow A} = (\beta'' \otimes \text{id}_M) \circ ob_{X,x;B \rightarrow A}. \tag{2.72}$$

Step 2: Factorization of obstruction maps

We claim that

$$(\gamma \otimes \text{id}_M) \circ \text{ob}_{B \rightarrow A} = \text{ob}_{X,x;B \rightarrow A} \quad (2.73)$$

holds for every small extension $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$. By assumptions we have a pullback diagram

$$\begin{array}{ccc} \text{Ob}_{X,x} \otimes M & \xrightarrow{\beta' \otimes \text{id}_M} & \text{Ob}_{X',x'} \otimes M \\ \downarrow \beta'' \otimes \text{id}_M & & \downarrow \alpha' \otimes \text{id}_M \\ \text{Ob}_{X'',x''} \otimes M & \xrightarrow{\alpha'' \otimes \text{id}_M} & L \otimes M. \end{array}$$

It remains a pullback in the category \mathfrak{Set} of sets. Therefore, in order to show Equation (2.73) it is enough to observe that there are equalities

$$(\beta'' \otimes \text{id}_M) \circ (\gamma \otimes \text{id}_M) \circ \text{ob}_{B \rightarrow A} \stackrel{(2.69)}{=} (\iota \otimes \text{id}_M) \circ \text{ob}_{B \rightarrow A} \stackrel{(2.72)}{=} (\beta'' \otimes \text{id}_M) \circ \text{ob}_{X,x;B \rightarrow A}$$

and

$$(\beta' \otimes \text{id}_M) \circ (\gamma \otimes \text{id}_M) \circ \text{ob}_{B \rightarrow A} \stackrel{(2.70)}{=} 0 \stackrel{(2.71)}{=} (\beta' \otimes \text{id}_M) \circ \text{ob}_{X,x;B \rightarrow A}.$$

Step 3: Verification of axiom 1 of tangent-obstruction theory

We verify that for every small extension $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ an element $\lambda \in D_{X,x}(A)$ lifts to $D_{X,x}(B)$ if and only if $\text{ob}_{B \rightarrow A}(\lambda)$ is zero.

The map $\gamma \otimes \text{id}_M$ is injective. Therefore, by Equation (2.73) we get $\text{ob}_{B \rightarrow A}(\lambda) = 0$ if and only if we have $\text{ob}_{X,x;B \rightarrow A}(\lambda) = 0$. This is equivalent to the existence of a lift of λ since the map $\text{ob}_{X,x;B \rightarrow A}$ is a part of the data of tangent-obstruction theory of $D_{X,x}$.

Step 4: Verification of axiom 3 of tangent-obstruction theory

Let

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow \varphi_M & & \downarrow \varphi_B & & \downarrow \varphi_A \\ 0 & \longrightarrow & M' & \longrightarrow & B' & \longrightarrow & A' \longrightarrow 0 \end{array}$$

be a commutative diagram whose rows are small extensions. We need to verify that the following equality holds

$$(\text{id}_{\ker \alpha''} \otimes \varphi_M) \circ \text{ob}_{B \rightarrow A} = \text{ob}_{B' \rightarrow A'} \circ D_{X,x}(\varphi_A).$$

Since $(\gamma \otimes \text{id}_{M'})$ is injective it is enough to observe that we have

$$\begin{aligned} (\gamma \otimes \text{id}_{M'}) \circ (\text{id}_{\ker \alpha''} \otimes \varphi_M) \circ \text{ob}_{B \rightarrow A} &= (\text{id}_{\text{Ob}_{X,x}} \otimes \varphi_M) \circ (\gamma \otimes \text{id}_M) \circ \text{ob}_{B \rightarrow A} \\ &\stackrel{(2.73)}{=} (\text{id}_{\text{Ob}_{X,x}} \otimes \varphi_M) \circ \text{ob}_{X,x;B \rightarrow A} = \text{ob}_{X,x;B' \rightarrow A'} \circ D_{X,x}(\varphi_A) \\ &\stackrel{(2.73)}{=} (\gamma \otimes \text{id}_{M'}) \circ \text{ob}_{B' \rightarrow A'} \circ D_{X,x}(\varphi_A). \end{aligned}$$

The third equality follows from the fact that the maps $\text{ob}_{X,x;B \rightarrow A}$ and $\text{ob}_{X,x;B' \rightarrow A'}$ are part of the data of a tangent-obstruction theory.

Step 5: Map of tangent-obstruction theories

Finally, we verify that $\iota: \ker \alpha'' \rightarrow \text{Ob}_{X'',x''}$ defines a map of tangent-obstruction theories. Let $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ be a small extension. We have

$$(\iota \otimes \text{id}_M) \circ \text{ob}_{B \rightarrow A} \stackrel{(2.72)}{=} (\beta'' \otimes \text{id}_M) \circ \text{ob}_{X,x;B \rightarrow A} = \text{ob}_{X'',x'';B \rightarrow A} \circ \eta''(A)$$

where the second equality follows from the fact that β'' induces a map of tangent-obstruction theories. \square

2.3.2 Tangent-obstruction theory of multigraded Hilbert schemes

Let $S = \mathbb{k}[\alpha_0, \alpha_1, \dots, \alpha_n]$ be a polynomial ring graded by an abelian group \mathbf{A} . Let $h: \mathbf{A} \rightarrow \mathbb{N}$ be a numerical function. In this subsection we study a tangent-obstruction theory of Hilb_S^h .

Remark 2.74. Suppose that $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ is a small extension and $\tilde{\mathcal{J}}$ is an element of $D_{\text{Hilb}_S^h, [J]}(A)$. A homogeneous ideal $\tilde{\mathcal{J}}' \subseteq S \otimes_{\mathbb{k}} B$ is a lift of $\tilde{\mathcal{J}}$ if and only if:

1. $(S \otimes_{\mathbb{k}} B)/\tilde{\mathcal{J}}'$ is B -flat;
2. there is an isomorphism $\tilde{\mathcal{J}}' \otimes_B A \cong \tilde{\mathcal{J}}$.

Indeed, since B is an Artin local ring, it follows from [80, Lemma 051G] that condition 1 implies that $(S_a \otimes_{\mathbb{k}} B)/\tilde{\mathcal{J}}'_a$ is a free B -module for every $a \in \mathbf{A}$. Thus, it is locally free of rank $h(a)$ since $((S_a \otimes_{\mathbb{k}} B)/\tilde{\mathcal{J}}'_a) \otimes_B A$ is a locally free A -module of rank $h(a)$.

Theorem 2.75. *Let $[J]$ be a closed point of Hilb_S^h . The deformation functor $D_{\text{Hilb}_S^h, [J]}$ has a tangent-obstruction theory with tangent space $\mathbf{T}_{\text{Hilb}_S^h, [J]} = \text{Hom}_S(J, S/J)_0$ and obstruction space $\text{Ob}_{\text{Hilb}_S^h, [J]} = \text{Ext}_S^1(J, S/J)_0$.*

Proof. For tangent space, see [46, Prop. 1.6]. We only sketch the construction of obstruction maps. For details, we refer to [35, Thm. 6.4.5] where the ungraded case is considered.

Let $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ be a small extension. Let $\tilde{\mathcal{J}} \in D_{\text{Hilb}_S^h, [J]}(A)$. We study the lifts of $\tilde{\mathcal{J}}$ to $\tilde{\mathcal{J}}' \in D_{\text{Hilb}_S^h, [J]}(B)$.

Consider the following commutative diagram with exact row and columns:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & \tilde{\mathcal{J}} \otimes_A M & & \tilde{\mathcal{J}} & & \\
& & \downarrow & \searrow \alpha & \downarrow & & \\
0 & \longrightarrow & S \otimes_{\mathbb{k}} M & \longrightarrow & S \otimes_{\mathbb{k}} B & \longrightarrow & S \otimes_{\mathbb{k}} A \longrightarrow 0 \\
& & \downarrow & & \searrow \beta & & \downarrow \\
& & ((S \otimes_{\mathbb{k}} A)/\tilde{\mathcal{J}}) \otimes_A M & & (S \otimes_{\mathbb{k}} A)/\tilde{\mathcal{J}} & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

By a diagram chase, we have a short exact sequence of $S \otimes_{\mathbb{k}} B$ -modules

$$0 \rightarrow ((S \otimes_{\mathbb{k}} A)/\tilde{\mathcal{J}}) \otimes_A M \rightarrow \ker \beta / \text{im } \alpha \rightarrow \tilde{\mathcal{J}} \rightarrow 0.$$

Furthermore, $M \cdot (\ker \beta)$ is contained in $\text{im } \alpha$. Thus, the above short exact sequence is a sequence of $S \otimes_{\mathbb{k}} A$ -modules. Finally, since $\mathfrak{m}_A M$ is zero, we have

$$((S \otimes_{\mathbb{k}} A)/\tilde{J}) \otimes_A M \cong ((S \otimes_{\mathbb{k}} A)/\tilde{J}) \otimes_A \mathbb{k} \otimes_{\mathbb{k}} M \cong S/J \otimes_{\mathbb{k}} M.$$

Therefore, we have associated with $\tilde{J} \in D_{\text{Hilb}_S^h, [J]}(A)$ a short exact sequence of $S \otimes_{\mathbb{k}} A$ -modules

$$0 \rightarrow S/J \otimes_{\mathbb{k}} M \rightarrow \ker \beta / \text{im } \alpha \rightarrow \tilde{J} \rightarrow 0. \quad (2.76)$$

We can consider the corresponding class $ob \in \text{Ext}_{S \otimes_{\mathbb{k}} A}^1(\tilde{J}, S/J \otimes_{\mathbb{k}} M)$. Since all morphisms were of degree 0, in fact ob is an element of $\text{Ext}_{S \otimes_{\mathbb{k}} A}^1(\tilde{J}, S/J \otimes_{\mathbb{k}} M)_0$.

Lifts of $\tilde{J} \in D_{\text{Hilb}_S^h, [J]}(A)$ to $\tilde{J}' \in D_{\text{Hilb}_S^h, [J]}(B)$ are in one-to-one correspondence with splittings $\xi: \tilde{J} \rightarrow \ker \beta / \text{im } \alpha$ of the short exact sequence (2.76) which are homogeneous of degree 0.

Finally, we have natural isomorphisms:

$$\text{Ext}_{S \otimes_{\mathbb{k}} A}^1(\tilde{J}, S/J \otimes_{\mathbb{k}} M)_0 \cong \text{Ext}_S^1(\tilde{J} \otimes_A \mathbb{k}, S/J \otimes_{\mathbb{k}} M)_0 \cong \text{Ext}_S^1(J, S/J)_0 \otimes_{\mathbb{k}} M. \quad (2.77)$$

□

We also use the following description of tangent-obstruction theory for flag multigraded Hilbert schemes.

Theorem 2.78. *Let $k: \mathbf{A} \rightarrow \mathbb{N}$ be another numerical function. Let $[K \subseteq J]$ be a closed point of $\text{Hilb}_S^{k,h}$ and assume that the natural map*

$$\text{Hom}_S(K, S/K)_0 \rightarrow \text{Hom}_S(K, S/J)_0$$

is surjective. There is a tangent-obstruction theory for the deformation functor $D_{\text{Hilb}_S^{k,h}, [K \subseteq J]}$ with tangent and obstruction spaces given by pullbacks

$$\begin{array}{ccc} \mathbf{T}_{\text{Hilb}_S^{k,h}, [K \subseteq J]} & \xrightarrow{\mathbf{T}_{\pi_1}} & \text{Hom}_S(K, S/K)_0 & & \text{Ob}_{\text{Hilb}_S^{k,h}, [K \subseteq J]} & \xrightarrow{\text{Ob}_{\pi_1}} & \text{Ext}_S^1(K, S/K)_0 \\ & & \downarrow & & \downarrow \text{Ob}_{\pi_2} & & \downarrow \\ \text{Hom}_S(J, S/J)_0 & \longrightarrow & \text{Hom}_S(K, S/J)_0 & & \text{Ext}_S^1(J, S/J)_0 & \longrightarrow & \text{Ext}_S^1(K, S/J)_0. \end{array}$$

Moreover, the maps \mathbf{T}_{π_1} , Ob_{π_1} , \mathbf{T}_{π_2} and Ob_{π_2} induce maps of tangent-obstruction theories.

Proof. Proof is analogous to the proof of [60, Thm 4.10]. Therefore, we only sketch the proof.

For tangent space, observe that the bijection from [46, Prop. 1.6]

$$\mathbf{T}_{\text{Hilb}_S^h, [J]} = D_{\text{Hilb}_S^h, [J]}(\mathbb{k}[\varepsilon]/(\varepsilon^2)) \leftrightarrow \text{Hom}_S(J, S/J)_0$$

is given explicitly by

$$\text{Hom}_S(J, S/J)_0 \ni \varphi \mapsto \tilde{J} = \{x + \varepsilon y \mid x \in J, y \in S \text{ such that } y + J = \varphi(x)\} \in D_{\text{Hilb}_S^h, [J]}(\mathbb{k}[\varepsilon]/(\varepsilon^2)).$$

See also [49, Prop. 2.3] for the proof of the analogous result in the ungraded case.

Therefore, an element of $D_{\text{Hilb}_S^{k,h}, [K \subseteq J]}(\mathbb{k}[\varepsilon]/(\varepsilon^2))$ is a pair of homomorphisms

$$\varphi \in \text{Hom}_S(J, S/J)_0 \text{ and } \psi \in \text{Hom}_S(K, S/K)_0$$

corresponding to ideals $\tilde{J}, \tilde{K} \subseteq \mathbb{k}[\varepsilon]/(\varepsilon^2) \otimes_{\mathbb{k}} S$. However, we need to consider only those pairs (φ, ψ) for which there is an inclusion $\tilde{K} \subseteq \tilde{J}$. Therefore, given $x \in K$ and $y \in S$ with $y + K = \psi(x)$ we want to have $\varphi(x) = y + J$. This means precisely, that the images of φ and ψ in $\text{Hom}_S(K, S/J)_0$ agree.

Now we proceed to the construction of obstruction maps. Let $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ be a small extension and let $(\tilde{K}, \tilde{J}) \in D_{\text{Hilb}_S^{k,h}}(A)$. There are obstructions $ob_J \in \text{Ext}_S^1(J, S/J)_0 \otimes_{\mathbb{k}} M$ for lifting \tilde{J} to an element $\tilde{J}' \in D_{\text{Hilb}_S^h}(B)$ and $ob_K \in \text{Ext}_S^1(K, S/K)_0 \otimes_{\mathbb{k}} M$ for lifting \tilde{K} to $\tilde{K}' \in D_{\text{Hilb}_S^k}(B)$.

Let

$$0 \rightarrow S/K \otimes_{\mathbb{k}} M \xrightarrow{\iota_K} \ker \beta_K / \text{im } \beta_K \rightarrow \tilde{K} \rightarrow 0 \quad (2.79)$$

and

$$0 \rightarrow S/J \otimes_{\mathbb{k}} M \rightarrow \ker \beta_J / \text{im } \beta_J \rightarrow \tilde{J} \rightarrow 0$$

be the extensions defining ob_K and ob_J , respectively (see the proof of Theorem 2.75). The images of ob_K and ob_J in $\text{Ext}_S^1(K, S/J)_0 \otimes_{\mathbb{k}} M \cong \text{Ext}_{S \otimes_{\mathbb{k}} A}^1(\tilde{K}, S/J \otimes_{\mathbb{k}} M)_0$ agree since they coincide with the class of the extension

$$0 \rightarrow S/J \otimes_{\mathbb{k}} M \rightarrow \ker \beta_K / \text{im } \alpha_J \rightarrow \tilde{K} \rightarrow 0.$$

Therefore, we have a well defined map $ob_{B \rightarrow A}: D_{\text{Hilb}_S^{k,h}}(A) \rightarrow \text{Ob}_{\text{Hilb}_S^{k,h}, [K \subseteq J]} \otimes_{\mathbb{k}} M$ where $\text{Ob}_{\text{Hilb}_S^{k,h}, [K \subseteq J]}$ is given by the pullback as in the statement. If $(\tilde{K}, \tilde{J}) \in D_{\text{Hilb}_S^{k,h}}(A)$ extends to $(\tilde{K}', \tilde{J}') \in D_{\text{Hilb}_S^{k,h}}(B)$ then in particular \tilde{J} and \tilde{K} lift to \tilde{J}' and \tilde{K}' . Thus, the obstructions ob_J and ob_K are zero.

Conversely, assume that we have $ob_J = 0$ and $ob_K = 0$. There are $\tilde{J}' \in D_{\text{Hilb}_S^h}(B)$ and $\tilde{K}' \in D_{\text{Hilb}_S^k}(B)$ lifting \tilde{J} and \tilde{K} , respectively. However, there is no reason to expect that \tilde{K}' is contained in \tilde{J}' .

Let $f: \tilde{K}' \rightarrow (S \otimes_{\mathbb{k}} B) / \tilde{J}'$ be the natural map. We describe how we can modify \tilde{K}' if necessary so that f becomes zero. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
& & K \otimes_{\mathbb{k}} M & \longrightarrow & S \otimes_{\mathbb{k}} M & \longrightarrow & S/K \otimes_{\mathbb{k}} M \\
& & \swarrow & & \downarrow & & \swarrow \\
& & J \otimes_{\mathbb{k}} M & \longrightarrow & S \otimes_{\mathbb{k}} M & \longrightarrow & S/J \otimes_{\mathbb{k}} M \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \tilde{K}' & \longrightarrow & S \otimes_{\mathbb{k}} B & \longrightarrow & (S \otimes_{\mathbb{k}} B) / \tilde{K}' \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \tilde{J}' & \longrightarrow & S \otimes_{\mathbb{k}} B & \longrightarrow & (S \otimes_{\mathbb{k}} B) / \tilde{J}' \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \tilde{K} & \longrightarrow & S \otimes_{\mathbb{k}} A & \longrightarrow & (S \otimes_{\mathbb{k}} A) / \tilde{K} \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \tilde{J} & \longrightarrow & S \otimes_{\mathbb{k}} A & \longrightarrow & (S \otimes_{\mathbb{k}} A) / \tilde{J}
\end{array}$$

By a diagram chase, we have $f \circ a = 0 = b \circ f$. Therefore, f is induced by a map $f': \widetilde{K} \rightarrow S/J \otimes_{\mathbb{k}} M$.

We have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{S \otimes_{\mathbb{k}} A}(\widetilde{K}, S/K \otimes_{\mathbb{k}} M)_0 & \longrightarrow & \mathrm{Hom}_{S \otimes_{\mathbb{k}} A}(\widetilde{K}, S/J \otimes_{\mathbb{k}} M)_0 \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_S(K, S/K)_0 \otimes_{\mathbb{k}} M & \longrightarrow & \mathrm{Hom}_S(K, S/J)_0 \otimes_{\mathbb{k}} M. \end{array}$$

By assumptions, the lower horizontal map is surjective. Therefore, so is the upper horizontal map. Thus, there is a map $g: \widetilde{K} \rightarrow S/K \otimes_{\mathbb{k}} M$ homogeneous of degree 0 such that it maps to f' under the upper horizontal map.

The ideal \widetilde{K}' is defined by a splitting $\xi_K: \widetilde{K} \rightarrow \ker \beta_K / \mathrm{im} \alpha_K$ of the short exact sequence (2.79). The map $\xi_K - \iota_K \circ g: \widetilde{K} \rightarrow \ker \beta_K / \mathrm{im} \alpha_K$ is another splitting of the short exact sequence (2.79) so it defines a lift $\widetilde{K}'' \in D_{\mathrm{Hilb}_S^g, [K]}(B)$ of \widetilde{K} . It can be checked by a diagram chase that the natural map $\widetilde{K}'' \rightarrow (S \otimes_{\mathbb{k}} B) / \widetilde{J}'$ is zero. Thus $\widetilde{K}'' \subseteq \widetilde{J}'$ is a lift of $\widetilde{K} \subseteq \widetilde{J}$. \square

Let S have the standard \mathbb{Z} -grading. Let f be the Hilbert function of S/I where I is a homogeneous ideal of S . Let d be a positive integer and $g: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by

$$g(a) = \begin{cases} f(a) & \text{for } a \geq d \\ \dim_{\mathbb{k}} S_a & \text{otherwise.} \end{cases}$$

Lemma 2.80. *Let $\pi: \mathrm{Hilb}_S^f \rightarrow \mathrm{Hilb}_S^g$ be the natural map given on closed points by $[I] \mapsto [I \cap \mathfrak{m}^d]$. Let $[I] \in \mathrm{Hilb}_S^f$ be a closed point. Let $\eta: D_{\mathrm{Hilb}_S^f, [I]} \rightarrow D_{\mathrm{Hilb}_S^g, [I \cap \mathfrak{m}^d]}$ be the morphism of deformation functors determined by π . The natural maps*

$$\mathrm{Hom}_S(I, S/I)_0 \rightarrow \mathrm{Hom}_S(I \cap \mathfrak{m}^d, S/I)_0 \xrightarrow{\cong} \mathrm{Hom}_S(I \cap \mathfrak{m}^d, S/I \cap \mathfrak{m}^d)_0$$

and

$$\mathrm{Ext}_S^1(I, S/I)_0 \rightarrow \mathrm{Ext}_S^1(I \cap \mathfrak{m}^d, S/I)_0 \xrightarrow{\cong} \mathrm{Ext}_S^1(I \cap \mathfrak{m}^d, S/I \cap \mathfrak{m}^d)_0$$

from exact sequence of Ext groups, define a morphism of tangent-obstruction theories as defined in Theorem 2.75.

Proof. Observe that both isomorphisms in the statement follow from the fact that $I \cap \mathfrak{m}^d$ has minimal generators of degree at least d , so the degree 0 parts of the Ext groups do not depend on $(S/I)_{<d}$. The claimed description of the map on tangent spaces follows from the bijection

$$\mathbf{T}_{[I]} \mathrm{Hilb}_S^f \leftrightarrow D_{\mathrm{Hilb}_S^f}(\mathbb{k}[\varepsilon]/(\varepsilon^2))$$

from [46, Prop. 1.6] which has been recalled at the beginning of the proof of Theorem 2.78.

Now we concentrate on obstruction spaces. Let $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ be a small extension, $\widetilde{I} \in D_{\mathrm{Hilb}_S^f, [I]}(A)$ and $\widetilde{I}_{\geq d} \in D_{\mathrm{Hilb}_S^g, [I \cap \mathfrak{m}^d]}(A)$ be its image under $\eta(A)$.

Let

$$0 \rightarrow S/I \otimes_{\mathbb{k}} M \rightarrow \ker \beta / \mathrm{im} \alpha \rightarrow \widetilde{I} \rightarrow 0$$

and

$$0 \rightarrow S/I_{\geq d} \otimes_{\mathbb{k}} M \rightarrow \ker \beta' / \operatorname{im} \alpha' \rightarrow \tilde{I}_{\geq d} \rightarrow 0$$

be as in Equation (2.76).

Let $ob_I \in \operatorname{Ext}_S^1(I, S/I)_0 \otimes_{\mathbb{k}} M$ and $ob_{I \cap \mathfrak{m}^d} \in \operatorname{Ext}_S^1(I \cap \mathfrak{m}^d, S/(I \cap \mathfrak{m}^d))_0 \otimes_{\mathbb{k}} M$ be obstructions corresponding to \tilde{I} and $\tilde{I}_{\geq d}$, respectively. As in the proof of Theorem 2.78 we consider the extension

$$0 \rightarrow S/I \otimes_{\mathbb{k}} M \rightarrow \ker \beta' / \operatorname{im} \alpha \rightarrow \tilde{I}_{\geq d} \rightarrow 0. \quad (2.81)$$

We see, that the images of ob_I and $ob_{I \cap \mathfrak{m}^d}$ in $\operatorname{Ext}_S^1(I \cap \mathfrak{m}^d, S/I)_0 \otimes_{\mathbb{k}} M$ agree since they coincide with the class of the extension given by Equation (2.81). Therefore, if we consider $D_{\operatorname{Hilb}_S^g, [I \cap \mathfrak{m}^d]}$ with obstruction space $\operatorname{Ext}_S^1(I \cap \mathfrak{m}^d, S/I)_0$ via the isomorphism $\operatorname{Ext}_S^1(I \cap \mathfrak{m}^d, S/I)_0 \cong \operatorname{Ext}_S^1(I \cap \mathfrak{m}^d, S/I \cap \mathfrak{m}^d)_0$, then the natural map of Ext groups as in the statement induces a map of obstruction theories. \square

2.4 Ranks and apolarity lemmas

In this section we recall various notions of rank and corresponding versions of apolarity lemma. Moreover, we define secant and cactus varieties. The apolarity lemma for border rank from [15] reflects a connection between ideals in $\operatorname{Slip}_{r,n}$ and the condition that a homogeneous polynomial in $n+1$ variables has border rank at most r . This is the main motivation to study the irreducible component $\operatorname{Slip}_{r,n}$. Observe that the connection from [15] works for more general toric varieties.

We present the theory of ranks of homogeneous subspaces instead of the more standard version of ranks of homogeneous polynomials. This is due to the fact that we need the general version in Chapter 5.

2.4.1 Apolarity action

Let n be a positive integer and $S = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ be a polynomial ring. Let $S^* = \mathbb{k}_{dp}[x_0, \dots, x_n]$ be the graded dual ring, where the index dp refers to the divided power structure of S^* . As a \mathbb{Z} -graded \mathbb{k} -vector space the ring S^* is given by

$$S^* = \bigoplus_{a \in \mathbb{Z}} \operatorname{Hom}_{\mathbb{k}}(S_a, \mathbb{k})$$

and the ring structure is the divided power structure. See [56, Appendix A] for basic properties of divided power ring S^* . For all $\mathbf{u} = (u_0, \dots, u_n) \in \mathbb{N}^{n+1}$ we define

$$\alpha^{\mathbf{u}} = \prod_{i=0}^n \alpha_i^{u_i}.$$

For an integer $a \in \mathbb{Z}$, the vector space S_a has a monomial basis, i.e.

$$S_a = \bigoplus_{\mathbf{u} \in \mathbb{N}^{n+1}, |\mathbf{u}|=a} \mathbb{k} \alpha^{\mathbf{u}}$$

where $|\mathbf{u}|$ denotes $\sum_{i=0}^n u_i$. Let $\{x^{[\mathbf{u}]} \mid \mathbf{u} \in \mathbb{N}^{n+1}, |\mathbf{u}| = a\}$ be the dual basis of S_a^* . We define the multiplication in S^* on divided power monomials by

$$x^{[\mathbf{u}]} \cdot x^{[\mathbf{v}]} = \frac{(\mathbf{u} + \mathbf{v})!}{\mathbf{u}!\mathbf{v}!} x^{[\mathbf{u}+\mathbf{v}]},$$

where $\mathbf{w}! = w_0! \cdots w_n!$ for all $\mathbf{w} \in \mathbb{N}^{n+1}$. We extend it by linearity to define a \mathbb{k} -algebra structure on S^* .

There is a natural action of S on S^* denoted by \lrcorner given on homogeneous elements $\theta \in S_a$ and $F \in S_b^*$ by

$$(\theta \lrcorner F)(\xi) = F(\theta\xi) \text{ for every } \xi \in S_{b-a}.$$

Using the monomial bases, the action of S on S^* can be written in the form

$$\alpha^{\mathbf{u}} \lrcorner x^{[\mathbf{v}]} = \begin{cases} x^{[\mathbf{v}-\mathbf{u}]} & \text{if } v_k \geq u_k \text{ for } k = 0, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

If \mathbb{k} has characteristic zero then we have $x^{[\mathbf{v}]} = \frac{x^{\mathbf{v}}}{\mathbf{v}!}$ and S^* is a polynomial ring.

Let d be a positive integer and $W \subseteq S_{\leq d}^*$ be a vector subspace. Using the action of S on S^* , we can associate with W a \mathbb{k} -algebra $\text{Apolar}(W)$.

Definition 2.82. Let d be a positive integer and $W \subseteq S_{\leq d}^*$ be a non-zero linear subspace. The *annihilator* of W is the ideal

$$\text{Ann}(W) = \{\theta \in S \mid \theta \lrcorner f = 0 \text{ for every } f \in W\}.$$

The *apolar algebra* of W is $\text{Apolar}(W) = S / \text{Ann}(W)$. If we have $W = \langle f \rangle$ for some $f \in S_{\leq d}^*$, we write $\text{Ann}(f)$ and $\text{Apolar}(f)$ instead of $\text{Ann}(\langle f \rangle)$ and $\text{Apolar}(\langle f \rangle)$.

Annihilator $\text{Ann}(W)$ of $W \subseteq S_{\leq d}^*$ plays a key role in apolarity lemmas which connect the properties of $\text{Ann}(W)$ and various notions of ranks of W .

2.4.2 (Border) rank, smoothable rank and (border) cactus rank

We keep the notation of Subsection 2.4.1. Let d be a positive integer. We recall various notions of rank of a subspace V of S_d^* . It is important to realize that the process of generalizing the definition of rank was not as straight-forward as it may seem from the short presentation that we give.

We start with introducing the rank of V . If $V = \langle F \rangle$ is one-dimensional, this is a classical notion that goes back to works of Sylvester. The general case was studied among others by Terracini [79] and Bronowski [8].

Definition 2.83. Let d be a positive integer and $V \subseteq S_d^*$ be a non-zero linear subspace. We define *rank* of V to be

$$r(V) = \min\{r \in \mathbb{Z} \mid \mathbb{P}V \subseteq \langle L_1^{[d]}, \dots, L_r^{[d]} \rangle \text{ for some } L_i \in S_1^*\},$$

where $\langle - \rangle$ denotes the projective linear span.

It is interesting to describe, for a positive integer k , the locus of all points $[V] \in \text{Gr}(k, S_d^*)$ with rank at most r . However, a more natural geometric object is the Zariski closure of this locus. Recall that $\nu_d: \mathbb{P}S_1^* \rightarrow \mathbb{P}S_d^*$ given by $[L] \mapsto [L^{[d]}]$ is the Veronese embedding.

Definition 2.84. Let d, k and r be positive integers. The r -th Grassmann secant variety is

$$\sigma_{r,k}(\nu_d(\mathbb{P}S_1^*)) = \overline{\{[V] \in \text{Gr}(k, S_d^*) \mid \text{r}(V) \leq r\}}.$$

If $k = 1$, we write $\sigma_r(\nu_d(\mathbb{P}S_1^*))$ instead of $\sigma_{r,1}(\nu_d(\mathbb{P}S_1^*))$ and call it the r -th secant variety.

With Grassmann secant varieties already defined, it is natural to introduce another variant of rank.

Definition 2.85. Let d and k be positive integers and $V \subseteq S_d^*$ be a k -dimensional linear subspace. The *border rank* of V is defined to be

$$\text{br}(V) = \min\{r \in \mathbb{Z} \mid [V] \in \sigma_{r,k}(\nu_d(\mathbb{P}S_1^*))\}.$$

The ranks and borders ranks of monomials have been studied in [64]. See also [63], [11] and [39] for some results concerning equations of secant varieties.

The definition of rank of $[V] \in \text{Gr}(k, S_d^*)$ can be restated as follows

$$\text{r}(V) = \min\{r \in \mathbb{Z} \mid \text{there exists a smooth zero-dimensional subscheme } R \subseteq \mathbb{P}S_1^* \text{ of length } r \text{ with } \mathbb{P}V \subseteq \langle \nu_d(R) \rangle\}.$$

One can consider different variants of this definition. The condition that R is smooth could be replaced by the condition that it is a limit of smooth schemes or it can be even skipped completely. These lead to the notions of smoothable rank and cactus rank. Let $\text{Hilb}_r^{\text{sm}}(\mathbb{P}S_1^*)$ be the smoothable component, i.e. the closure of the locus of all points of $\text{Hilb}_r(\mathbb{P}S_1^*)$ corresponding to smooth subschemes.

Definition 2.86. Let d and k be positive integers and $[V] \in \text{Gr}(k, S_d^*)$. The *smoothable rank* of V is defined to be

$$\text{sr}(V) = \min\{r \in \mathbb{Z} \mid \mathbb{P}V \subseteq \langle \nu_d(R) \rangle \text{ for some } [R] \in \text{Hilb}_r^{\text{sm}}(\mathbb{P}S_1^*)\}.$$

We are interested only in the smoothable rank of a non-zero homogeneous polynomial $F \in S_d^*$. We have $\text{br}(F) \leq \text{sr}(F)$. Following [12] we make a definition capturing the cases where the inequality is strict.

Definition 2.87. Let d be a positive integer. A non-zero polynomial $F \in S_d^*$ is *wild* if we have $\text{br}(F) < \text{sr}(F)$.

Considering the whole Hilbert scheme $\text{Hilb}_r(\mathbb{P}S_1^*)$ instead of its smoothable component leads to the definition of cactus rank.

Definition 2.88. Let d and k be positive integers and $[V] \in \text{Gr}(k, S_d^*)$. The *cactus rank* of V is defined to be

$$\text{cr}(V) = \min\{r \in \mathbb{Z} \mid \mathbb{P}V \subseteq \langle \nu_d(R) \rangle \text{ for some } [R] \in \text{Hilb}_r(\mathbb{P}S_1^*)\}.$$

Similarly, as in the case of rank, we can consider the Zariski closure in $\text{Gr}(k, S_d^*)$ of those $[V]$ which have cactus rank at most r .

Definition 2.89. Let d, k and r be positive integers. The r -th Grassmann cactus variety is

$$\kappa_{r,k}(\nu_d(\mathbb{P}S_1^*)) = \overline{\{[V] \in \text{Gr}(k, S_d^*) \mid \text{cr}(V) \leq r\}}.$$

If $k = 1$, we write $\kappa_r(\nu_d(\mathbb{P}S_1^*))$ instead of $\kappa_{r,1}(\nu_d(\mathbb{P}S_1^*))$ and call it the r -th cactus variety.

Cactus varieties have been introduced in [11]. The name might be slightly confusing since in general cactus varieties are not irreducible. An example when the cactus variety is reducible is presented in [40, Thm. 1.4, 1.5]. The cactus rank has been studied for instance in [74] and [4].

Finally, we define the border cactus rank in a way analogous to the definition of border rank.

Definition 2.90. Let d and k be positive integers and $V \subseteq S_d^*$ be a k -dimensional linear subspace. The *border cactus rank* of V is defined to be

$$\text{bcr}(V) = \min\{r \in \mathbb{Z} \mid [V] \in \kappa_{r,k}(\nu_d(\mathbb{P}S_1^*))\}.$$

2.4.3 Apolarity lemmas

The ranks of a subspace $V \subseteq S_d^*$ can be computed by apolarity lemmas. We state only the versions of apolarity lemma that we use. We start with the one related to the cactus rank.

Proposition 2.91 (Cactus apolarity lemma). *Let d be a positive integer and $V \subseteq S_d^*$ be a non-zero subspace. For a positive integer r we have $\text{cr}(V) \leq r$ if and only if there is a saturated, homogeneous ideal $I \subseteq \text{Ann}(V)$ such that S/I has Hilbert polynomial r .*

For a proof, see [78, Thm. 4.7].

Next, we deal with the border rank. The version for a polynomial $F \in S_d^*$ is a special case of a recent result by Buczyńska and Buczyński [15]. It is the main motivation for studying the irreducible component $\text{Slip}_{r,n}$ of the multigraded Hilbert scheme $\text{Hilb}_S^{h_r,n}$. The following version for subspaces is presented in [40, Prop. 2.3] and follows from the proof of [13, Thm. 1.3].

Proposition 2.92 (Border apolarity lemma). *Let d be a positive integer and $V \subseteq S_d^*$ be a non-zero subspace. For a positive integer r we have $\text{br}(V) \leq r$ if and only if there exists a homogeneous ideal $I \subseteq \text{Ann}(V)$ such that $[I]$ belongs to $\text{Slip}_{r,n} \subseteq \text{Hilb}_S^{h_r,n}$.*

Finally, we present a weak version of apolarity lemma for border cactus rank. Following [40] we introduce the following definition.

Definition 2.93. For positive integers r and n , a function $h: \mathbb{Z} \rightarrow \mathbb{Z}$ is called an $(r, n + 1)$ -standard Hilbert function if it satisfies the following conditions:

- (i) the inequality $h(d) \leq h(d + 1)$ holds for all $d \in \mathbb{Z}$;
- (ii) if there is an equality $h(d) = h(d + 1)$ for some $d \geq 0$, then we have $h(e) = r$ for all $e \geq d$;
- (iii) the inequality $0 \leq h(d) \leq h_{r,n}(d)$ holds for all $d \in \mathbb{Z}$.

Proposition 2.94 (Weak border cactus apolarity lemma). *Let d be a positive integer and $V \subseteq S_d^*$ be a non-zero subspace. If we have $\text{bcr}(V) \leq r$ for some positive integer r , then there exists a homogeneous ideal $I \subseteq \text{Ann}(V)$ such that S/I has an $(r, n + 1)$ -standard Hilbert function.*

See [13, Thm. 1.1] for a proof.

Chapter 3

Criteria for projective space

In this chapter we present conditions for a point $[I]$ in the multigraded Hilbert scheme $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,n}}$ to be in the irreducible component $\text{Slip}_{r,n}$.

Section 3.1 contains a necessary condition based on bounding the degrees of minimal generators of saturated ideals with Hilbert function $h_{r,n}$.

The criterion from Section 3.2 is based on the properties of the Hilbert function of a power of a radical ideal with Hilbert function $h_{r,n}$ that were established in Proposition 2.19.

In Section 3.3 we show that the locus of all points of Hilb_S^f corresponding to saturated ideals is smooth and irreducible when $n = 2$ and f is the Hilbert function of a zero-dimensional subscheme of \mathbb{P}^2 . We also show that in $\mathcal{Hilb}_r(\mathbb{P}^2)$ the locus of all points corresponding to subschemes with fixed Hilbert function is irreducible. In characteristic 0 this has been shown by Gotzmann [42].

In Section 3.4 we present a sufficient condition for $[I]$ as above to be in $\text{Slip}_{r,n}$ for $n = 2$.

Sections 3.5 and 3.7 contain some examples. In particular, in Section 3.7 we present the full set-theoretic description of $\text{Slip}_{r,2}$ for all $r \in \{4, 5, 6\}$.

Section 3.6 presents a necessary condition (Theorem 3.42). This criterion has three technical assumptions, one is based on smoothness of the Hilbert scheme at a prescribed point while two are about surjectivity of some maps of spaces of homomorphisms. In Subsections 3.6.1, 3.6.2 and 3.6.3 we study some cases in which the assumptions of Theorem 3.42 are fulfilled. In Subsections 3.6.4 and 3.6.5 we present some nice applications of Theorem 3.42.

The main results of this chapter are criteria for $[I] \in \text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,n}}$ to be in $\text{Slip}_{r,n}$:

- Proposition 3.1 which is an example of a small tangent space argument;
- Theorem 3.5 which shows that if $[I]$ is in $\text{Slip}_{r,n}$ then H_{S/I^k} is large enough in large degrees;
- Theorem 3.13 which is a sufficient condition in the case $n = 2$. It states that if the Hilbert function of S/\bar{I} differs from $h_{r,2}$ in exactly one degree then $[I]$ belongs to $\text{Slip}_{r,2}$;
- Theorem 3.42 and its applications Theorems 3.66 and 3.75.

Notation

Throughout this chapter, r and n are positive integers and $S = S[\mathbb{P}^n] = \mathbb{k}[\alpha_0, \dots, \alpha_n]$ is a polynomial ring over a fixed algebraically closed field \mathbb{k} . Recall that $h_{r,n}: \mathbb{Z} \rightarrow \mathbb{Z}$ is defined by

$$h_{r,n}(a) = \min\{\dim_{\mathbb{k}} S_a, r\}.$$

3.1 Criterion based on degrees of minimal generators

Our first criterion is obtained by bounding the degrees of minimal generators of saturated ideals of points. The proof is an illustration of a small tangent space method: if the dimension of the tangent space to a \mathbb{k} -scheme X at a point x is smaller than the dimension of an irreducible closed subset Y , then we have $x \notin Y$. See [55] for a classical application of this argument.

Proposition 3.1. *Let $I \subseteq S$ be a homogeneous ideal such that S/I has Hilbert function $h_{r,n}$. Let $e = \min\{a \in \mathbb{Z} \mid h_{r,n}(a) = r\}$ and $d \geq e + 2$. If the inequality $\dim_{\mathbb{k}} \text{Hom}_S(I + \mathfrak{m}^d, S/(I + \mathfrak{m}^d))_0 < rn$ holds, then we have $[I] \notin \text{Slip}_{r,n}$.*

Proof. Define $g: \mathbb{Z} \rightarrow \mathbb{Z}$ by

$$g(a) = \begin{cases} h_{r,n}(a) & \text{if } a < d \\ 0 & \text{otherwise.} \end{cases}$$

Let $\pi: \text{Hilb}_S^{h_{r,n}} \rightarrow \text{Hilb}_S^g$ be the morphism defined on closed points by $[J] \mapsto [J + \mathfrak{m}^d]$. We show that

$$\text{if } J \text{ and } J' \text{ are saturated ideals and } \pi([J]) = \pi([J']), \text{ then we have } J = J'. \quad (3.2)$$

It is enough to show that every saturated ideal J of S such that S/J has Hilbert function $h_{r,n}$ is generated in degrees at most $d - 1$. Since J is saturated, we have $\text{depth } S/J \geq 1$ by Lemma 2.9(i). The quotient algebra S/J has Krull dimension 1. It follows that S/J is Cohen-Macaulay. Furthermore, by the Auslander-Buchsbaum Theorem [70, Thm. 15.3] the projective dimension of S/J is n . Therefore, we get $\text{reg}(S/J) = e$ from [32, Thm. 4.2]. Consequently, $\beta_{1,a}(S/J)$ is zero for all $a \geq e + 2$. Thus, J is generated in degrees at most $e + 1 \leq d - 1$.

The irreducible component $\text{Slip}_{r,n}$ has dimension rn . Therefore, by (3.2) the irreducible closed subset $\pi(\text{Slip}_{r,n})$ is also of dimension rn . Consequently, if $[I]$ belongs to $\text{Slip}_{r,n}$ then we have

$$rn \leq \dim_{\mathbb{k}} \mathbf{T}_{[I + \mathfrak{m}^d]} \text{Hilb}_S^g = \dim_{\mathbb{k}} \text{Hom}_S(I + \mathfrak{m}^d, S/(I + \mathfrak{m}^d))_0,$$

where the equality follows from Theorem 2.75. \square

Example 3.3. Let $S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$ and consider $I = (\alpha_0^3, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \alpha_1^6)$. The point $[I]$ is in $\text{Hilb}_S^{h_{6,2}}$ and we claim that it is not in $\text{Slip}_{6,2}$. Let $J = I + (\alpha_0, \alpha_1, \alpha_2)^5$. We have

$$\dim_{\mathbb{k}} \text{Hom}_S(J, S/J)_0 = 8 < 12.$$

The claim follows from Proposition 3.1.

3.2 Criterion based on a power of ideal

The criterion presented in this section is based on Proposition 2.19. There we computed the Hilbert polynomial of a power of a homogeneous radical ideal defining a zero-dimensional subscheme of projective space. Moreover, we bounded the degree from which this Hilbert polynomial agrees with Hilbert function. In Theorem 3.5, using the semicontinuity of the dimensions of the fibers of a sheaf of modules (see [48, Ex. II.5.8]), we obtain a criterion for $[I] \in \text{Hilb}_S^{h_{r,n}}$ to be in $\text{Slip}_{r,n}$.

In Subsection 2.2.3 we introduced subsets $\text{Sip}_{r,n}$ and $\text{Slip}_{r,n}$ of the multigraded Hilbert scheme $\text{Hilb}_S^{h_{r,n}}$. Here we generalize this for some functions $h: \mathbb{Z} \rightarrow \mathbb{Z}$ more general than $h_{r,n}$.

Definition 3.4. Assume that $h: \mathbb{Z} \rightarrow \mathbb{Z}$ is the Hilbert function of a zero-dimensional closed subscheme of \mathbb{P}^n . We denote by $\text{Sip}_{h,n}$ the locus of all closed points of Hilb_S^h corresponding to radical ideals. Moreover, let $\text{Slip}_{h,n}$ be the closure of $\text{Sip}_{h,n}$ in Hilb_S^h .

The following theorem provides a necessary condition for a closed point of Hilb_S^h to be in $\text{Slip}_{h,n}$, where $h: \mathbb{Z} \rightarrow \mathbb{Z}$ is as in Definition 3.4.

Theorem 3.5. *Let $h: \mathbb{Z} \rightarrow \mathbb{Z}$ be the Hilbert function of a zero-dimensional length r closed subscheme of \mathbb{P}^n . Define $e = \min\{a \in \mathbb{Z} \mid h(a) = r\}$ and let $[I] \in \text{Hilb}_S^h$ be a closed point. If the point $[I]$ belongs to $\text{Slip}_{h,n}$, then $H_{S/I^k}(d) \geq r \cdot \dim_{\mathbb{k}} S_{k-1}$ holds for every positive integer k and for every $d \geq ke + k$.*

Proof. Let \mathcal{I} be the universal ideal sheaf on $\text{Hilb}_S^h \times \mathbb{A}^{n+1}$. Consider the quotient \mathcal{P} of

$$\mathcal{O}_{\text{Hilb}_S^h \times \mathbb{A}^{n+1}} \cong \mathcal{O}_{\text{Hilb}_S^h}[\alpha_0, \dots, \alpha_n]$$

by \mathcal{I}^k and let \mathcal{Q} be the pushforward of \mathcal{P} under the projection morphism $\pi: \text{Hilb}_S^h \times \mathbb{A}^{n+1} \rightarrow \text{Hilb}_S^h$. The sheaf $\mathcal{Q} = \bigoplus_d \mathcal{Q}_d$ is a quasi-coherent sheaf on Hilb_S^h with \mathcal{Q}_d a coherent sheaf for every $d \in \mathbb{Z}$. Therefore, for every $d \in \mathbb{Z}$, the rank function $\varphi_d: \text{Hilb}_S^h \rightarrow \mathbb{Z}$ given by $\varphi_d(x) = \dim_{\kappa(x)}(\mathcal{Q}_d)_x \otimes_{\mathcal{O}_{\text{Hilb}_S^h, x}} \kappa(x)$ is upper semicontinuous (see [48, Ex. II.5.8]). We claim that for a closed point $P = [K] \in \text{Hilb}_S^h$ we have $\varphi_d(P) = H_{S/K^k}(d)$.

This can be checked affine locally, so we can replace Hilb_S^h by an affine open subset $U = \text{Spec } A$ containing $[K]$. Let J be the ideal in $A[\alpha_0, \dots, \alpha_n]$ defining the restriction of \mathcal{I} to $\pi^{-1}(U)$. Let $[K]$ in U correspond to the maximal ideal \mathfrak{n} of A . In what follows, we consider \mathbb{k} with A -module structure given by $A \rightarrow A_{\mathfrak{n}}/\mathfrak{n}A_{\mathfrak{n}} \cong \mathbb{k}$. By the definition of universal ideal sheaf we have $(A[\alpha_0, \dots, \alpha_n]/J) \otimes_A \mathbb{k} \cong S/K$. Therefore, from the universal property of kernel, there is an induced map $J \otimes_A \mathbb{k} \rightarrow K$ fitting into the commutative diagram

$$\begin{array}{ccccccc} J \otimes_A \mathbb{k} & \longrightarrow & S & \longrightarrow & S/(J \otimes_A \mathbb{k}) & \longrightarrow & 0 \\ & & \vdots & & \parallel & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & S & \longrightarrow & S/K \longrightarrow 0 \end{array}$$

whose rows are exact. It follows from snake lemma that the map $J \otimes_A \mathbb{k} \rightarrow K$ is surjective. Hence also the map $J^k \otimes_A \mathbb{k} \rightarrow K^k$ is surjective. The snake lemma applied to the diagram

$$\begin{array}{ccccccc} J^k \otimes_A \mathbb{k} & \longrightarrow & S & \longrightarrow & S/(J^k \otimes_A \mathbb{k}) & \longrightarrow & 0 \\ & & \downarrow & & \parallel & & \vdots \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K^k & \longrightarrow & S & \longrightarrow & S/K^k \longrightarrow 0 \end{array}$$

implies that the dotted arrow induced by the universal property of cokernel is injective. Since it is clearly surjective, it is an isomorphism. Thus we have

$$\varphi_d([K]) = \dim_{\mathbb{k}}(S/(J^k \otimes_A \mathbb{k}))_d = \dim_{\mathbb{k}}(S/K^k)_d = H_{S/K^k}(d)$$

and the claim of the theorem follows from Proposition 2.19. \square

Example 3.6. Let $S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$ and consider the ideal

$$I' = (\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_2^6).$$

The point $[I']$ is in $\text{Hilb}_S^{h_{6,2}}$ and we claim that it is not in $\text{Slip}_{6,2}$. We have $H_{S/(I')^2}(9) = 17 < 18$. Thus, the claim follows from Theorem 3.5.

Observe that $\dim_{\mathbb{k}} \text{Hom}_S(I' + \mathfrak{m}^d, S/(I' + \mathfrak{m}^d)) \geq 12$ holds for all $d \geq 4$, so the criterion from Proposition 3.1 cannot be applied to deduce that $[I']$ does not belong to $\text{Slip}_{6,2}$.

Consider again the ideal $I = (\alpha_0^3, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \alpha_2^6)$ from Example 3.3. We have $H_{S/I^2}(d) \geq 18$ for all $d \geq 6$. Hence we cannot use the criterion from Theorem 3.5 (with $k = 2$) to deduce that $[I]$ is not in $\text{Slip}_{6,2}$. We summarize this in the following table.

Ideal	Proposition 3.1	Theorem 3.5
$(\alpha_0^3, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \alpha_2^6) \subseteq \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$	✓	?
$(\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_2^6) \subseteq \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$?	✓

Table 3.1: Examples of points that are not in the irreducible component Slip .

Here, for both ideals, the corresponding point of $\text{Hilb}_S^{h_{6,2}}$ is outside the irreducible component $\text{Slip}_{6,2}$. The check mark (✓) indicates that a given criterion shows that a given point is not in $\text{Slip}_{6,2}$. The question mark (?) shows that a given criterion is inconclusive.

3.3 Smoothness and irreducibility of the locus of saturated ideals of points in projective plane

Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be the Hilbert function of a zero-dimensional subscheme of \mathbb{P}^2 of length r . Let V_f be the locally closed subset of $\mathcal{H}ilb_r(\mathbb{P}^2)$ whose points correspond to the subschemes of \mathbb{P}^2 with Hilbert function f . If the field \mathbb{k} has characteristic 0 then V_f with the reduced structure is smooth and irreducible [42].

In this section we show that in fact V_f is irreducible for any characteristic of \mathbb{k} . Furthermore, let $E_f \subseteq \text{Hilb}_S^f$ be the open subset whose points correspond to saturated ideals. We show that E_f is irreducible and smooth.

In this section we consider polynomial rings $S = \mathbb{k}[\alpha_0, \alpha_1, \dots, \alpha_n]$ and $T = \mathbb{k}[\alpha_0, \alpha_1, \dots, \alpha_{n-1}]$ for a positive integer n . We eventually restrict our attention to the case $n = 2$, but we do not make this assumption when the proofs work more generally.

We need the following result on the behavior of Ext groups under flat base change.

Lemma 3.7. *Let $R \rightarrow S$ be a flat ring homomorphism with R a Noetherian ring. For every finitely generated R -module M and any R -module N , the natural map*

$$\text{Ext}_R^i(M, N) \otimes_R S \rightarrow \text{Ext}_S^i(M \otimes_R S, N \otimes_R S)$$

is an isomorphism for all integers i .

Proof. Since R is Noetherian and M is a finitely generated R -module, there exists a projective resolution P_\bullet of M by finitely generated free R -modules. The R -module S is flat. Hence we obtain a projective resolution $P_\bullet \otimes_R S$ of $M \otimes_R S$.

Finally, since M is finitely presented and S is a flat R -module, the natural map

$$\mathrm{Hom}_R(P_\bullet, N) \otimes_R S \rightarrow \mathrm{Hom}_S(P_\bullet \otimes_R S, N \otimes_R S)$$

is an isomorphism of chain complexes of S -modules by [31, Prop. 2.10]. Therefore, we have natural isomorphisms of S -modules

$$\begin{aligned} \mathrm{Ext}_R^i(M, N) \otimes_R S &= H^i(\mathrm{Hom}_R(P_\bullet, N)) \otimes_R S = H^i(\mathrm{Hom}_R(P_\bullet, N) \otimes_R S) \\ &= H^i(\mathrm{Hom}_S(P_\bullet \otimes_R S, N \otimes_R S)) = \mathrm{Ext}_S^i(M \otimes_R S, N \otimes_R S). \end{aligned} \quad \square$$

The following lemma gives a condition under which $E_f \subseteq \mathrm{Hilb}_S^f$ is smooth. Observe that this condition is satisfied for $n = 2$ since $\mathcal{Hilb}_r(\mathbb{P}^2)$ is smooth.

Lemma 3.8. *Let f be the Hilbert function of a zero-dimensional subscheme of \mathbb{P}^n of length r . Let $E_f \subseteq \mathrm{Hilb}_S^f$ be the open subset whose points correspond to saturated ideals. If $[\mathrm{Proj} S/I]$ is a smooth point of $\mathcal{Hilb}_r(\mathbb{P}^n)$ for every point $[I] \in E_f$ such that I is a saturated Borel-fixed ideal, then E_f is smooth.*

Proof. Let $[J] \in \mathrm{Hilb}_S^f$ be a closed point such that J is a saturated ideal. Let J' be the generic initial ideal (see [31, §15.9]) of J with respect to the grevlex order with $\alpha_0 > \alpha_1 > \dots > \alpha_n$. It is enough to show that $[J'] \in \mathrm{Hilb}_S^f$ is a smooth point.

The ideal J' is saturated by Corollary 2.10. Moreover, it is Borel-fixed by [31, Thm. 15.20]. Thus, $[\mathrm{Proj} S/J']$ is a smooth point of $\mathcal{Hilb}_r(\mathbb{P}^n)$ by assumptions.

Therefore, by Lemma 2.67, it is enough to show that the natural map of deformation functors $D_{\mathrm{Hilb}_S^f, [J']} \rightarrow D_{\mathcal{Hilb}_r(\mathbb{P}^n), [\mathrm{Proj} S/J']}$ induced by the map $\mathrm{Hilb}_S^f \rightarrow \mathcal{Hilb}_r(\mathbb{P}^n)$ admits a map of tangent-obstruction theories that is injective on obstruction spaces. By Lemmas 2.44 and 2.80 there is a map of tangent-obstruction theories which on obstruction spaces is the natural map

$$\mathrm{Ext}_S^1(J', S/J')_0 \rightarrow \mathrm{Ext}_S^1(J' \cap \mathfrak{m}^r, S/J')_0$$

from the exact sequence of Ext groups. It suffices to show that

$$\mathrm{Ext}_S^1(J'/J' \cap \mathfrak{m}^r, S/J')_0 \rightarrow \mathrm{Ext}_S^1(J', S/J')_0 \quad (3.9)$$

is the zero map. Since J' is a saturated and Borel-fixed ideal, it is an extended ideal from the polynomial ring $T = \mathbb{k}[\alpha_0, \dots, \alpha_{n-1}]$ by Lemma 2.11. Therefore, by Lemma 3.7 we get $\mathrm{Ext}_S^1(J', S/J') \cong \mathrm{Ext}_T^1(\mathfrak{a}, T/\mathfrak{a}) \otimes_T S$ where \mathfrak{a} is the contracted ideal $T \cap J'$. In particular, multiplication by α_n is injective on $\mathrm{Ext}_S^1(J', S/J')$. On the other hand, since multiplication by α_n^r is zero on $J'/J' \cap \mathfrak{m}^r$, it is zero on $\mathrm{Ext}_S^1(J'/J' \cap \mathfrak{m}^r, S/J')$ as well. It follows that the map from Equation (3.9) is the zero map. \square

Finally we present the main results of this section.

Proposition 3.10. *Let $S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$ and $T = \mathbb{k}[\alpha_0, \alpha_1]$. Let f be the Hilbert function of a zero-dimensional length r subscheme of \mathbb{P}^2 . Let E_f be the open subset of Hilb_S^f whose points correspond to saturated ideals. The subscheme E_f is smooth and irreducible.*

Proof. Smoothness of E_f follows from Lemma 3.8 since the Hilbert scheme $\mathcal{Hilb}_r(\mathbb{P}^2)$ is smooth for every positive integer r .

Now we show that E_f is connected. Since it is also smooth this would finish the proof. Given a point $[I] \in E_f$ we may connect it to the point $[I']$ corresponding to the generic initial ideal I' of I with respect to the grevlex order with $\alpha_0 > \alpha_1 > \alpha_2$. The ideal I' is saturated by Corollary 2.10 and it is Borel-fixed by [31, Thm. 15.20]. Therefore, it is enough to find a connected subset of E_f that contains all points corresponding to Borel-fixed saturated ideals.

Let $g: \mathbb{Z} \rightarrow \mathbb{Z}$ be given by $g(a) = f(a) - f(a - 1)$ for every $a \in \mathbb{Z}$. We have a natural map $\text{Hilb}_T^g \rightarrow \text{Hilb}_S^f$. The scheme Hilb_T^g is irreducible by [65]. Therefore, its image Z in Hilb_S^f is irreducible. By construction, Z is contained in E_f . Furthermore, it contains all saturated Borel-fixed ideals by Lemma 2.11. \square

Proposition 3.11. *Let $S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$. Let f be the Hilbert function of a zero-dimensional length r subscheme of \mathbb{P}^2 . Let V_f be the locally closed subset of $\mathcal{Hilb}_r(\mathbb{P}^2)$ whose points correspond to the subschemes with Hilbert function f . The subset V_f is irreducible.*

Proof. As before, let E_f be the open subset of Hilb_S^f whose points correspond to the saturated ideals. The natural map $\text{Hilb}_S^f \rightarrow \mathcal{Hilb}_r(\mathbb{P}^2)$ induces a map $E_f \rightarrow V_f$ which is bijective on \mathbb{k} -points. Therefore, V_f is homeomorphic with E_f by Lemma 2.29. It follows from Proposition 3.10 that V_f is irreducible. \square

3.4 Sufficient condition for projective plane

In this section we assume that $n = 2$ so $S = S[\mathbb{P}^2] = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$ is the homogeneous coordinate ring of projective plane. We show that a closed points $[I] \in \text{Hilb}_S^{h_{r,2}}$ for which S/\bar{I} has Hilbert function differing from $h_{r,2}$ in only one degree is in $\text{Slip}_{r,2}$.

For ease of reference we write this condition more precisely. We consider functions $f: \mathbb{Z} \rightarrow \mathbb{Z}$ that satisfy the following condition:

$$\text{there exist } e, d \in \mathbb{Z}_{>0} \text{ with } f(a) = \begin{cases} h_{r,2}(a) & \text{if } a \neq e \\ d & \text{if } a = e. \end{cases} \quad (3.12)$$

The main result of this section is the following theorem which gives a sufficient condition for a closed point $[I] \in \text{Hilb}_S^{h_{r,2}}$ to be in $\text{Slip}_{r,2}$.

Theorem 3.13. *Let $[I]$ be a closed point of the multigraded Hilbert scheme $\text{Hilb}_S^{h_{r,2}}$. If the Hilbert function of S/\bar{I} satisfies condition (3.12), then we have $[I] \in \text{Slip}_{r,2}$.*

Before proving Theorem 3.13 we need a few lemmas. The first one enables us to consider a more restrictive condition (3.15) instead of condition (3.12).

Lemma 3.14. *Let $[I] \in \text{Hilb}_S^{h_{r,2}}$. If the Hilbert function f of S/\bar{I} satisfies condition (3.12) for some integers d and e then we have $[I] \in \text{Slip}_{r,2}$ unless the following holds:*

$$f \text{ satisfies condition (3.12) and } \dim_{\mathbb{k}} S_{e-1} < d < r < \dim_{\mathbb{k}} S_e. \quad (3.15)$$

Proof. Assume that $[I]$ is a point of $\text{Hilb}_S^{h_{r,2}}$ that does not belong to $\text{Slip}_{r,2}$ and the Hilbert function of S/\bar{I} satisfies condition (3.12) for some integers d and e .

Suppose that d equals r . We have $f = h_{r,2}$, so $[I]$ is in $\text{Slip}_{r,2}$ by Remark 2.47 since $\mathcal{Hilb}_r(\mathbb{P}^2)$ is irreducible. Thus, by Lemma 2.9(ii), we may assume that $d < r$ holds. Moreover, if there

is an inequality $\dim_{\mathbb{k}} S_e \leq r$, then $[I]$ is in $\text{Slip}_{r,2}$ since in that case $[I] = [\bar{I} \cap \mathfrak{m}^{e+1}]$ is the unique closed point of the fiber over $\varphi_{r,\mathbb{P}^2}([I])$ of the natural map $\varphi_{r,\mathbb{P}^2}: \text{Hilb}_S^{h_{r,2}} \rightarrow \mathcal{H}ilb_r(\mathbb{P}^2)$ from Remark 2.47. Therefore, we may assume that the inequality $r < \dim_{\mathbb{k}} S_e$ holds. We claim that it is enough to consider the case that we have $f(e-1) = h_{r,2}(e-1) = \dim_{\mathbb{k}} S_{e-1}$. Indeed, otherwise we get $f(e-1) = h_{r,2}(e-1) = r$ and this contradicts Lemma 2.9(ii) since we have $f(e) = d < r = f(e-1)$. Using Lemma 2.9(ii) we obtain $\dim_{\mathbb{k}} S_{e-1} \leq d$ and moreover by Lemma 2.9(iii) this inequality is strict, since we have $r = f(e+1) > d$. \square

For a fixed positive integer r , let

$$\Omega_r = \{f: \mathbb{Z} \rightarrow \mathbb{Z} \mid f \text{ satisfies condition (3.15) and there exists } [I] \in \text{Hilb}_S^{h_{r,2}} \text{ such that } S/\bar{I} \text{ has Hilbert function } f\}.$$

By virtue of the following lemma, in order to obtain a proof of Theorem 3.13, it is enough to find a point $[I_f] \in \text{Slip}_{r,2}$, for every $f \in \Omega_r$ that satisfies the properties:

1. the Hilbert function of S/\bar{I}_f is f ;
2. $[I_f]$ is a smooth point of $\text{Hilb}_S^{h_{r,2}}$.

Lemma 3.16. *Let $f \in \Omega_r$ for some positive integer r . The locus of all closed points $[I]$ of $\text{Hilb}_S^{h_{r,2}}$ such that S/\bar{I} has Hilbert function f is irreducible.*

Proof. Denote this locus by U_f . Let $V_f \subseteq \mathcal{H}ilb_r(\mathbb{P}^2)$ be the locally closed subset whose closed points correspond to the closed subschemes of \mathbb{P}^2 with Hilbert function f . By definition, U_f is the set of all closed points of the preimage of V_f under the map

$$\varphi_{r,\mathbb{P}^2}: \text{Hilb}_S^{h_{r,2}} \rightarrow \mathcal{H}ilb_r(\mathbb{P}^2).$$

The locus V_f is irreducible by Proposition 3.11. Furthermore, we have $h_{r,2}(a) = f(a) = \dim_{\mathbb{k}} S_a$ for every $a < e$ and $h_{r,n}(a) = f(a)$ for every $a > e$. Therefore, U_f is irreducible by [81, 11.4.C] and Lemmas 2.43 and 2.44. \square

Fix a positive integer r and a function $f \in \Omega_r$, or equivalently, a pair of integers d and e corresponding to f . To simplify the notation let $s := \dim_{\mathbb{k}} S_{e-1}$ and we define $A_i = \alpha_0^i \alpha_1^{e-i}$ for all $0 \leq i \leq e$, $B_i = \alpha_0^i \alpha_1^{e+1-i}$ for all $0 \leq i \leq e+1$ and $C_i = \alpha_0^i \alpha_1^{e+2-i}$ for all $0 \leq i \leq e+2$ to make it easier to distinguish between the generators of different degrees. We define the ideals

$$J_f = (A_e, A_{e-1}, \dots, A_{d-s}, B_{d-s-1}, B_{d-s-2}, \dots, B_{r-d}, C_{r-d-1}, C_{r-d-2}, \dots, C_0) \quad (3.17)$$

and

$$I_f = (A_e, \dots, A_{r-s}, A_{r-s-1}\alpha_1, A_{r-s-1}\alpha_2, \dots, A_{d-s}\alpha_1, A_{d-s}\alpha_2, B_{d-s-1}, \dots, B_{r-d}, C_{r-d-1}, \dots, C_0). \quad (3.18)$$

Note that there are equalities $A_i\alpha_1 = B_i$ but we have written I_f in the form as above since it is more convenient in the proof of the following lemma.

Lemma 3.19. *The Hilbert function of S/I_f is $h_{r,2}$ and the Hilbert function of S/\bar{I}_f is f . Moreover, J_f is the saturation of I_f .*

Proof. Let $T = \mathbb{k}[\alpha_0, \alpha_1]$. We start with showing that J_f is a saturated ideal and that the Hilbert function of S/J_f is f . Indeed, J_f is an extension of the ideal $\mathfrak{a}_f = J_f \cap T$ in T so it is saturated with respect to α_2 and thus, is saturated with respect to \mathfrak{m} . Moreover, we have $H_{S/J_f}(a) = \sum_{b=0}^a H_{T/\mathfrak{a}_f}(b)$ and the latter sum can be computed from the staircase diagram of \mathfrak{a}_f .

It follows from the generators of I_f and J_f displayed above, that the saturation of I_f contains J_f . Therefore, J_f is the saturation of I_f since J_f is saturated. The quotient algebra S/J_f has Hilbert function f . Furthermore, we have $(J_f)_{>e} = (I_f)_{>e}$ and $(J_f)_{<e} = (I_f)_{<e} = 0$. Therefore, it follows from Equations (3.17) and (3.18) that the Hilbert function of S/I_f is $h_{r,2}$. \square

We now find a saturated ideal K_f such that the initial ideal of K_f with respect to an appropriate monomial order is I_f . Let

$$K_f = (A_e, \dots, A_{r-s}, A_{r-s-1}\alpha_1, A_{r-s-1}\alpha_2 + B_{r-d-1}, \dots, A_{d-s}\alpha_1, A_{d-s}\alpha_2 + B_0, \\ B_{d-s-1}, \dots, B_{r-d}, C_{r-d-1}, \dots, C_0). \quad (3.20)$$

Lemma 3.21. *The initial ideal of K_f with respect to the lex order $>$ with $\alpha_2 > \alpha_1 > \alpha_0$ is I_f . In particular, S/K_f has Hilbert function $h_{r,2}$.*

Proof. All S -polynomials of the generators displayed in Equation (3.20) belong to the ideal $\mathfrak{b} := (C_{e+2}, \dots, C_0)$. Let $\mathfrak{c} := (A_e, \dots, A_{r-s}, B_{r-s-1}, \dots, B_{r-d}, C_{r-d-1}, \dots, C_0)$. We have $\mathfrak{b} \subseteq \mathfrak{c} \subseteq K_f$. It follows that the set of generators from Equation (3.20) satisfies the Büchberger criterion (see [27, Thm. 6 in Ch. 2 §6]). Hence it is a Gröbner basis. In particular, the initial ideal $\text{in}_{<}(K_f)$ is I_f so S/K_f has Hilbert function $h_{r,2}$ by Lemma 3.19. \square

Next, we verify that K_f is a saturated ideal.

Lemma 3.22. *The ideal K_f is saturated. In particular, we have $[I_f] \in \text{Slip}_{r,2}$.*

Proof. Let $>$ be the lex order with $\alpha_2 > \alpha_1 > \alpha_0$. From Lemmas 2.7, 3.19 and 3.21 we obtain

$$\text{in}_{<}(\overline{K_f}) \subseteq \overline{\text{in}_{<}(K_f)} = \overline{I_f} = J_f. \quad (3.23)$$

Suppose that $\overline{K_f} \neq K_f$. We have $I_f = \text{in}_{<}(K_f) \subsetneq \text{in}_{<}(\overline{K_f}) \subseteq J_f$. Since I_f and J_f differ only in degree e , it follows that there is an element $g \in S_e \cap \overline{K_f}$ such that $\text{in}_{<}(g)$ does not belong to the set of monomial generators of I_f of degree e . However, $\text{in}_{<}(g)$ is an element of $(J_f)_e$ by Equation (3.23). Therefore, by the choice of the monomial order and Equation (3.17) we get $g = \sum_{i=d-s}^e a_i A_i$ for some $a_i \in \mathbb{k}$. We assumed that $\text{in}_{<}(g)$ is not in I_f . Thus, by Equation (3.18), we have $a_i \neq 0$ for some $i \in \{d-s, \dots, r-s-1\}$. Furthermore, we may assume that a_i is zero for all $i = r-s, \dots, e$ by Equation (3.20). Multiplying g by α_2^2 and using the generators of K_f given in Equation (3.20) we obtain

$$g' := -\alpha_2^2 g + \sum_{i=d-s}^{r-s-1} a_i \alpha_2 (A_i \alpha_2 + B_{i+s-d}) = \sum_{i=d-s}^{r-s-1} a_i B_{i+s-d} \alpha_2 \in \overline{K_f}.$$

We claim that it is not possible. By Equation (3.23), it is enough to show that no monomial of the form $B_j \alpha_2$ for some $j \in \{0, \dots, r-d-1\}$ is in J_f . This is clear since monomials of degree $e+2$ in J_f that are divisible by α_2 are also divisible by α_0^{r-d} .

Since K_f is saturated, it follows from Remark 2.47 that $[K_f]$ belongs to $\text{Slip}_{r,2}$. Therefore, $[I_f]$ is in $\text{Slip}_{r,2}$ by Lemma 3.21. \square

Finally, we show that $[I_f]$ is a smooth point of $\text{Hilb}_S^{h_{r,2}}$. It is enough to show that we have

$$\dim_{\mathbb{k}} \text{Hom}_S(I_f, S/I_f)_0 = \dim_{\mathbb{k}} \mathbf{T}_{[I_f]} \text{Hilb}_S^{h_{r,2}} \leq \dim \text{Slip}_{r,2} = 2r,$$

where the first equality follows from Theorem 2.75. Lemmas 3.24, 3.26, 3.27 and 3.33 are devoted to this calculation.

Lemma 3.24. *We have $\dim_{\mathbb{k}} \text{Hom}_S(J_f/I_f, S/I_f)_0 = (r-d)^2$.*

Proof. Since J_f/I_f is isomorphic to $\mathbb{k}(-e)^{\oplus(r-d)}$ we have

$$\begin{aligned} \dim_{\mathbb{k}} \text{Hom}_S(J_f/I_f, J_f/I_f)_0 &= (r-d)^2 \dim_{\mathbb{k}} \text{Hom}_S(\mathbb{k}(-e), \mathbb{k}(-e))_0 \\ &= (r-d)^2 \dim_{\mathbb{k}} \text{Hom}_S(\mathbb{k}, \mathbb{k})_0 = (r-d)^2. \end{aligned}$$

It follows from equalities $(J_f: \alpha_2) = J_f$ and $\alpha_2 \cdot J_f/I_f = 0$ that $\text{Hom}_S(J_f/I_f, S/J_f)_0$ is zero. Therefore, we get

$$\dim_{\mathbb{k}} \text{Hom}_S(J_f/I_f, S/I_f)_0 = \dim_{\mathbb{k}} \text{Hom}_S(J_f/I_f, J_f/I_f)_0 = (r-d)^2$$

from the long exact sequence obtained by applying $\text{Hom}_S(J_f/I_f, -)_0$ to the short exact sequence

$$0 \rightarrow J_f/I_f \rightarrow S/I_f \rightarrow S/J_f \rightarrow 0. \quad (3.25)$$

\square

Lemma 3.26. *We have $\dim_{\mathbb{k}} \text{Ext}_S^1(J_f/I_f, S/I_f)_0 = (r-d)^2$.*

Proof. We claim that $\dim_{\mathbb{k}} \text{Ext}_S^i(J_f/I_f, J_f/I_f)_0 = 0$ holds for all $i \in \{1, 2\}$. It is enough to show that $\dim_{\mathbb{k}} \text{Ext}_S^i(\mathbb{k}, \mathbb{k})_0$ is zero since we have $\text{Ext}_S^i(J_f/I_f, J_f/I_f)_0 = (\text{Ext}_S^i(\mathbb{k}, \mathbb{k})_0)^{\oplus(r-d)^2}$. Therefore, the claim follows from Lemma 2.28.

Applying the functor $\text{Hom}_S(J_f/I_f, -)_0$ to the short exact sequence (3.25) we obtain

$$\dim_{\mathbb{k}} \text{Ext}_S^1(J_f/I_f, S/I_f)_0 = \dim_{\mathbb{k}} \text{Ext}_S^1(J_f/I_f, S/J_f)_0.$$

We have $\text{Ext}_S^1(J_f/I_f, S/J_f)_0 \cong (\text{Ext}_S^1(\mathbb{k}, S/J_f)_e)^{\oplus(r-d)}$ so it is enough to compute the dimension of $\text{Ext}_S^1(\mathbb{k}, S/J_f)_e$ as a \mathbb{k} -vector space.

Applying the functor $\text{Hom}_S(-, S/J_f)_e$ to the Koszul resolution of \mathbb{k} we obtain the following complex:

$$(S/J_f)_e \xrightarrow{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}} (S/J_f)_{e+1}^{\oplus 3} \xrightarrow{\begin{bmatrix} -\alpha_1 & \alpha_0 & 0 \\ -\alpha_2 & 0 & \alpha_0 \\ 0 & -\alpha_2 & \alpha_1 \end{bmatrix}} (S/J_f)_{e+2}^{\oplus 3} \xrightarrow{\begin{bmatrix} \alpha_2 & -\alpha_1 & \alpha_0 \end{bmatrix}} (S/J_f)_{e+3}.$$

We need to show that the cohomology at $(S/J_f)_{e+1}^{\oplus 3}$ is an $(r-d)$ -dimensional \mathbb{k} -vector space. We denote the map $(S/J_f)_e \rightarrow (S/J_f)_{e+1}^{\oplus 3}$ by d_0 and the map $(S/J_f)_{e+1}^{\oplus 3} \rightarrow (S/J_f)_{e+2}^{\oplus 3}$ by d_1 . Let $h_0, h_1, h_2 \in S_{e+1}$ be such that $d_1(\overline{h_0}, \overline{h_1}, \overline{h_2})$ is zero, where $\overline{h_i}$ is the class of h_i in the quotient ring S/J_f . Let $h_2 = \alpha_2 h'_2(\alpha_0, \alpha_1, \alpha_2) + h''_2(\alpha_0, \alpha_1)$. We have $(-\alpha_1 h_0 + \alpha_0 h_1, \alpha_2(-h_0 + \alpha_0 h'_2) +$

$\alpha_0 h_2'', \alpha_2(-h_1 + \alpha_1 h_2'') + \alpha_1 h_2'' \subseteq J_f$. The ideal J_f is monomial. Therefore, since $(J_f: \alpha_2)$ is equal to J_f and h_2'' does not depend on α_2 , we get $(h_0 - \alpha_0 h_2'', h_1 - \alpha_1 h_2'') \subseteq J_f$. Thus, there is an equality $(\overline{h_0}, \overline{h_1}, \overline{h_2}) = (0, 0, \overline{h_2'') + d_0(\overline{h_2'')}$. We claim that $(0, 0, \overline{h_2'')$ is in $\ker d_1$ for every degree $e + 1$ homogeneous polynomial $h_2'' \in \mathbb{k}[\alpha_0, \alpha_1]$. Indeed, we have $\alpha_0 h_1'', \alpha_1 h_1'' \in J_f$ as there is an equality $(J_f)_{e+2} \cap \mathbb{k}[\alpha_0, \alpha_1] = \mathbb{k}[\alpha_0, \alpha_1]_{e+2}$. We get

$$\dim_{\mathbb{k}} \text{Ext}_S^1(\mathbb{k}, S/J_f)_e = \dim_{\mathbb{k}} \mathbb{k}[\alpha_0, \alpha_1]_{e+1} / (J_f \cap \mathbb{k}[\alpha_0, \alpha_1])_{e+1} = (r - d). \quad \square$$

Lemma 3.27. *We have $\dim_{\mathbb{k}} \text{Hom}_S(J_f, S/I_f)_0 = 2r$.*

Proof. Observe that $\text{Ext}_S^1(J_f, J_f/I_f)_0 \cong (\text{Ext}_S^1(J_f, \mathbb{k})_{-e})^{\oplus(r-d)}$ is zero by Lemma 2.28. Indeed, since $(J_f)_{e-1}$ is zero so is $\beta_{1,e}(J_f) = \beta_{2,e}(S/J_f)$.

Therefore, applying the functor $\text{Hom}_S(J_f, -)_0$ to the short exact sequence (3.25) we obtain a short exact sequence

$$0 \rightarrow \text{Hom}_S(J_f, J_f/I_f)_0 \rightarrow \text{Hom}_S(J_f, S/I_f)_0 \rightarrow \text{Hom}_S(J_f, S/J_f)_0 \rightarrow 0. \quad (3.28)$$

Since $(J_f)_{\leq e-1}$ is zero we have

$$\begin{aligned} \dim_{\mathbb{k}} \text{Hom}_S(J_f, J_f/I_f)_0 &= (r - d) \cdot \dim_{\mathbb{k}} \text{Hom}_S(J_f, \mathbb{k}(-e))_0 = (r - d) \cdot \dim_{\mathbb{k}}(J_f)_e \\ &= (r - d)(\dim_{\mathbb{k}} S_e - d). \end{aligned} \quad (3.29)$$

Finally, let $\mathfrak{a}_f = J_f \cap T$, where T is the polynomial ring $\mathbb{k}[\alpha_0, \alpha_1]$. The equality $J_f = \mathfrak{a}_f \cdot S$ implies that we have

$$\dim_{\mathbb{k}} \text{Hom}_S(J_f, S/J_f)_0 = \sum_{i \leq 0} \dim_{\mathbb{k}} \text{Hom}_T(\mathfrak{a}_f, T/\mathfrak{a}_f)_i. \quad (3.30)$$

Since $\text{Spec}(T/\mathfrak{a}_f)$ corresponds to a point of the Hilbert scheme $\mathcal{Hilb}_r(\mathbb{A}^2)$ which is smooth and $2r$ -dimensional we get

$$\sum_{i \leq 0} \dim_{\mathbb{k}} \text{Hom}_T(\mathfrak{a}_f, T/\mathfrak{a}_f)_i = 2r - \sum_{i > 0} \dim_{\mathbb{k}} \text{Hom}_T(\mathfrak{a}_f, T/\mathfrak{a}_f)_i \quad (3.31)$$

by [49, Prop. 2.3]. By Equation (3.17) the minimal generators of \mathfrak{a}_f appear in degrees $e, e + 1$ and $e + 2$. Furthermore, $H_{T/\mathfrak{a}_f}(e + 2)$ is zero. Therefore, by Proposition 2.22 we have

$$\sum_{i > 0} \dim_{\mathbb{k}} \text{Hom}_T(\mathfrak{a}_f, T/\mathfrak{a}_f)_i = \beta_{1,e}(T/\mathfrak{a}_f) \cdot H_{T/\mathfrak{a}_f}(e+1) = (e+1-d+s) \cdot (r-d) = (\dim_{\mathbb{k}} S_e - d)(r-d). \quad (3.32)$$

The exact sequence (3.28) and Equations (3.29), (3.30), (3.31) and (3.32) imply the equality

$$\dim_{\mathbb{k}} \text{Hom}_S(J_f, S/I_f)_0 = 2r. \quad \square$$

Lemma 3.33. *We have $\dim_{\mathbb{k}} \text{Hom}_S(I_f, S/I_f)_0 \leq 2r$.*

Proof. From the long exact sequence obtained by applying the functor $\text{Hom}_S(-, S/I_f)_0$ to the

short exact sequence $0 \rightarrow I_f \rightarrow J_f \rightarrow J_f/I_f \rightarrow 0$ we get

$$\begin{aligned} \dim_{\mathbb{k}} \operatorname{Hom}_S(I_f, S/I_f)_0 &\leq \dim_{\mathbb{k}} \operatorname{Hom}_S(J_f, S/I_f)_0 + \dim_{\mathbb{k}} \operatorname{Ext}_S^1(J_f/I_f, S/I_f)_0 \\ &\quad - \dim_{\mathbb{k}} \operatorname{Hom}_S(J_f/I_f, S/I_f)_0. \end{aligned}$$

Using Lemmas 3.24, 3.26 and 3.27 we conclude that the inequality $\dim_{\mathbb{k}} \operatorname{Hom}_S(I_f, S/I_f)_0 \leq 2r$ holds. \square

We summarize the above considerations to obtain a proof of Theorem 3.13.

Proof of Theorem 3.13. Let f be the Hilbert function of S/\bar{I} . By Lemma 3.14 we may assume that f satisfies condition (3.15), i.e. that f belongs to Ω_r . Let U_f be the locus of those closed points $[I']$ of $\operatorname{Hilb}_S^{h_{r,2}}$ for which S/\bar{I}' has Hilbert function f . We show that U_f is contained in $\operatorname{Slip}_{r,2}$. Locus U_f is irreducible by Lemma 3.16. We claim that it is enough to find a point $[I''] \in \operatorname{Slip}_{r,2} \cap U_f$ with $\dim_{\mathbb{k}} \mathbf{T}_{[I'']} \operatorname{Hilb}_S^{h_{r,2}} = 2r$. Indeed, then by Lemma 2.30 every irreducible component of the intersection $\bar{U}_f \cap \operatorname{Slip}_{r,2}$ passing through $[I'']$ has dimension at least $\dim U_f + \dim \operatorname{Slip}_{r,2} - 2r = \dim U_f$. It follows that \bar{U}_f is a subset of $\operatorname{Slip}_{r,2}$.

We claim that we may take $I'' = I_f$ as defined by Equation (3.18). We have $[I_f] \in U_f \cap \operatorname{Slip}_{r,2}$ by Lemmas 3.19 and 3.22. Moreover, $[I_f]$ is a smooth point of $\operatorname{Hilb}_S^{h_{r,2}}$ by Lemma 3.33 and Theorem 2.75. \square

We illustrate Theorem 3.13 with the following example.

Example 3.34. Let $S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$ be a polynomial ring and $J = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0\alpha_2^2, \alpha_1^4)$. The point $[J]$ is in $\operatorname{Hilb}_S^{h_{5,2}}$ and we claim that it is in $\operatorname{Slip}_{5,2}$. The Hilbert function of S/\bar{J} is $(1, 3, 4, 5, 5, \dots)$ so it satisfies condition (3.12). Thus, the claim follows from Theorem 3.13.

3.4.1 The analogue of the sufficient condition does not hold in general for projective space

For fixed positive integers r and n , condition (3.12) can be generalized as follows:

$$\text{there exist } e, d \in \mathbb{Z}_{>0} \text{ with } f(a) = \begin{cases} h_{r,n}(a) & \text{if } a \neq e \\ d & \text{if } a = e. \end{cases} \quad (3.35)$$

For all $n \geq 3$ and r large enough, the Hilbert scheme $\operatorname{Hilb}_r(\mathbb{P}^n)$ is reducible (see [54]). Therefore, it cannot be expected that a naive analogue of Theorem 3.13 holds in \mathbb{P}^n . The following remark gives a counterexample.

Remark 3.36. Assume that \mathbb{k} has characteristic zero. By [21, Thm. B and Thm. 6.17], there are non-smoothable closed subschemes of \mathbb{A}^6 corresponding to Gorenstein local algebras with (local) Hilbert function $(1, 6, 6, 1)$. See also [59, Thm. 1.1] for a description of the corresponding irreducible component of the Gorenstein locus of the Hilbert scheme. By embedding \mathbb{A}^6 in \mathbb{P}^6 as a complement of a hyperplane we conclude that there are non-smoothable closed subschemes of \mathbb{P}^6 with Hilbert function

$$(1, 7, 13, 14, 14, \dots).$$

Let R be such a subscheme and $I = I(R)$ be its homogeneous ideal. Choose a 14-dimensional subspace V of I_2 and construct an ideal $J = V \oplus I_{\geq 3}$. The point $[J]$ is in $\operatorname{Hilb}_{S[\mathbb{P}^6]}^{h_{14,6}}$ and the

Hilbert function of $S[\mathbb{P}^6]/\bar{J}$ satisfies condition (3.35) (with $r = 14$ and $n = 6$). However, $[J]$ does not belong to $\text{Slip}_{14,6}$.

The above remark may suggest, that existence of more irreducible components of $\mathcal{Hilb}_r(\mathbb{P}^n)$ is the only obstacle. The following example shows that the Hilbert function of S/\bar{I} may satisfy condition (3.35) for some $[I] \in \text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,n}}$ that is not in the closure of the locus of all points corresponding to saturated ideals.

Remark 3.37. Consider the ideal

$$I = (\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0\alpha_2, \alpha_0\alpha_3, \alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_2^4) \in S[\mathbb{P}^3] = \mathbb{k}[\alpha_0, \dots, \alpha_3].$$

The point $[I]$ is in $\text{Hilb}_{S[\mathbb{P}^3]}^{h_{6,3}}$ and the Hilbert function of S/\bar{I} satisfies condition (3.35) with $r = 6$ and $n = 3$. Suppose that $[I]$ is in the closure of the locus of all points corresponding to saturated ideals. Since, $\mathcal{Hilb}_6(\mathbb{P}^3)$ is irreducible (see [20, Thm. 1.1]) it follows that $[I]$ is in $\text{Slip}_{6,3}$. This contradicts Theorem 3.5 since we have $H_{S/I^2}(6) = 23 < r(n+1) = 24$.

3.5 Points on projective space—examples, part I

3.5.1 Initial cases

Proposition 3.38. *Let r and n be positive integers. We have $\text{Hilb}_{S[\mathbb{P}^1]}^{h_{r,1}} \cong \mathbb{P}^r$ and $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{1,n}} \cong \mathbb{P}^n$. In particular, $\text{Slip}_{r,1}$ and $\text{Slip}_{1,n}$ are equal to $\text{Hilb}_{S[\mathbb{P}^1]}^{h_{r,1}}$ and $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{1,n}}$, respectively.*

Proof. We have $h_{r,1} = f_{r,1}$ and $h_{1,n} = f_{1,n}$ where $f_{r',n'}$ for all positive integers r' and n' is as in Lemma 2.44. It follows that there are isomorphisms $\text{Hilb}_{S[\mathbb{P}^1]}^{h_{r,1}} \cong \mathcal{Hilb}_r(\mathbb{P}^1)$ and $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{1,n}} \cong \mathcal{Hilb}_1(\mathbb{P}^n)$. The scheme $\mathcal{Hilb}_r(\mathbb{P}^1)$ is isomorphic to \mathbb{P}^r (see [35, pp. 111–112]) and the scheme $\mathcal{Hilb}_1(\mathbb{P}^n)$ is isomorphic to \mathbb{P}^n (see [35, Ex. 7.3.1]). \square

Proposition 3.39. *Let n be a positive integer. The scheme $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{2,n}}$ is a \mathbb{P}^2 -bundle over*

$$\text{Gr}(n-1, S[\mathbb{P}^n]_1).$$

In particular, we have $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{2,n}} = \text{Slip}_{2,n}$.

Proof. It follows from [20, Prop. 3.1] that the Hilbert scheme $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{2,n}}$ is a $\text{Hilb}_{S[\mathbb{P}^1]}^{h_{2,1}}$ -bundle over $\text{Gr}(n-1, S[\mathbb{P}^n]_1)$. Furthermore, $\text{Hilb}_{S[\mathbb{P}^1]}^{h_{2,1}}$ is isomorphic to \mathbb{P}^2 by Proposition 3.38. \square

Proposition 3.40. *Let n be a positive integer. The scheme $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{3,n}}$ is irreducible. In particular, we have $\text{Slip}_{3,n} = \text{Hilb}_{S[\mathbb{P}^n]}^{h_{3,n}}$.*

Proof. By [20, Prop. 3.1] we may restrict to the case that we have $n = 1$ or $n = 2$. In the first case, the claim follows from Proposition 3.38.

Assume that n equals 2. Let $[I]$ be a closed point of $\text{Hilb}_{S[\mathbb{P}^2]}^{h_{3,2}}$. If I is saturated, then $[I]$ is in $\text{Slip}_{3,2}$ by Remark 2.47. On the other hand, if \bar{I} is not equal to I , then S/\bar{I} has Hilbert function $h_{3,1}$ by Lemma 2.9(ii), (iii). Therefore, we have $[I] \in \text{Slip}_{3,2}$ by Theorem 3.13. \square

3.5.2 Example of a singular point in the interior of Slip for projective plane

Since $\mathcal{Hilb}_r(\mathbb{P}^2)$ is smooth and $\text{Slip}_{r,2}$ is related to $\mathcal{Hilb}_r(\mathbb{P}^2)$ by the natural morphism

$$\varphi_{r,\mathbb{P}^2} : \text{Hilb}_{S[\mathbb{P}^2]}^{h_{r,2}} \rightarrow \mathcal{Hilb}_r(\mathbb{P}^2)$$

from Remark 2.47, it could be expected that the only singular points of $\text{Hilb}_{S[\mathbb{P}^2]}^{h_{r,2}}$ in $\text{Slip}_{r,2}$ are the points that lie in another irreducible component. We apply Theorem 3.13 to give an example of a singular point in the interior of $\text{Slip}_{8,2}$.

We start with introducing some notation. Let Θ_8 be the set of all functions $f: \mathbb{Z} \rightarrow \mathbb{Z}$ such that f is the Hilbert function of a saturated homogeneous ideal of S defining a zero-dimensional closed subscheme of \mathbb{P}^2 of length 8. Given $f \in \Theta_8$, let V_f be the locally closed subset of $\mathcal{Hilb}_r(\mathbb{P}^2)$ defined by those closed points that correspond to subschemes of \mathbb{P}^2 with Hilbert function f . These sets with varying $f \in \Theta_8$ form a stratification of $\mathcal{Hilb}_r(\mathbb{P}^2)$ by locally closed irreducible subsets (see Proposition 3.11). Let U_f be the set-theoretic inverse image of V_f under φ_{r,\mathbb{P}^2} . In particular, $\text{Slip}_{8,2}$ is the closure of $U_{h_{8,2}}$. Also, we say that $f \leq g$ for some $f, g: \mathbb{Z} \rightarrow \mathbb{Z}$ if for every $a \in \mathbb{Z}$ we have $f(a) \leq g(a)$. This gives a partial order on Θ_8 .

Let $f_1, f_2: \mathbb{Z} \rightarrow \mathbb{Z}$ be given by

$$f_1(a) = \begin{cases} \dim_{\mathbb{k}} S_a & \text{for } a < 3 \\ 7 & \text{for } a = 3 \\ 8 & \text{for } a > 3 \end{cases} \quad \text{and} \quad f_2(a) = \begin{cases} \dim_{\mathbb{k}} S_a & \text{for } a < 2 \\ 5 & \text{for } a = 2 \\ 7 & \text{for } a = 3 \\ 8 & \text{for } a > 3 \end{cases}$$

or, in brief form, $f_1 = (1, 3, 6, 7, 8, 8, \dots)$ and $f_2 = (1, 3, 5, 7, 8, 8, \dots)$.

Proposition 3.41. *Let $J = (\alpha_1^3, \alpha_1^2\alpha_2, \alpha_1^2\alpha_0^2, \alpha_1\alpha_2^3, \alpha_2^5)$. The point $[J] \in \text{Hilb}_S^{h_{8,2}}$ is a singular point that belongs to the irreducible component $\text{Slip}_{8,2}$ and no other.*

Proof. Let $I = (\alpha_1^2\alpha_2, \alpha_0\alpha_1^2 + \alpha_1\alpha_2^2, \alpha_1^4, \alpha_1\alpha_2^3, \alpha_2^5)$. The Hilbert function of S/\bar{I} is f_1 so $[I]$ belongs to $\text{Slip}_{8,2}$ by Theorem 3.13. Hence also its initial ideal with respect to lex order ($\alpha_0 > \alpha_1 > \alpha_2$), i.e. $I' = (\alpha_0\alpha_1^2, \alpha_1^2\alpha_2, \alpha_1^4, \alpha_1\alpha_2^3, \alpha_2^5)$, is in $\text{Slip}_{8,2}$. Since we have $\dim_{\mathbb{k}} \mathbf{T}_{[I]} \text{Hilb}_{S[\mathbb{P}^2]}^{h_{8,2}} = 16 = \dim \text{Slip}_{8,2}$, it follows that every irreducible closed subset of $\text{Hilb}_{S[\mathbb{P}^2]}^{h_{8,2}}$ passing through $[I]$ is contained in $\text{Slip}_{8,2}$. In particular $[J]$ belongs to $\text{Slip}_{8,2}$ since $[I']$ and $[J]$ lie in U_{f_2} which is irreducible by Lemma 2.43.

We have $\dim_{\mathbb{k}} \mathbf{T}_{[J]} \text{Hilb}_{S[\mathbb{P}^2]}^{h_{8,2}} = 17 > \dim \text{Slip}_{8,2}$. Let Z be an irreducible component of $\text{Hilb}_{S[\mathbb{P}^2]}^{h_{8,2}}$ containing $[J]$. We show that Z is equal to $\text{Slip}_{8,2}$. Let η be the generic point of Z . If η does not belong to $U_{h_{8,2}} \cup U_{f_1} \cup U_{f_2}$ then $[J]$ is not in $\overline{\{\eta\}}$ since U_{f_2} is open in $\text{Hilb}_{S[\mathbb{P}^2]}^{h_{8,2}} \setminus (U_{h_{8,2}} \cup U_{f_1})$ as f_2 is the greatest element of $\Theta_8 \setminus \{h_{8,2}, f_1\}$. Therefore, $Z = \overline{\{\eta\}}$ is contained in $\overline{U_{f_1}} \cup \overline{U_{f_2}} \cup \text{Slip}_{8,2}$. As shown above, we have $U_{f_1} \cup U_{f_2} \subseteq \text{Slip}_{8,2}$. The equality $Z = \text{Slip}_{8,2}$ follows. \square

3.6 Criterion based on smoothness

In this section we give another criterion for a closed point $[I] \in \text{Hilb}_S^{h_{r,n}}$ to be in $\text{Slip}_{r,n}$. The criterion is based on smoothness of a point in a certain related multigraded Hilbert scheme. It

is first stated in a general form (Theorem 3.42) but later we impose additional assumptions to guarantee that conditions 1–3 from the theorem are fulfilled.

Subsections 3.6.1–3.6.3 are concerned with describing some situations in which assumptions 1–3 are fulfilled (each subsection deals with one assumption). The main results of these subsections are:

- Proposition 3.46 which implies condition 1 of Theorem 3.42
- Proposition 3.54 which implies condition 2 of Theorem 3.42
- Lemma 3.57 which implies condition 3 of Theorem 3.42.

Moreover, in Subsections 3.6.4 and 3.6.5 we present two applications of Theorem 3.42: Theorems 3.66 and 3.75. In the proof of the first of them we use Proposition 3.46 and Lemma 3.57. The proof of the second result, i.e. Theorem 3.75 is based on Propositions 3.46, 3.54 and Lemma 3.57.

Notation

In this section $[I] \in \text{Hilb}_S^{hr,n}$ is a closed point corresponding to an ideal that is not saturated. By d we denote a positive integer with $I_d \neq \bar{I}_d$. We define $J = \bar{I} \cap \mathfrak{m}^d$ and $K = I \cap \mathfrak{m}^d$.

Now we present the main result of this section.

Theorem 3.42. *When the following hold:*

1. *the natural map $\text{Hom}_S(J, S/J)_0 \rightarrow \text{Hom}_S(K, S/J)_0$ is surjective;*
2. *$[J] \in \text{Hilb}_S^h$ is a smooth point where h is the Hilbert function of S/J ;*
3. *the natural map $\text{Hom}_S(K, S/K)_0 \rightarrow \text{Hom}_S(K, S/J)_0$ is surjective,*

there is no $[I'] \in \text{Slip}_{r,n}$ with $I'_{\geq d} = I_{\geq d}$. In particular, we have $[I] \notin \text{Slip}_{r,n}$.

Proof. Let k be the Hilbert function of S/K . Consider the multigraded flag Hilbert scheme $\text{Hilb}_S^{k,h}$ (see Subsection 2.2.4) and natural morphisms $\pi_k: \text{Hilb}_S^{k,h} \rightarrow \text{Hilb}_S^k$ and $\pi_h: \text{Hilb}_S^{k,h} \rightarrow \text{Hilb}_S^h$.

We first show that π_k induces isomorphism on tangent spaces $\mathbf{T}_{[K \subseteq J]} \text{Hilb}_S^{k,h} \rightarrow \mathbf{T}_{[K]} \text{Hilb}_S^k$. This map on tangent spaces is the upper horizontal map in the pullback diagram

$$\begin{array}{ccc} \mathbf{T}_{[K \subseteq J]} \text{Hilb}_S^{k,h} & \longrightarrow & \text{Hom}_S(K, S/K)_0 \\ \downarrow & & \downarrow \\ \text{Hom}_S(J, S/J)_0 & \longrightarrow & \text{Hom}_S(K, S/J)_0 \end{array}$$

in which the right vertical and the lower horizontal maps are natural maps of Hom groups (see Theorem 2.78). By assumption 1, the lower horizontal map is surjective. Moreover, $\text{Hom}_S(J/K, S/J)_0$ is zero by Lemma 2.8 since we have $J = \bar{J}_{\geq d}$ and $\mathfrak{m}^r \cdot J/K = 0$. Therefore, the lower horizontal map is bijective. Thus, so is the upper horizontal map since the diagram is a pullback.

Now we show that the natural transformation $D_{\text{Hilb}_S^{k,h}, [K \subseteq J]} \rightarrow D_{\text{Hilb}_S^k, [K]}$ of deformation functors induced by π_k admits a map of tangent-obstruction theories which is injective on obstruction spaces. By Theorem 2.75, there are tangent-obstruction theories for $D_{\text{Hilb}_S^h, [J]}$ and

$D_{\text{Hilb}_S^k, [K]}$ with obstruction spaces $\text{Ext}_S^1(J, S/J)_0$ and $\text{Ext}_S^1(K, S/K)_0$, respectively. Moreover, by Theorem 2.78 there is a tangent-obstruction theory for $D_{\text{Hilb}_S^{k,h}, [K \subseteq J]}$ with obstruction space given by the pullback diagram

$$\begin{array}{ccc} \text{Ob}_{[K \subseteq J]} \text{Hilb}_S^{k,h} & \xrightarrow{\beta} & \text{Ext}_S^1(K, S/K)_0 \\ \downarrow \alpha & & \downarrow \\ \text{Ext}_S^1(J, S/J)_0 & \longrightarrow & \text{Ext}_S^1(K, S/J)_0, \end{array}$$

where the lower horizontal and the right vertical maps are maps from long exact sequences of Ext groups. Furthermore, α and β induce maps of tangent-obstruction theories. Here we have used assumption 3. Since $[J] \in \text{Hilb}_S^h$ is a smooth point, we can use Lemma 2.68, to change the tangent-obstruction theory of $D_{\text{Hilb}_S^{k,h}, [K \subseteq J]}$ so that $D_{\text{Hilb}_S^{k,h}, [K \subseteq J]} \rightarrow D_{\text{Hilb}_S^k, [K]}$ admits a map of tangent-obstruction theories which is injective on obstruction spaces.

It follows from Corollary 2.66 that the map π_k is étale at $[K \subseteq J]$. In particular, there is an open subset U of $\text{Hilb}_S^{k,h}$ containing $[K \subseteq J]$ that is mapped onto an open subset V of Hilb_S^k containing $[K]$. If there is a point $[I'] \in \text{Slip}_{r,n}$ with $I_{\geq d} = I'_{\geq d}$, then there is a saturated ideal I'' with $[I''] \in \text{Hilb}_S^{h,r,n}$ and $[I'' \cap \mathfrak{m}^d] \in V$. Therefore, there is an ideal $[J''] \in \text{Hilb}_S^h$ with $I'' \cap \mathfrak{m}^d \subseteq J''$. This gives a contradiction since we have $I'' = \overline{J''}$ and $I''_d \subsetneq J''_d$. \square

Remark 3.43. It seems that assumption 2 of the above theorem is both the most restrictive and potentially the hardest to check in practice. On the other hand, if we have $n = 2$ and

$$d = \min\{a \in \mathbb{Z} \mid \bar{I}_a \neq I_a\}$$

then condition 2 always holds (see Proposition 3.10) since the usual Hilbert scheme $\text{Hilb}_r(\mathbb{P}^2)$ is smooth. Furthermore, for $d = \max\{a \in \mathbb{Z} \mid \bar{I}_a \neq I_a\}$ condition 3 is satisfied (see Lemma 3.57). Even in the case of \mathbb{P}^2 and $d = \max\{a \in \mathbb{Z} \mid \bar{I}_a \neq I_a\}$ it is not clear, in how general setups can we expect condition 2 of Theorem 3.42 to hold. We present one specific situation when this holds in Proposition 3.54.

In the following subsections we study some situations in which conditions 1–3 of Theorem 3.42 hold.

3.6.1 About condition 1

The main result of this subsection is Proposition 3.46 which describes a situation in which condition 1 of Theorem 3.42 holds.

We keep the notation of Theorem 3.42. Let $R = S/\bar{I}$ and pick a linear form $L \in S_1$ that is a nonzerodivisor on R . This is possible by Lemma 2.9(i). By a linear change of variables, we may and do assume that we have $L = \alpha_0$.

We start with the following simple observation.

Lemma 3.44. *Let \mathfrak{b} be a homogeneous ideal of S that is generated in degrees at most d for a positive integer d . Let $\mathfrak{a} = \frac{\mathfrak{b} + \bar{I}}{\bar{I}} \subseteq R$. If we have $\dim_{\mathbb{k}} \mathfrak{a}_d = \dim_{\mathbb{k}} \mathfrak{a}_{d+1}$ then the equality*

$$\mathfrak{a}_{a'} = \alpha_0^{a'-a} \mathfrak{a}_a$$

holds for every $a' \geq a \geq d$.

Proof. Let $\iota: R \rightarrow R$ be the multiplication by α_0 . This induces an injective map $\mathfrak{a}_a \rightarrow \mathfrak{a}_{a+1}$ for every integer a . It is enough to show that for all $a \geq d$ this map is surjective. We prove this by induction. The case $a = d$ follows from the assumption that we have $\dim_{\mathbb{k}} \mathfrak{a}_d = \dim_{\mathbb{k}} \mathfrak{a}_{d+1}$. Let $a_0 > d$ and suppose that there is an equality $\mathfrak{a}_{a_0} = \alpha_0 \mathfrak{a}_{a_0-1}$. Let $g \in \mathfrak{a}_{a_0+1}$. Since \mathfrak{b} is generated in degrees at most d we obtain

$$g = \sum_{i=0}^n \alpha_i f_i \text{ for some } f_i \in \mathfrak{a}_{a_0}.$$

By induction there are $h_i \in \mathfrak{a}_{a_0-1}$ with $f_i = \alpha_0 h_i$ for all $i = 0, 1, \dots, n$. Therefore, we get $g = \alpha_0 (\sum_{i=0}^n \alpha_i h_i)$. \square

The following lemma is used in the proof of Proposition 3.46 to extend a homomorphism $\varphi \in \text{Hom}_S(K, S/J)_0$ to an element $\psi \in \text{Hom}_S(J, S/J)_0$ under some additional assumptions.

Lemma 3.45. *Let $d = \max\{a \in \mathbb{Z} \mid \bar{I}_a \neq I_a\}$. Let $\mathfrak{b} \subseteq S$ be a homogeneous ideal that is generated in degrees at most d . Let $\mathfrak{a} = \frac{\mathfrak{b} + \bar{I}}{\bar{I}} \subseteq S/\bar{I} = R$. Assume that we have $\dim_{\mathbb{k}} \mathfrak{a}_d = \dim_{\mathbb{k}} \mathfrak{a}_{d+1}$.*

Let $F \in J_d$ and assume that it is of the form $F = fg$ for some homogeneous $f \in \mathfrak{b}$ and $g \in \bar{I}$. Let $\varphi \in \text{Hom}_S(K, S/J)_0$. There is $h \in (S/J)_d$ with $\varphi(\alpha_0 F) = \alpha_0 h$.

Proof. Observe that $\alpha_0 F$ belongs to $\bar{I}_{d+1} = K_{d+1}$, so $\varphi(\alpha_0 F)$ is a well-defined element of R_{d+1} . Let $a = \deg(f)$. We have

$$\alpha_0^a \varphi(\alpha_0 F) = \alpha_0^a \varphi(\alpha_0 fg) = f \varphi(\alpha_0^{a+1} g).$$

Hence $\alpha_0^a \varphi(\alpha_0 F)$ is in \mathfrak{a}_{d+a+1} . Therefore, by Lemma 3.44 there is $h \in \mathfrak{a}_d \subseteq R_d = (S/J)_d$ with $\alpha_0^a \varphi(\alpha_0 F) = \alpha_0^{a+1} h$. Since α_0 is a nonzerodivisor on R , the equality $\varphi(\alpha_0 F) = \alpha_0 h$ follows. \square

Now we can present the main result of this subsection.

Proposition 3.46. *Let $d = \max\{a \in \mathbb{Z} \mid \bar{I}_a \neq I_a\}$. Let $\mathfrak{b} \subseteq S$ be a homogeneous ideal that is generated in degrees at most d . Let $\mathfrak{a} = \frac{\mathfrak{b} + \bar{I}}{\bar{I}} \subseteq S/\bar{I} = R$. Assume that we have $\dim_{\mathbb{k}} \mathfrak{a}_d = \dim_{\mathbb{k}} \mathfrak{a}_{d+1}$.*

Let $F_1, \dots, F_s \in J_d$ be elements whose classes form a basis of J_d/K_d . Assume that for all $i \in \{1, \dots, s\}$ the polynomial F_i equals $f_i g_i$ for some homogeneous $f_i \in \mathfrak{b}$ and $g_i \in \bar{I}$. Let $\varphi \in \text{Hom}_S(K, S/J)_0$. There exists $\psi \in \text{Hom}_S(J, S/J)_0$ with $\psi|_K = \varphi$. Thus, condition 1 from Theorem 3.42 is fulfilled.

Proof. Let $\{p_1, \dots, p_t\}$ be a minimal set of homogeneous generators of J containing $\{F_1, \dots, F_s\}$. We may and do assume that p_i belongs to K if $p_i \notin \{F_1, \dots, F_s\}$. By Lemma 3.45, there are $h_i \in (S/J)_d$ with

$$\alpha_0 h_i = \varphi(\alpha_0 F_i) \text{ for all } i \in \{1, 2, \dots, s\}. \quad (3.47)$$

We define ψ on generators $\{p_1, \dots, p_t\}$ of J by

$$\psi(p_i) = \begin{cases} \varphi(p_i) & \text{if } p_i \in K \\ h_j & \text{if } p_i = F_j \text{ for some } j \in \{1, 2, \dots, s\}. \end{cases}$$

We claim that ψ is a well-defined element of $\text{Hom}_S(J, S/J)_0$. Indeed, let

$$\{p_1, \dots, p_t\} = \{F_1, \dots, F_s, Q_{s+1}, \dots, Q_t\}$$

and assume that $G_i, H_i \in S$ are such that we have

$$\sum_{i=1}^s G_i F_i + \sum_{i=s+1}^t H_i Q_i = 0.$$

We need to show that the following holds

$$\sum_{i=1}^s G_i \psi(F_i) + \sum_{i=s+1}^t H_i \psi(Q_i) = \sum_{i=1}^s G_i h_i + \sum_{i=s+1}^t H_i \varphi(Q_i) = 0.$$

Since α_0 is a nonzerodivisor on R , it is enough to observe that there is a chain of equalities

$$\begin{aligned} \alpha_0 \left(\sum_{i=1}^s G_i h_i + \sum_{i=s+1}^t H_i \varphi(Q_i) \right) &\stackrel{(3.47)}{=} \sum_{i=1}^s G_i \varphi(\alpha_0 F_i) + \alpha_0 \sum_{i=s+1}^t H_i \varphi(Q_i) \\ &= \varphi \left(\alpha_0 \left(\sum_{i=1}^s G_i F_i + \sum_{i=s+1}^t H_i Q_i \right) \right) = \varphi(0) = 0. \quad \square \end{aligned}$$

3.6.2 About condition 2

In this subsection we show in Proposition 3.54 that if n equals 2 and under some additional assumptions, condition 2 from Theorem 3.42 is fulfilled.

In the notation of Theorem 3.42, assume that d is equal to $\min\{a \in \mathbb{Z} \mid \bar{I}_a \neq I_a\}$. We claim that we have $d = \min\{a \in \mathbb{Z} \mid \bar{I}_a \neq 0\}$ and therefore, that J is equal to \bar{I} . Let $d' = \min\{a \in \mathbb{Z} \mid \bar{I}_a \neq 0\}$. If \bar{I}_a is zero, then there is an equality $I_a = \bar{I}_a$ so we get $d' \leq d$. On the other hand, if we have $d' < d$, then $I_{d'} = \bar{I}_{d'}$ is non-zero. Thus, $H_{S/I}(a) = H_{S/\bar{I}}(a) = r$ holds for every $a \geq d'$ by Lemma 2.9(ii) and the definition of $h_{r,n}$. We obtain $I_{\geq d'} = \bar{I}_{\geq d'}$ which contradicts the definition of d and proves the claim.

The smoothness of $\mathcal{H}ilb_r(\mathbb{P}^2)$ and its consequence, Proposition 3.10, play a key role in our approach to condition 2 of Theorem 3.42. However, Proposition 3.46 requires $d = \max\{a \in \mathbb{Z} \mid \bar{I}_a \neq I_a\}$ while Proposition 3.10 corresponds to the case $d = \min\{a \in \mathbb{Z} \mid \bar{I}_a \neq I_a\}$, when we have $[J] = [\bar{I} \cap \mathfrak{m}^d] = [\bar{I}]$. Therefore, we would like to show that, under some additional assumptions, the condition that $[\bar{I} \cap \mathfrak{m}^d]$ is a smooth point of $\mathcal{H}ilb_S^h$ holds also for $d = \max\{a \in \mathbb{Z} \mid \bar{I}_a \neq I_a\}$. This is achieved in Proposition 3.54.

Lemma 3.48. *Let $I \subseteq S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$ be a homogeneous ideal. We have*

$$\dim_{\mathbb{k}} \text{Ext}_S^2(\mathbb{k}, S/I)_a = \beta_{1,a+3}(S/I)$$

for every $a \geq 0$.

Proof. Consider the short exact sequence

$$0 \rightarrow I \rightarrow S \rightarrow S/I \rightarrow 0.$$

Applying the functor $\text{Hom}_S(\mathbb{k}, -)_a$ to the above short exact sequence we obtain the exact sequence

$$\cdots \rightarrow \text{Ext}_S^2(\mathbb{k}, S)_a \rightarrow \text{Ext}_S^2(\mathbb{k}, S/I)_a \rightarrow \text{Ext}_S^3(\mathbb{k}, I)_a \rightarrow \text{Ext}_S^3(\mathbb{k}, S)_a \rightarrow \cdots \quad (3.49)$$

We claim that $\text{Ext}_S^i(\mathbb{k}, S)_a = 0$ holds for all $i \in \{2, 3\}$.

Consider the Koszul complex

$$0 \rightarrow S(-3) \xrightarrow{\begin{bmatrix} \alpha_2 \\ -\alpha_1 \\ \alpha_0 \end{bmatrix}} S(-2)^{\oplus 3} \xrightarrow{\begin{bmatrix} -\alpha_1 & -\alpha_2 & 0 \\ \alpha_0 & 0 & -\alpha_2 \\ 0 & \alpha_0 & \alpha_1 \end{bmatrix}} S(-1)^{\oplus 3} \xrightarrow{\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{bmatrix}} S \rightarrow 0. \quad (3.50)$$

The Ext groups $\text{Ext}_S^i(\mathbb{k}, S)_a$ for all $i \in \{2, 3\}$ can be computed as the cohomology groups at $S_{a+2}^{\oplus 3}$ and S_{a+3} of the complex

$$0 \rightarrow S_a \xrightarrow{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}} S_{a+1}^{\oplus 3} \xrightarrow{\begin{bmatrix} -\alpha_1 & \alpha_0 & 0 \\ -\alpha_2 & 0 & \alpha_0 \\ 0 & -\alpha_2 & \alpha_1 \end{bmatrix}} S_{a+2}^{\oplus 3} \xrightarrow{\begin{bmatrix} \alpha_2 & -\alpha_1 & \alpha_0 \end{bmatrix}} S_{a+3} \rightarrow 0$$

obtained from the Koszul complex (3.50) by applying the functor $\text{Hom}_S(-, S)_a$. These groups are trivial.

Thus, we have

$$\dim_{\mathbb{k}} \text{Ext}_S^2(\mathbb{k}, S/I)_a = \dim_{\mathbb{k}} \text{Ext}_S^3(\mathbb{k}, I)_a$$

by exact sequence (3.49). Applying the functor $\text{Hom}_S(-, S/I)_a$ to the Koszul complex (3.50) we get that $\text{Ext}_S^3(\mathbb{k}, I)_a$ is the cokernel of the map

$$I_{a+2}^{\oplus 3} \xrightarrow{\begin{bmatrix} \alpha_2 & -\alpha_1 & \alpha_0 \end{bmatrix}} I_{a+3}.$$

The equality $\dim_{\mathbb{k}} \text{Ext}_S^3(\mathbb{k}, I)_a = \beta_{1, a+3}(S/I)$ follows. \square

Lemma 3.51. *Let f be the Hilbert function of a zero-dimensional length r subscheme of \mathbb{P}^n . Let \bar{I} be a saturated ideal of S with $[\bar{I}] \in \text{Hilb}_S^f$. Let $m = \min\{a \in \mathbb{Z} \mid \bar{I}_a \neq 0\}$ and let $d \geq m$ be a positive integer. Let h be the Hilbert function of $S/\bar{I} \cap \mathfrak{m}^d$. If $[\bar{I}]$ is a smooth point and the natural map $\text{Hom}_S(\bar{I} \cap \mathfrak{m}^a, S/\bar{I})_0 \rightarrow \text{Hom}_S(\bar{I} \cap \mathfrak{m}^{a+1}, S/\bar{I})_0$ is bijective for every $m \leq a < d$, then $[\bar{I} \cap \mathfrak{m}^d] \in \text{Hilb}_S^h$ is a smooth point. Moreover, we have $\dim_{\mathbb{k}} \mathbf{T}_{[\bar{I} \cap \mathfrak{m}^d]} \text{Hilb}_S^h = \dim W$ where $W \subseteq \text{Hilb}_S^h$ is the locally closed subset whose closed points correspond to ideals defining subschemes of \mathbb{P}^n with Hilbert function f .*

Proof. For all integers $m \leq a \leq d$ let g_a be the Hilbert function of $S/\bar{I} \cap \mathfrak{m}^a$. In particular, we have $g_m = f$ and $g_d = h$. Let W_a be the locally closed subset of $\text{Hilb}_S^{g_a}$ whose closed points correspond to ideals defining subschemes of \mathbb{P}^n with Hilbert function f . In particular, W_d is equal to W . For all $m \leq a \leq d-1$ let $\pi_a: \text{Hilb}_S^{g_a} \rightarrow \text{Hilb}_S^{g_{a+1}}$ be the natural map given on closed points by $[I'] \mapsto [I' \cap \mathfrak{m}^{a+1}]$. The map π_a induces a homeomorphism $W_a \cong W_{a+1}$ by Lemma 2.29.

We assumed that the natural map $\text{Hom}_S(\bar{I} \cap \mathfrak{m}^a, S/\bar{I})_0 \rightarrow \text{Hom}_S(\bar{I} \cap \mathfrak{m}^{a+1}, S/\bar{I})_0$ is bijective for every $m \leq a \leq d-1$. Therefore, by Lemma 2.80 the map π_a induces an isomorphism of tangent spaces

$$\mathbf{T}_{[\bar{I} \cap \mathfrak{m}^a]} \text{Hilb}_S^{g_a} \cong \mathbf{T}_{[\bar{I} \cap \mathfrak{m}^{a+1}]} \text{Hilb}_S^{g_{a+1}} \quad (3.52)$$

for every $m \leq a \leq d-1$.

Now we show that $[\bar{I} \cap \mathfrak{m}^d] \in \text{Hilb}_S^h$ is a smooth point. Observe that $W_m \subseteq \text{Hilb}_S^f$ is open

and $[\bar{I} \cap \mathfrak{m}^m] = [\bar{I}] \in \text{Hilb}_S^f$ is a smooth point by assumption. Therefore, we have

$$\dim_{\mathbb{k}} \mathbf{T}_{[\bar{I} \cap \mathfrak{m}^d]} \text{Hilb}_S^h = \dim_{\mathbb{k}} \mathbf{T}_{[\bar{I}]} \text{Hilb}_S^f = \dim W_m = \dim W_d = \dim W \quad (3.53)$$

where the first equality follows from Equation (3.52), the second from the fact that $[\bar{I}]$ is a smooth point and $W_m \subseteq \text{Hilb}_S^f$ is open and the third equality follows from the homeomorphism $W_m \cong W_d$. Equation (3.53) implies that $[\bar{I} \cap \mathfrak{m}^d]$ is a smooth point of Hilb_S^h since $[\bar{I} \cap \mathfrak{m}^d]$ belongs to W . \square

Finally, we present the main result of this subsection.

Proposition 3.54. *In the notation of Theorem 3.42, assume that n equals 2. Let*

$$m = \min\{a \in \mathbb{Z} \mid \bar{I}_a \neq I_a\} = \min\{a \in \mathbb{Z} \mid \bar{I}_a \neq 0\}.$$

Let f be the Hilbert function of S/\bar{I} and assume that

$$f(a) - 3f(a+1) + 3f(a+2) - f(a+3) = \beta_{1,a+3}(S/\bar{I}) \quad (3.55)$$

holds for every $m \leq a \leq d-1$. The point $[J]$ is smooth in Hilb_S^h , i.e. condition 2 from Theorem 3.42 is fulfilled. Moreover, we have $\dim_{\mathbb{k}} \text{Hom}_S(J, S/J)_0 = \dim W$ where $W \subseteq \text{Hilb}_S^h$ is the locally closed subset whose closed points correspond to ideals defining subschemes of \mathbb{P}^2 with Hilbert function f .

Proof. The point $[\bar{I} \cap \mathfrak{m}^m] = [\bar{I}] \in \text{Hilb}_S^f$ is a smooth point by Proposition 3.10. Therefore, by Lemma 3.51 it is enough to show that the natural map

$$\text{Hom}_S(\bar{I} \cap \mathfrak{m}^a, S/\bar{I})_0 \rightarrow \text{Hom}_S(\bar{I} \cap \mathfrak{m}^{a+1}, S/\bar{I})_0 \quad (3.56)$$

coming from the exact sequence of Hom groups is bijective for every $m \leq a < d$. Fix $m \leq a \leq d-1$ and let $Q = \bar{I} \cap \mathfrak{m}^a / \bar{I} \cap \mathfrak{m}^{a+1}$. Since we have $\mathfrak{m} \cdot Q = 0$ and $(\bar{I} : \mathfrak{m}) = \bar{I}$, we conclude by Lemma 2.8 that $\text{Hom}_S(Q, S/\bar{I})_0$ is zero. Thus, the map from Equation (3.56) is injective.

We have $Q = \mathbb{k}(-a)^s$ for some integer s . Therefore, by Lemma 2.27 we have

$$s \left(f(a) - 3f(a+1) + 3f(a+2) - f(a+3) \right) = \sum_{i=0}^3 (-1)^i \dim_{\mathbb{k}} \text{Ext}_S^i(Q, S/\bar{I})_0.$$

Since $\dim_{\mathbb{k}} \text{Hom}_S(Q, S/\bar{I})_0$ is zero, by Equation (3.55) and Lemma 3.48 we get

$$\dim_{\mathbb{k}} \text{Ext}_S^1(Q, S/\bar{I})_0 + \dim_{\mathbb{k}} \text{Ext}_S^3(Q, S/\bar{I})_0 = 0.$$

Hence there is an equality $\text{Ext}_S^1(Q, S/\bar{I})_0 = 0$. Thus, the map from Equation (3.56) is surjective. \square

3.6.3 About condition 3

Lemma 3.57. *In the notation of Theorem 3.42, assume that we have $d = \max\{a \in \mathbb{Z} \mid \bar{I}_a \neq I_a\}$. The natural map $\text{Hom}_S(K, S/K)_0 \rightarrow \text{Hom}_S(K, S/J)_0$ is surjective, i.e. condition 3 from Theorem 3.42 is fulfilled.*

Proof. It is enough to establish that $\text{Ext}_S^1(K, J/K)_0$ is zero. Let P_\bullet be a minimal graded free resolution of K . The group $\text{Ext}_S^1(K, J/K)_0$ is a subquotient of $\text{Hom}_S(P_1, J/K)_0$ so it is enough to show that the latter group is trivial. This holds, since the minimal generators of P_1 are of degree at least $d + 1$ and we have $(J/K)_{\geq d+1} = 0$. \square

3.6.4 Application one: subschemes contained in a line

In this subsection we consider ideals defining subschemes contained in a line. The statement of Theorem 3.66 is a generalization of [67, Thm. 2.7] for fields of arbitrary characteristic. We provide a proof in the setting of Theorem 3.42.

We start with a lemma. It is stated in a general version since we also use it in Subsection 3.6.5. We introduce some notation. Let f be the Hilbert function of a zero-dimensional, length r subscheme of $\text{Proj } S$. Assume that we have $f \neq h_{r,n}$. Let

$$e = \max\{a \in \mathbb{Z} \mid f(a) \neq h_{r,n}(a)\}.$$

Let $h: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by

$$h(a) = \begin{cases} \dim_{\mathbb{k}} S_a & \text{for } a < e; \\ f(a) & \text{for } a \geq e \end{cases}$$

and $k: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by

$$k(a) = \begin{cases} \dim_{\mathbb{k}} S_a & \text{for } a < e; \\ h_{r,n}(a) & \text{for } a \geq e. \end{cases}$$

Let $\pi: \text{Hilb}_S^{h_{r,n}} \rightarrow \text{Hilb}_S^k$, $\pi_h: \text{Hilb}_S^h \rightarrow \mathcal{Hilb}_r(\mathbb{P}^n)$ and $\pi_k: \text{Hilb}_S^k \rightarrow \mathcal{Hilb}_r(\mathbb{P}^n)$ be the natural morphisms.

Let V be the locally closed subset of $\mathcal{Hilb}_r(\mathbb{P}^n)$ consisting of points corresponding to subschemes with Hilbert function f . Let W be the set-theoretic inverse image of V under π_h and let W' be the set-theoretic inverse image of V under π_k .

Lemma 3.58. *If the following hold:*

1. $f(e)$ equals $r - 1$;
2. $V \subseteq \mathcal{Hilb}_r(\mathbb{P}^n)$ is irreducible;
3. there exists an irreducible closed $(rn - 1)$ -dimensional subset $U \subseteq W'$ with $\pi(\text{Slip}_{r,n}) \cap W' \subseteq U$, set-theoretically;
4. there is an equality $\dim_{\mathbb{k}} \text{Hom}_S(J, S/J)_0 = \dim W$ for every $[J] \in W \subseteq \text{Hilb}_S^h$;
5. $\dim_{\mathbb{k}} \text{Ext}_S^1(\mathbb{k}, S/J)_e \leq 1$ holds for every $[J] \in W \subseteq \text{Hilb}_S^h$,

then we have $\pi(\text{Slip}_{r,n}) \cap W' = U$, set-theoretically.

Proof. From assumption 1 and the fact that $h_{r,n}(e)$ is not equal to $f(e)$ we get $h_{r,n}(e) = r$. Let $N = \dim V$. It follows from Lemmas 2.43, 2.44, [81, 11.4.C] and assumption 2 that W and W' are irreducible and their dimensions are $\dim W = N$ and $\dim W' = N + (\dim_{\mathbb{k}} S_e - (r - 1) - 1) = N + \dim_{\mathbb{k}} S_e - r$.

We have $\dim \pi(\text{Slip}_{r,n}) = rn$. We claim that $V \cap \mathcal{Hilb}_r^{sm}(\mathbb{P}^n)$ is non-empty. Let $[R] \in V$ and let I' be the generic initial ideal of the ideal $I(R)$ with respect to the grevlex order with

$\alpha_0 > \alpha_1 > \dots > \alpha_n$. We get $[\text{Proj } S/I'] \in V$ by Corollary 2.10 and $[\text{Proj } S/I'] \in \mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$ by [20, Prop. 4.15] since I' is an extended ideal from $\mathbb{k}[\alpha_0, \dots, \alpha_{n-1}]$ by Lemma 2.11. Since we have $\pi_k \circ \pi(\text{Slip}_{r,n}) = \mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$ and $V \cap \mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$ is non-empty, $\pi(\text{Slip}_{r,n}) \cap W'$ is non-empty.

Now we show that

$$\dim_{\mathbb{k}} \text{Hom}_S(K, S/K)_0 \leq N + \dim_{\mathbb{k}} S_e - r + 1 = \dim W' + 1 \quad (3.59)$$

holds for every $[K] \in W' \subseteq \text{Hilb}_S^k$.

Let $[K] \in W'$ and $J = \overline{K} \cap \mathfrak{m}^e$. Consider the exact sequences

$$0 \rightarrow \text{Hom}_S(J/K, S/J)_0 \rightarrow \text{Hom}_S(J, S/J)_0 \rightarrow \text{Hom}_S(K, S/J)_0 \rightarrow \text{Ext}_S^1(J/K, S/J)_0 \quad (3.60)$$

and

$$0 \rightarrow \text{Hom}_S(K, J/K)_0 \rightarrow \text{Hom}_S(K, S/K)_0 \rightarrow \text{Hom}_S(K, S/J)_0 \rightarrow \text{Ext}_S^1(K, J/K)_0. \quad (3.61)$$

By assumption 1, there is an equality $J/K = \mathbb{k}(-e)$. It follows from Lemma 2.28 that we have

$$\dim_{\mathbb{k}} \text{Hom}_S(K, J/K)_0 = \beta_{1,e}(S/K) = \dim_{\mathbb{k}} S_e - r \quad (3.62)$$

and

$$\dim_{\mathbb{k}} \text{Ext}_S^1(K, J/K)_0 = 0. \quad (3.63)$$

Thus, by Equations (3.61) and (3.63) we get

$$\dim_{\mathbb{k}} \text{Hom}_S(K, S/K)_0 = \dim_{\mathbb{k}} \text{Hom}_S(K, J/K)_0 + \dim_{\mathbb{k}} \text{Hom}_S(K, S/J)_0. \quad (3.64)$$

Moreover, $\text{Hom}_S(J/K, S/J)_0$ is zero by Lemma 2.8. Therefore, it follows from Equations (3.60) and (3.64) that we have

$$\dim_{\mathbb{k}} \text{Hom}_S(K, S/K)_0 \leq \dim_{\mathbb{k}} \text{Hom}_S(K, J/K)_0 + \dim_{\mathbb{k}} \text{Hom}_S(J, S/J)_0 + \dim_{\mathbb{k}} \text{Ext}_S^1(\mathbb{k}, S/J)_e.$$

Equation (3.59) follows from Equation (3.62) and assumptions 4 and 5.

Let $[K] \in W' \cap \pi(\text{Slip}_{r,n})$ and let Z be an irreducible component of $\overline{W'} \cap \pi(\text{Slip}_{r,n})$ passing through $[K]$. By Theorem 2.75, Lemma 2.30 and Equation (3.59) we get

$$\dim Z \geq \dim \pi(\text{Slip}_{r,n}) + \dim W' - (\dim W' + 1) = rn - 1.$$

Moreover, W' is open in $\overline{W'}$ since W' is locally closed. Therefore, the generic point of Z belongs to W' . As a result, Z is contained in \overline{U} by assumption 3 and in fact there is an equality $Z = \overline{U}$ since we have $\dim Z \geq rn - 1 = \dim \overline{U}$. Since U is closed in W' we get $U = W' \cap Z \subseteq W' \cap \pi(\text{Slip}_{r,n})$ and therefore, $W' \cap \pi(\text{Slip}_{r,n})$ is equal to U by assumption 3. \square

Lemma 3.65. *Let J be a homogeneous ideal of S such that S/J has Hilbert function $h_{r,1}$. We have*

$$\dim_{\mathbb{k}} \text{Ext}_S^1(\mathbb{k}, S/J)_a = 0$$

for all $1 \leq a < r - 2$ and

$$\dim_{\mathbb{k}} \text{Ext}_S^1(\mathbb{k}, S/J)_{r-2} = 1.$$

Proof. Up to a linear change of variables, we may and do assume that J is equal to

$$(\alpha_0, \dots, \alpha_{n-2}, \theta_r(\alpha_{n-1}, \alpha_n)).$$

The Ext groups from the statement are the middle cohomology groups of the complex

$$(S/J)_a \xrightarrow{\delta_0} (S/J)_{a+1}^{n+1} \xrightarrow{\delta_1} (S/J)_{a+2}^{\binom{n+1}{2}}$$

obtained from the Koszul complex. Here we have

$$\delta_0([f]) = ([\alpha_i f])_{i=0, \dots, n} \text{ and } \delta_1([f_i]_{i=0, \dots, n}) = ([\alpha_i f_j - \alpha_j f_i]_{0 \leq i < j \leq n}).$$

Let $a \in \{1, \dots, r-3\}$ and assume that $\delta_1([f_i]_{i=0, \dots, n})$ is zero. Observe that $[\alpha_j f_i]$ is zero for every $i \in \{0, \dots, n\}$ and $j \in \{0, \dots, n-2\}$. Furthermore, multiplications by α_{n-1} or α_n give injective maps $(S/J)_a \rightarrow (S/J)_{a+1}$. It follows that $[f_j]$ is zero for all $j \leq n-2$. Moreover, we have $[\alpha_n f_{n-1}] = [\alpha_{n-1} f_n]$. There are unique representatives f_{n-1} and f_n of the classes $[f_{n-1}]$ and $[f_n]$ in S/J which are polynomials in variables α_{n-1} and α_n . From the equality $J_{a+2} = (\alpha_0, \dots, \alpha_{n-2})_{a+2}$, it follows that there is a polynomial g in variables α_{n-1} and α_n with $g\alpha_{n-1} = f_{n-1}$ and $g\alpha_n = f_n$. Hence we have $\delta_0([g]) = ([f_i]_{i=0, \dots, n})$. Consequently, $\text{Ext}_S^1(\mathbb{k}, S/J)_a$ is trivial.

Now assume that a equals $r-2$. As in the previous case, we get $[f_i] = 0$ for all $i \leq n-2$. Lift $[f_{n-1}]$ and $[f_n]$ to unique representatives which are polynomials in α_{n-1} and α_n . Now from the equality $[\alpha_n f_{n-1}] = [\alpha_{n-1} f_n]$ we deduce that there is a constant c with

$$\alpha_n f_{n-1} = \alpha_{n-1} f_n + c\theta_r.$$

The space

$$\Theta = \{(f_{n-1}, f_n) \in (\mathbb{k}[\alpha_{n-1}, \alpha_n]_{r-1})^2 \mid \alpha_n f_{n-1} - \alpha_{n-1} f_n \text{ is divisible by } \theta_r\}$$

is r -dimensional. It follows that we have

$$\dim_{\mathbb{k}} \text{Ext}_S^1(\mathbb{k}, S/J)_{r-2} = r - \dim_{\mathbb{k}} \delta_0\left((S/J)_{r-2}\right) = r - (r-1) = 1. \quad \square$$

Finally, we can present the criterion for ideals defining subschemes contained in a line.

Theorem 3.66. *Let $[I] \in \text{Hilb}_S^{h_r, n}$ be a closed point corresponding to an ideal I such that S/\bar{I} has Hilbert function $h_{r,1}$. There exists $[I'] \in \text{Slip}_{r,n}$ with $I_{\geq r-2} = I'_{\geq r-2}$ if and only if $(\bar{I}^2)_{r-2}$ is contained in I_{r-2} .*

Proof. We may assume that we have $n \geq 2$ and $r \geq 4$ since otherwise both conditions are trivially satisfied. Indeed, if we have $n = 1$ or $r < 4$ then $\text{Hilb}_S^{h_r, n}$ is irreducible (see Propositions 3.38–3.40) so we may take $I' = I$. On the other hand, if n equals 1 then I is saturated while if we have $r \leq 3$ then $(\bar{I}^2)_{r-2}$ is zero.

We use the notation of the beginning of this subsection with $f = h_{r,1}$. Let U be the locus of those points $[K]$ from W' that satisfy

$$(\bar{K}^2)_{r-2} \subseteq K_{r-2}.$$

In this notation, we need to prove that we have $\pi(\text{Slip}_{r,n}) \cap W' = U$, set-theoretically.

We start with showing that $\pi(\text{Slip}_{r,n}) \cap W'$ is a subset of U . Let $[I''] \in \text{Hilb}_S^{h_{r,n}}$ satisfy $[I'' \cap \mathfrak{m}^{r-2}] \in W' \setminus U$. We use Theorem 3.42 with $d = r - 2$ and $I = I''$ to show that $[I'' \cap \mathfrak{m}^{r-2}]$ does not belong to $\pi(\text{Slip}_{r,n})$. By Lemma 3.57 condition 3 of Theorem 3.42 holds. Let $\mathfrak{b} = ((\overline{I''})_1) \subseteq S$ and $\mathfrak{a} = \frac{\mathfrak{b} + \overline{I''}}{\overline{I''}} \subseteq S/\overline{I''}$. The ideal \mathfrak{a} is the zero ideal. In particular, there are equalities $\dim_{\mathbb{k}} \mathfrak{a}_{r-2} = \dim_{\mathbb{k}} \mathfrak{a}_{r-1} = 0$. Furthermore, since $[I'' \cap \mathfrak{m}^{r-2}]$ is not in U we have $(\overline{I''})_{r-2} = I''_{r-2} + \text{lin}\{F\}$ for some $F \in ((\overline{I''})^2)_{r-2} = ((\overline{I''})_1 \cdot \overline{I''})_{r-2} = (\mathfrak{b} \cdot \overline{I''})_{r-2}$. Thus, by Proposition 3.46 condition 1 of Theorem 3.42 is fulfilled. We are left with proving that $[\overline{I''} \cap \mathfrak{m}^{r-2}]$ is a smooth point of Hilb_S^h . Note that the following proof of this fact uses only the assumption that $S/\overline{I''}$ has Hilbert function $h_{r,1}$.

By Lemma 2.42 and Proposition 3.38, $[\overline{I''}] = [\overline{I''} \cap \mathfrak{m}]$ is a smooth point. Therefore, by Lemma 3.51 it is enough to show that the natural map

$$\text{Hom}_S(\overline{I''} \cap \mathfrak{m}^a, S/\overline{I''})_0 \rightarrow \text{Hom}_S(\overline{I''} \cap \mathfrak{m}^{a+1}, S/\overline{I''})_0$$

is bijective for every $1 \leq a < r - 2$. Let $1 \leq a \leq r - 3$ and $Q = (\overline{I''} \cap \mathfrak{m}^a)/(\overline{I''} \cap \mathfrak{m}^{a+1})$. We have an exact sequence

$$0 \rightarrow \text{Hom}_S(Q, S/\overline{I''})_0 \rightarrow \text{Hom}_S(\overline{I''} \cap \mathfrak{m}^a, S/\overline{I''})_0 \rightarrow \text{Hom}_S(\overline{I''} \cap \mathfrak{m}^{a+1}, S/\overline{I''})_0 \rightarrow \text{Ext}_S^1(Q, S/\overline{I''})_0.$$

Since $\text{Hom}_S(Q, S/\overline{I''})_0$ is zero by Lemma 2.8, it follows from Lemma 3.65 that we have

$$\text{Hom}_S(\overline{I''} \cap \mathfrak{m}^a, S/\overline{I''})_0 \cong \text{Hom}_S(\overline{I''} \cap \mathfrak{m}^{a+1}, S/\overline{I''})_0. \quad (3.67)$$

Thus, $[\overline{I''} \cap \mathfrak{m}^{r-2}] \in \text{Hilb}_S^h$ is a smooth point by Lemma 3.51. By Theorem 3.42 we conclude that $[I'' \cap \mathfrak{m}^{r-2}]$ does not belong to $\pi(\text{Slip}_{r,n})$. Hence there is an inclusion

$$\pi(\text{Slip}_{r,n}) \cap W' \subseteq U.$$

Now we show the opposite inclusion using Lemma 3.58. Assumption 1 is satisfied. Moreover, assumption 4 holds by Lemma 3.51 and Equation (3.67). We have shown above that $\pi(\text{Slip}_{r,n}) \cap W'$ is contained in U . Assumption 5 is fulfilled by Lemma 3.65. In our case, V is homeomorphic to a $\text{Hilb}_{S_{[\mathbb{P}^1]}}^{h_{r,1}}$ -bundle over $\text{Gr}(n-1, S_1)$ by [20, Prop. 3.1]. In particular, V is irreducible of dimension $2(n-1) + r$.

We need to show that U is irreducible of dimension $rn - 1$. Consider the natural map $U \rightarrow V$. The fiber over every closed point is irreducible of dimension $(n-1)(r-2) - 1$. Indeed, up to a linear change of variables we may assume that we have $[\text{Proj } S/\overline{I}] \in V$ with $\overline{I} = (\alpha_0, \dots, \alpha_{n-2}, \theta_r(\alpha_{n-1}, \alpha_n))$. The fiber over $[\text{Proj } S/\overline{I}]$ is the set of all codimension one subspaces of

$$\text{lin}\{\alpha_i \alpha_{n-1}^a \alpha_n^b \mid i \in \{0, \dots, n-2\}, a + b = r - 3, a, b \geq 0\}.$$

It follows from [81, 11.4.C] that U is irreducible of dimension

$$(n-1)(r-2) - 1 + 2(n-1) + r = rn - 1. \quad \square$$

We end this subsection with two examples of applications of Theorem 3.66. In the first of them, we show that a certain point does not belong to $\text{Slip}_{r,n}$.

Example 3.68. Let $S = \mathbb{k}[\alpha_0, \dots, \alpha_3]$ be a polynomial ring and let

$$I'' = (\alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_2, \alpha_0\alpha_3, \alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_0^3, \alpha_2^4).$$

The point $[I'']$ is in $\text{Hilb}_S^{h_{4,3}}$ and we claim that $[I'']$ does not belong to $\text{Slip}_{4,3}$. Indeed, we have $\overline{I}'' = (\alpha_0, \alpha_1, \alpha_2^4)$ but $\alpha_0^2 \in (\overline{I}''^2)_2 \setminus I''_2$. Thus, the claim follows from Theorem 3.66.

Observe that we have $H_{S/(I'')^2}(6) = 15 < 16$. Thus, we could have deduced that $[I'']$ is not in $\text{Slip}_{4,3}$ from Theorem 3.5. In fact, the proof of Theorem 3.66 presented in [67, Thm. 2.7] is based on the criterion from Theorem 3.5. Here we presented another proof that fits into the bigger picture (Theorem 3.42).

On the other hand $\dim_{\mathbb{k}} \text{Hom}_S(I'' + \mathfrak{m}^d, S/(I'' + \mathfrak{m}^d))_0 \geq 12$ holds for every $d \geq 3$. It follows that Proposition 3.1 cannot be used to deduce that $[I'']$ does not belong to $\text{Slip}_{4,3}$.

Consider again ideals I from Example 3.3 and I' from Example 3.6. We have $(\overline{I}^2)_4 = (\overline{I'}^2)_4 = (\alpha_0^2)_4$. Moreover, there are inclusions $(\alpha_0^2)_4 \subseteq I_4$ and $(\alpha_0^2)_4 \subseteq I'_4$. Thus, Theorem 3.66 cannot be used to deduce that $[I]$ and $[I']$ are not in $\text{Slip}_{6,2}$.

The following table summarizes Examples 3.3, 3.6 and 3.68.

Ideal	Prop. 3.1	Thm. 3.5	Thm. 3.66
$(\alpha_0^3, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \alpha_1^6)$	✓	?	?
$(\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_2^6)$?	✓	?
$(\alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_2, \alpha_0\alpha_3, \alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_0^3, \alpha_2^4)$?	✓	✓

Table 3.2: Examples of points that are not in the irreducible component Slip .

The ideals in the first and second row are in $\mathbb{k}[\alpha_0, \dots, \alpha_2]$. The ideal from the bottom row is in $\mathbb{k}[\alpha_0, \dots, \alpha_3]$.

In the second example, we use Theorem 3.66 to show that a given point belongs to $\text{Slip}_{r,n}$.

Example 3.69. Let $S = \mathbb{k}[\alpha_0, \dots, \alpha_3]$ be a polynomial ring and

$$J' = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_2, \alpha_1\alpha_2, \alpha_0\alpha_3, \alpha_1\alpha_3^2, \alpha_2^4).$$

The point $[J']$ is in $\text{Hilb}_S^{h_{4,3}}$ and we claim that $[J']$ belongs to $\text{Slip}_{4,3}$. We have $\overline{J}' = (\alpha_0, \alpha_1, \alpha_2^4)$ and $(\overline{J}'^2)_2 \subseteq J'_2$. Therefore, by Theorem 3.66 there exists $[K] \in \text{Slip}_{4,3}$ with $K_{\geq 2} = J'_{\geq 2}$. Since $K_1 = J'_1$ is zero, we conclude that $[J'] = [K]$ is in $\text{Slip}_{4,3}$.

3.6.5 Application two: constant growth on projective plane

In this subsection we have $S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$. Let $[I] \in \text{Hilb}_S^{h_{r,2}}$ and let f be the Hilbert function of S/\overline{I} . Let $m = \min\{a \in \mathbb{Z} \mid \overline{I}_a \neq 0\}$. Assume that there exist positive integers t and $e > m$ such that f is given by

$$f(a) = \begin{cases} \dim_{\mathbb{k}} S_a & \text{for } a < m \\ r - (e + 1 - a)t & \text{for } a \in \{m, m + 1, \dots, e\} \\ r & \text{for } a \geq e + 1 \end{cases} \quad (3.70)$$

Observe that for all $r \geq 4$ the function $h_{r,1}$ is of the above form with $m = 1, e = r - 2$ and $t = 1$. Another example of such Hilbert function is $(1, 3, 5, 7, 9, 11, 11, \dots)$. Here we have $m = 2, e = 4$ and $t = 2$.

Thus, in this subsection we consider more general Hilbert functions of S/\bar{I} than in Subsection 3.6.4 but we require that n equals 2.

The goal of this subsection is Theorem 3.75. There, we give a necessary condition for $[I]$ to be in $\text{Slip}_{r,2}$.

In the following lemma, we show that $[\bar{I} \cap \mathfrak{m}^e]$ is a smooth point. Thus, we verify condition 2 from Theorem 3.42.

Lemma 3.71. *We have*

$$\beta_{1,a}(S/\bar{I}) = 0 \text{ for all } e + 2 \neq a \geq m + 2, \quad (3.72)$$

and

$$\beta_{1,e+2}(S/\bar{I}) = t. \quad (3.73)$$

As a result, $[\bar{I} \cap \mathfrak{m}^e] \in \text{Hilb}_S^h$ is a smooth point where h is the Hilbert function of $S/(\bar{I} \cap \mathfrak{m}^e)$. Moreover, we have $\dim_{\mathbb{k}} \text{Hom}_S(\bar{I} \cap \mathfrak{m}^e, S/(\bar{I} \cap \mathfrak{m}^e))_0 = \dim W$ where $W \subseteq \text{Hilb}_S^h$ is the locally closed subset whose closed points correspond to the ideals defining subschemes of \mathbb{P}^2 with Hilbert function f .

Proof. Recall that the Hilbert function f of S/\bar{I} satisfies Equation (3.70). Let $e + 2 \neq a \geq m + 2$. By Lemma 2.13 we obtain $\beta_{1,a}(S/\bar{I}) \leq 2f(a - 1) - f(a) - f(a - 2) = 0$. Similarly, we get $\beta_{1,e+2}(S/\bar{I}) \leq 2f(e + 1) - f(e + 2) - f(e) = t$.

We show that we have $\beta_{1,e+2}(S/\bar{I}) \geq t$. Since \bar{I} is saturated, $\dim_{\mathbb{k}} \text{Hom}_S(\mathbb{k}, S/\bar{I})$ is zero. Therefore, by Lemma 2.27 we get

$$\begin{aligned} & -\dim_{\mathbb{k}} \text{Ext}_S^1(\mathbb{k}, S/\bar{I})_{e-1} + \dim_{\mathbb{k}} \text{Ext}_S^2(\mathbb{k}, S/\bar{I})_{e-1} - \dim_{\mathbb{k}} \text{Ext}_S^3(\mathbb{k}, S/\bar{I})_{e-1} \\ & = H_{S/\bar{I}}(e - 1) - 3H_{S/\bar{I}}(e) + 3H_{S/\bar{I}}(e + 1) - H_{S/\bar{I}}(e + 2) \\ & = (r - 2t) - 3(r - t) + 3r - r = t. \end{aligned}$$

It follows from Lemma 3.48 that $\beta_{1,e+2}(S/\bar{I}) \geq t$ holds.

Having calculated the Betti numbers, we proceed to proving the second part of the lemma. By Proposition 3.54 it is enough to show that there is an equality

$$H_{S/\bar{I}}(a) - 3H_{S/\bar{I}}(a + 1) + 3H_{S/\bar{I}}(a + 2) - H_{S/\bar{I}}(a + 3) = \beta_{1,a+3}(S/\bar{I}) \quad (3.74)$$

for every $a \in \{m, m + 1, \dots, e - 1\}$. By direct calculation, the left-hand side of Equation (3.74) equals zero for all $a \in \{m, \dots, e - 2\}$ and equals t for $a = e - 1$. The same is true for the right-hand side by Equations (3.72) and (3.73). \square

We present the main result of this subsection.

Theorem 3.75. *In the notation of Theorem 3.42 assume additionally that n equals 2. Let $m = \min\{a \in \mathbb{Z} \mid \bar{I}_a \neq 0\}$ and $e = \max\{a \in \mathbb{Z} \mid \bar{I}_a \neq I_a\}$. Assume that $e > m$ holds and that there exists a positive integer t such that S/\bar{I} has Hilbert function f as in Equation (3.70).*

(i) *There exists $\theta \in S_t$ with $\bar{I}_e \subseteq (\theta)_e$;*

- (ii) Let θ be as in part (i). If we have $\bar{I}_e = I_e + (\theta \cdot \bar{I})_e$ then there is no $[I'] \in \text{Slip}_{r,2}$ with $I'_{\geq e} = I_{\geq e}$. In particular, $[I]$ does not belong to $\text{Slip}_{r,2}$;
- (iii) If t equals 1 and $\theta \in S_1$ is as in part (i), then there exists $[I'] \in \text{Slip}_{r,2}$ with $I'_{\geq e} = I_{\geq e}$ if and only if $(\theta \cdot \bar{I})_e$ is contained in I_e .

Proof.

- (i) By Lemma 2.9(i) there is an element $L \in S_1$ that is a nonzerodivisor on S/\bar{I} . Moreover, we can take for L any general linear form. Let $T = S/(L)$ and

$$\mathfrak{a} = (\bar{I} + (L))/(L) \subseteq T.$$

We have $H_{S/\bar{I}}(a) - H_{S/\bar{I}}(a-1) = H_{T/\mathfrak{a}}(a)$ for any positive integer a . In particular, \mathfrak{a}_m is non-zero. Therefore, we have $t = H_{T/\mathfrak{a}}(m+1) \leq H_{T/\mathfrak{a}}(m) \leq m$, where the first inequality follows from Lemma 2.12 and the second from the fact that \mathfrak{a}_m is non-zero. We get $t = H_{T/\mathfrak{a}}(e+1) = H_{T/\mathfrak{a}}(e) < e$. Hence \mathfrak{a} has maximal growth in degree e . By [5, Lem. 1.4], we get that $I' = (\bar{I}_{\leq e})$ is a saturated ideal. Moreover, $\beta_{1,e+1}(S/\bar{I})$ is zero by Lemma 3.71. Thus, S/I' has Hilbert polynomial $P(a) = at + (r - e - 1)$ by Gotzmann's persistence theorem [10, Thm. 4.3.3] applied to $T/(\mathfrak{a}_{\leq e})$. Since P is of degree 1 and its leading coefficient is t it follows from [48, Prop. I.7.6] that the subscheme of \mathbb{P}^2 defined by I' contains a curve of degree t . Hence there exists $\theta \in S_t$ with $\bar{I}_e \subseteq (\theta)_e$.

- (ii) We want to use Theorem 3.42 with $d = e$. Assumption 3 of the theorem is fulfilled by Lemma 3.57. Lemma 3.71 implies that assumption 2 is satisfied.

Finally, we address assumption 1. Let $\mathfrak{b} = (\theta)$ and $\mathfrak{a} = \frac{(\theta) + \bar{I}}{\bar{I}} \subseteq S/\bar{I}$. We assumed that we have $\bar{I}_e = I_e + (\theta \cdot \bar{I})_e$. Therefore, by Proposition 3.46 it is enough to show that there is an equality $\dim_{\mathbb{k}} \mathfrak{a}_e = \dim_{\mathbb{k}} \mathfrak{a}_{e+1}$. By part (i) and Equation (3.72) we get $\bar{I}_{\leq e+1} \subseteq (\theta)$. It follows that there is a chain of equalities

$$\begin{aligned} \dim_{\mathbb{k}} \mathfrak{a}_{e+1} - \dim_{\mathbb{k}} \mathfrak{a}_e &= (\dim_{\mathbb{k}}(\theta)_{e+1} - \dim_{\mathbb{k}}(\theta)_e) - (\dim_{\mathbb{k}} \bar{I}_{e+1} - \dim_{\mathbb{k}} \bar{I}_e) \\ &= (\dim_{\mathbb{k}} S_{e-t+1} - \dim_{\mathbb{k}} S_{e-t}) - (\dim_{\mathbb{k}} S_{e+1} - H_{S/\bar{I}}(e+1) - \dim_{\mathbb{k}} S_e + H_{S/\bar{I}}(e)) \\ &= (e+2-t) - (e+2-t) = 0. \end{aligned}$$

- (iii) Assume that there exists $[I'] \in \text{Slip}_{r,2}$ with $I'_{\geq e} = I_{\geq e}$. We show that $(\theta \cdot \bar{I})_e$ is contained in I_e . It follows from $t = 1$ that I_e is of codimension 1 in \bar{I}_e . Therefore, if $(\theta \cdot \bar{I})_e$ is not contained in I_e then we have $(\theta \cdot \bar{I})_e + I_e = \bar{I}_e$. Thus, we obtain a contradiction with part (ii).

We proceed to the proof of the other implication. Let h, k, V, W and W' be defined as in the beginning of Subsection 3.6.4 with $n = 2$ and f being the Hilbert function of S/\bar{I} .

Let U be the locus of those points $[K]$ of W' that satisfy

$$(\theta \cdot \bar{K})_e \subseteq K_e$$

where $\theta \in S_1$ is the common divisor of \bar{K}_e . We need to show that U is contained in $\pi(\text{Slip}_{r,2}) \cap W'$ set-theoretically. We use Lemma 3.58. Assumption 1 is clear and assumption 2 follows from Proposition 3.11. Assumption 4 is a consequence of Lemma 3.71.

Part (ii) implies that we have $\pi(\text{Slip}_{r,2}) \cap W' \subseteq U$, set-theoretically. Therefore, we need to show that

- (a) U is irreducible of dimension $2r - 1$;
- (b) $\text{Ext}_S^1(\mathbb{k}, S/J)_e \leq 1$ holds for every $[J] \in W$.

We start with (a). By [81, 11.4.C] the subset U is an irreducible subset of Hilb_S^k of dimension

$$\dim V + \left(\dim_{\mathbb{k}} S_e - r - (\dim_{\mathbb{k}} S_{e-1} - (r-2)) \right) = \dim V + e - 1.$$

Indeed, the fiber over a closed point $[\text{Proj } S/I'] \in V$ of the natural map $U \rightarrow V$ is the set of those codimension 1 subspaces of $(\overline{I'})_e$ that contain $(\theta \cdot \overline{I'})_e$ where $\theta \in S_1$ is the common divisor of $(\overline{I'})_e$. Now, it suffices to establish the equality $\dim V = 2r - e$.

Let $E_f \subseteq \text{Hilb}_S^f$ be the open subset whose points correspond to saturated ideals. It is homeomorphic with V by Lemma 2.29 applied to the natural map $\text{Hilb}_S^f \rightarrow \mathcal{Hilb}_r(\mathbb{P}^2)$.

Let $I'' = (\alpha_0^m, \alpha_0^{m-1}\alpha_1, \dots, \alpha_0^s\alpha_1^{m-s}, \alpha_0^{s-1}\alpha_1^{m-s+2}, \dots, \alpha_0\alpha_1^m, \alpha_1^{e+2}) \subseteq S$ where we have $m+1-s = \dim_{\mathbb{k}} S_m - f(m)$. The algebra S/I'' has Hilbert function f and I'' is saturated. Therefore, by Proposition 3.10 and Theorem 2.75, it is enough to show that the equality

$$\dim_{\mathbb{k}} \text{Hom}_S(I'', S/I'')_0 = 2r - e$$

holds. Let $T = \mathbb{k}[\alpha_0, \alpha_1]$ and $\mathfrak{a} = T \cap I''$. We have

$$\dim_{\mathbb{k}} \text{Hom}_S(I'', S/I'')_0 = \dim_{\mathbb{k}} \text{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{\leq 0} = 2r - \dim_{\mathbb{k}} \text{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{> 0},$$

where the last equality follows from the fact that $[\text{Spec } T/\mathfrak{a}]$ is a point of the smooth $2r$ -dimensional scheme $\mathcal{Hilb}_r(\mathbb{A}^2)$ and [49, Prop. 2.3]. By Proposition 2.22 the dimension of $\text{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{> 0}$ can be computed from the staircase diagram of \mathfrak{a} .

Observe that \mathfrak{a} can have minimal generators only in degrees $m, m+1$ and $e+2$. Furthermore, $\beta_{2,a}(T/\mathfrak{a})$ can be non-zero only for some $a \in \{m+1, m+2, e+3\}$. Let $A = (m+1-s) = \beta_{1,m}(T/\mathfrak{a})$. We have $\beta_{1,m+1}(T/\mathfrak{a}) = m-A$, $\beta_{2,m+1}(T/\mathfrak{a}) = A-1$ and $\beta_{2,m+2}(T/\mathfrak{a}) = m-A$. Therefore, by Proposition 2.22 we get

$$\begin{aligned} \text{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{> 0} &= \beta_{1,m}(T/\mathfrak{a})(e-m+1) + \beta_{1,m+1}(T/\mathfrak{a})(e-m) \\ &\quad - \beta_{2,m+1}(T/\mathfrak{a})(e-m) - \beta_{2,m+2}(T/\mathfrak{a})(e-m-1) \\ &= A(e-m+1) + (m-A)(e-m) - [(A-1)(e-m) + (m-A)(e-m-1)] = e. \end{aligned}$$

This concludes the proof of (a).

Let $[J] \in W$. Using Lemmas 3.48 and 3.71 we get

$$\dim_{\mathbb{k}} \text{Ext}_S^2(\mathbb{k}, S/J)_e = \dim_{\mathbb{k}} \text{Ext}_S^2(\mathbb{k}, S/\overline{J})_e = \beta_{1,e+3}(S/\overline{J}) = 0.$$

Furthermore, Lemma 2.27 and Equation (3.70) (with $t = 1$) imply that we have

$$\begin{aligned} & - \dim_{\mathbb{k}} \text{Ext}_S^1(\mathbb{k}, S/J)_e + \dim_{\mathbb{k}} \text{Ext}_S^2(\mathbb{k}, S/J)_e - \dim_{\mathbb{k}} \text{Ext}_S^3(\mathbb{k}, S/J)_e \\ & = H_{S/J}(e) - 3H_{S/J}(e+1) + 3H_{S/J}(e+2) - H_{S/J}(e+3) = -1. \end{aligned}$$

The inequality $\dim_{\mathbb{k}} \text{Ext}_S^1(\mathbb{k}, S/J)_e \leq 1$ follows. Thus, (b) holds. \square

We end this subsection with two more examples. In the first, we use Theorem 3.75 to show that a certain point is outside of $\text{Slip}_{6,2}$.

Example 3.76. Let $S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$ be a polynomial ring and let

$$I''' = (\alpha_0^2\alpha_1, \alpha_0^2\alpha_2, \alpha_0\alpha_1^2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_1^5).$$

The point $[I''']$ is in $\text{Hilb}_S^{h_{6,2}}$ and we claim that $[I''']$ does not belong to $\text{Slip}_{6,2}$. We have $\overline{I'''} = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^5)$. Thus, the Hilbert function of $S/\overline{I'''}$ is as in Equation (3.70) with $r = 6, m = 2, e = 3$ and $t = 1$. We have $(\overline{I'''})_3 \subseteq (\alpha_0)_3$ but $\alpha_0^3 \in (\alpha_0 \cdot \overline{I'''})_3 \setminus I'''_3$. It follows from Theorem 3.75 that $[I''']$ does not belong to $\text{Slip}_{6,2}$.

On the other hand, criteria from Proposition 3.1 and Theorem 3.5 (with $k = 2$) do not show that $[I''']$ is not in $\text{Slip}_{6,2}$. Furthermore, criterion from Theorem 3.66 cannot be applied to $[I''']$ since we have $H_{S/\overline{I'''}}(1) \neq 2$.

We summarize Examples 3.3, 3.6, 3.68 and 3.76 in the following table.

Ideal	Prop. 3.1	Thm. 3.5	Thm. 3.66	Thm. 3.75
$(\alpha_0^3, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \alpha_1^6)$	✓	?	?	?
$(\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1^6)$?	✓	?	?
$(\alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_2, \alpha_0\alpha_3, \alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_0^3, \alpha_2^4)$?	✓	✓	NA
$(\alpha_0^2\alpha_1, \alpha_0^2\alpha_2, \alpha_0\alpha_1^2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_1^5)$?	?	NA	✓

Table 3.3: Examples of points that are not in the irreducible component Slip .

Here ideals from first, second and fourth rows are in $\mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$, while ideal in the third row is in the ring $\mathbb{k}[\alpha_0, \alpha_1, \alpha_2, \alpha_3]$. The symbol NA means that a certain criterion cannot be applied to a given point since the assumptions are not fulfilled. As before, the question mark means that a given necessary condition is satisfied by the given point, i.e. the criterion is inconclusive.

In the second example, we use Theorem 3.75(iii) to deduce that a certain point belongs to the irreducible component $\text{Slip}_{6,2}$.

Example 3.77. Let $S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$ and $J'' = (\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^3, \alpha_1^5)$. The point $[J'']$ is in $\text{Hilb}_S^{h_{6,2}}$ and we claim that it belongs to $\text{Slip}_{6,2}$. The Hilbert function of $S/\overline{J''}$ is of the form given by Equation (3.70) with $m = 2, e = 3, t = 1$ and $r = 6$. Furthermore, $(\overline{J''})_{\leq 3}$ is contained in $(\alpha_0)_3$. We have $(\alpha_0 \cdot \overline{J''})_3 = (\alpha_0^2\alpha_1, \alpha_0^2\alpha_2)_3 \subseteq J''_3$. Therefore, by Theorem 3.75(iii), there exists $[K] \in \text{Slip}_{6,2}$ with $K_{\geq 3} = J''_{\geq 3}$. Since $K_{\leq 2} = J''_{\leq 2}$ is zero, we conclude that $[J''] = [K]$ is in $\text{Slip}_{6,2}$.

We summarize Examples 3.34, 3.69 and 3.77 in the following table.

Ideal	Theorem 3.13	Theorem 3.66	Theorem 3.75
$(\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0\alpha_2^2, \alpha_1^4)$	✓	NA	NA
$(\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_2, \alpha_1\alpha_2, \alpha_0\alpha_3, \alpha_1\alpha_3^2, \alpha_2^4)$	NA	✓	NA
$(\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^3, \alpha_1^5)$	NA	NA	✓

Table 3.4: Examples of points that are in the irreducible component Slip .

Here the ideal in the second row is in $\mathbb{k}[\alpha_0, \dots, \alpha_3]$ while the other two ideals are in $\mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$. As before, NA means that a certain criterion cannot be applied to the given ideal. Observe that we cannot use Theorem 3.75(iii) for J since, in the notation of that theorem, we have $m = e = 2$.

3.7 Points on projective space—examples, part II

From Fogarty's result [36] on $\mathcal{H}ilb_r(\mathbb{P}^2)$ it may seem that $\text{Hilb}_{S[\mathbb{P}^2]}^{h_{r,2}}$ should be smooth, or at least not too complicated. We show that this is not the case. Speculating a bit, we may say that Fogarty's result concerns the case of codepth two, while we work in nonsaturated setting, hence in codepth three. Thus, the correct parallel would be $\mathcal{H}ilb_r(\mathbb{P}^3)$, where almost nothing is known about the principal component.

3.7.1 4 points on projective space

In this subsection, we describe the closed points of $\text{Slip}_{4,n}$ for a positive integer n .

Proposition 3.78. *Let $I \subseteq S[\mathbb{P}^n]$ be a homogeneous ideal such that $S[\mathbb{P}^n]/I$ has Hilbert function $h_{4,n}$. The point $[I]$ is in $\text{Slip}_{4,n}$ if and only if $(\bar{I}^2)_2 \subseteq I_2$.*

Proof. Condition $(\bar{I}^2)_2 \subseteq I_2$ holds trivially for $[I] \in \text{Hilb}_{S[\mathbb{P}^n]}^{h_{4,n}}$ if I is saturated. On the other hand, $\mathcal{H}ilb_4(\mathbb{P}^n)$ is irreducible by [20, Thm. 1.1]. Thus, by Remark 2.47, it is enough to consider ideals that are not saturated. Furthermore, by [20, Prop. 3.1], we may assume that we have $n \leq 3$.

If n equals 1, then every closed point $[I] \in \text{Hilb}_{S[\mathbb{P}^1]}^{h_{4,1}}$ corresponds to a saturated ideal.

If n equals 2 and $[I]$ is a point of $\text{Hilb}_{S[\mathbb{P}^2]}^{h_{4,2}}$ with $I \neq \bar{I}$, then S/\bar{I} has Hilbert function $h_{4,1}$. This follows from Lemma 2.9. Therefore, by Theorem 3.66 we have

$$(\bar{I}^2)_2 \subseteq I_2 \Leftrightarrow \text{there exists } [J] \in \text{Slip}_{4,2} \text{ with } J_{\geq 2} = I_{\geq 2} \Leftrightarrow [I] \in \text{Slip}_{4,2}.$$

The latter equivalence follows from the fact that $J_{\leq 1} = I_{\leq 1}$ is zero.

Assume that n equals 3 and $[I] \in \text{Hilb}_{S[\mathbb{P}^3]}^{h_{4,3}}$ satisfies $I \neq \bar{I}$. The algebra S/\bar{I} has Hilbert function $h_{4,2}$ or $h_{4,1}$. In the first case the condition $(\bar{I}^2)_2 \subseteq I_2$ holds. We claim that $[I]$ belongs to $\text{Slip}_{4,3}$. Indeed, we have $[\text{Proj } S/I] \in \mathcal{H}ilb_4^{sm}(\mathbb{P}^3) = \mathcal{H}ilb_4(\mathbb{P}^3)$, so there exists an ideal $[J] \in \text{Slip}_{4,3}$ with $\bar{J} = \bar{I}$. However, we have $J_1 = I_1 = 0$ and $J_{\geq 2} = \bar{I}_{\geq 2} = I_{\geq 2}$, so $[I] = [J]$ is in $\text{Slip}_{4,3}$.

Finally, assume that $[I] \in \text{Hilb}_{S[\mathbb{P}^3]}^{h_{4,3}}$ is such that S/\bar{I} has Hilbert function $h_{4,1}$. As in the case $n = 2$, by Theorem 3.66 we get

$$(\bar{I}^2)_2 \subseteq I_2 \Leftrightarrow \text{there exists } [J] \in \text{Slip}_{4,3} \text{ with } J_{\geq 2} = I_{\geq 2} \Leftrightarrow [I] \in \text{Slip}_{4,3}. \quad \square$$

As a corollary, we obtain an example of a reducible multigraded Hilbert scheme.

Corollary 3.79. *The scheme $\text{Hilb}_{S[\mathbb{P}^2]}^{h_{4,2}}$ is reducible. In fact, we have $[(\alpha_0\alpha_1, \alpha_0\alpha_2, \alpha_0^3, \alpha_1^4)] \notin \text{Slip}_{4,2}$.*

We stress the fact, that there is a point outside of $\text{Slip}_{4,2}$ that corresponds to a *monomial* ideal since it was conjectured that it is impossible.

Remark 3.80. The comment after [15, Cor. 6.3] puts forward a conjecture that conditions (i) and (iii) of [15, Cor. 6.3] imply condition (iv). Corollary 3.79 shows that this is not true.

3.7.2 5 points on projective plane

In this subsection, we describe the closed points of $\text{Slip}_{5,2}$ (see Proposition 3.89). Let $S = S[\mathbb{P}^2] = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$ and $[I]$ be a closed point of $\text{Hilb}_S^{h_{5,2}}$. By Lemma 2.9, the Hilbert function of S/\bar{I} is one of the three: $h_{5,2}, g$ or $h_{5,1}$, where $g: \mathbb{Z} \rightarrow \mathbb{Z}$ is given by

$$g(a) = \begin{cases} \dim_{\mathbb{k}} S_a & \text{for } a \leq 0; \\ a + 2 & \text{for } a = 1, 2; \\ 5 & \text{for } a \geq 3; \end{cases}$$

or informally, $g = (1, 3, 4, 5, 5, \dots)$.

The cases when S/\bar{I} has Hilbert function $h_{5,2}$ or g are easy.

Lemma 3.81. *Let $[I] \in \text{Hilb}_S^{h_{5,2}}$ be a closed point such that S/\bar{I} has Hilbert function $h_{5,2}$ or g . The point $[I]$ is in $\text{Slip}_{5,2}$.*

Proof. If I is saturated, then the claim follows from Remark 2.47 since $\mathcal{Hilb}_5(\mathbb{P}^2)$ is irreducible. Assume that S/\bar{I} has Hilbert function g . Observe that we have $g(a) \neq h_{5,2}(a)$ if and only if a equals 2. Thus, the claim follows from Theorem 3.13. \square

We study those points $[I] \in \text{Hilb}_S^{h_{5,2}}$ for which S/\bar{I} has Hilbert function $h_{5,1}$. We introduce some more notation. Let $h: \mathbb{Z} \rightarrow \mathbb{Z}$ be given by $h = (1, 3, 6, 5, 5, \dots)$ or more formally,

$$h(a) = \begin{cases} \dim_{\mathbb{k}} S_a & \text{for } a \leq 2; \\ 5 & \text{for } a \geq 3. \end{cases}$$

Let $\pi: \text{Hilb}_S^{h_{5,2}} \rightarrow \text{Hilb}_S^h$ and $\pi': \text{Hilb}_S^h \rightarrow \mathcal{Hilb}_5(\mathbb{P}^2)$ be the natural morphisms. Let $V \subseteq \mathcal{Hilb}_r(\mathbb{P}^2)$ be the closed subset whose closed points correspond to the subschemes with Hilbert function $h_{5,1}$.

Let $W \subseteq \text{Hilb}_S^h$ be the set-theoretic inverse image of V under π' and let U be the set-theoretic inverse image of W under π . We show that these subsets are irreducible and we calculate their dimensions. We start with V , but we state it in greater generality since we need this also in Subsection 3.7.3.

Lemma 3.82. *Let r be a positive integer. Let V be the closed subset of $\mathcal{Hilb}_r(\mathbb{P}^2)$, whose closed points correspond to subschemes with Hilbert function $h_{r,1}$. The set V is irreducible and $(r+2)$ -dimensional.*

Proof. The scheme $\text{Hilb}_S^{h_{r,1}}$ is irreducible, smooth and $(r+2)$ -dimensional by Proposition 3.38 and [20, Prop. 3.1]. Therefore, the natural morphism $\text{Hilb}_S^{h_{r,1}} \rightarrow \mathcal{Hilb}_r(\mathbb{P}^2)$ factors through V (with reduced subscheme structure). It follows from Lemma 2.29 that V is homeomorphic to $\text{Hilb}_S^{h_{r,1}}$. In particular, it is irreducible and $(r+2)$ -dimensional. \square

Lemma 3.83. *The subset $U \subseteq \text{Hilb}_S^{h_{5,2}}$ is irreducible and 11-dimensional.*

Proof. By Lemma 3.82, the locus V is irreducible and 7-dimensional. Thus, in order to show that U is irreducible and 11-dimensional, it is enough to show that the fiber of $\pi' \circ \pi$ over every closed point of V is irreducible and 4-dimensional (see [81, 11.4.C]). Choose a closed point $[\text{Proj } S/\bar{I}] \in V$. Denote the fiber of $\pi' \circ \pi$ over this point by X . We have a pullback diagram

$$\begin{array}{ccc}
X & \longrightarrow & \text{Fl}(3, 5, \bar{I}_3) \\
\downarrow & & \downarrow \\
\text{Gr}(1, \bar{I}_2) & \longrightarrow & \text{Gr}(3, \bar{I}_3),
\end{array}$$

where $\text{Fl}(3, 5, \bar{I}_3)$ is the flag variety parametrizing pairs of linear subspaces $A_3 \subseteq A_5$ of \bar{I}_3 with $\dim A_i = i$ for all $i \in \{3, 5\}$. The lower horizontal morphism maps $\text{lin}\{\ell\}$ to $\text{lin}\{\alpha_0\ell, \alpha_1\ell, \alpha_2\ell\}$. Since the fibers of the right vertical map are irreducible and of dimension 2, it follows that X is irreducible and of dimension $\dim X = \dim \text{Gr}(1, \bar{I}_2) + 2 = 4$. \square

Let U' be the subset of U whose closed points $[I]$ satisfy $(\bar{I}^2)_3 \subseteq I_3$.

Lemma 3.84. *The subset U' of U is closed.*

Proof. Consider the diagram

$$\begin{array}{ccc}
& & \text{Hilb}_S^{h_{5,2}} \\
& & \downarrow d \\
\text{Fl}(3, 5, S_3) & \xrightarrow{c} & \text{Gr}(5, S_3) \\
\downarrow b & & \\
\text{Gr}(1, S_1) & \xrightarrow{a} & \text{Gr}(3, S_3)
\end{array}
,$$

where a maps $\text{lin}(\ell)$ to $\text{lin}\{\alpha_0\ell^2, \alpha_1\ell^2, \alpha_2\ell^2\}$ and b, c, d are the natural maps. We have $U' = U \cap d^{-1}(c(b^{-1}(a(\text{Gr}(1, S_1)))))$, so it is closed. \square

Let $W' \subseteq \text{Hilb}_S^h$ be the set-theoretic image $\pi(U')$. It is closed by Lemma 3.84 since π is a morphism of projective schemes by Theorem 2.36.

Lemma 3.85. *The subset $W' \subseteq \text{Hilb}_S^h$ is irreducible and 9-dimensional.*

Proof. Consider the natural map $W' \rightarrow V$. By Lemma 3.82 and [81, 11.4.C], it is enough to show that the fibers are irreducible and 2-dimensional. Let $[\text{Proj } S/\bar{I}] \in V$. We may assume that \bar{I} is equal to $(\alpha_0, \theta_5(\alpha_1, \alpha_2))$. We have $(\alpha_0^2)_3 \subseteq I_3$ for every $[I] \in W'$ that is in the fiber over $[\text{Proj } S/\bar{I}]$. Thus, the fiber of $W' \rightarrow V$ is $\text{Gr}(2, \text{lin}\{\alpha_0\alpha_1^2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^2\})$. \square

Let

$$W'_i = \{[K] \in W' \mid \dim_{\mathbb{k}}(K : \mathfrak{m})_2 = i\}$$

for all $i \in \{1, 2\}$. Observe that we have $W' = W'_1 \cup W'_2$. Indeed, let $[K] \in W'$ and assume that there is an equality $\bar{K}_1 = (\alpha_0)_1$. We have $\alpha_0^2 \in (K : \mathfrak{m})_2$ by the definition of W' . On the other hand, if $\dim_{\mathbb{k}}(K : \mathfrak{m})_2 \geq 3$ holds then $(\alpha_0)_2$ is contained in $(K : \mathfrak{m})_2$. It follows that $(\alpha_0)_3$ is a subset of K_3 . This contradicts the assumption that $H_{S/K}(3)$ equals 5.

Lemma 3.86. *The closed subset W'_2 of W' is irreducible and 8-dimensional.*

Proof. We have a natural map $W'_2 \rightarrow V$. By Lemma 3.82 and [81, 11.4.C], it is enough to show that its fibers are irreducible and 1-dimensional. Consider the point $[\text{Proj } S/(\alpha_0, \theta_5(\alpha_1, \alpha_2))] \in V$. Fiber over this point is

$$\{[K_3] \in \text{Gr}(5, (\alpha_0)_3) \mid (\alpha_0^2)_3 \subseteq K_3 \text{ and } \dim_{\mathbb{k}}((K_3) : \mathfrak{m})_2 = 2\}.$$

We have $\alpha_0^2 \in ((K_3): \mathfrak{m})_2$ for every $[K_3]$ in the fiber. Therefore, the fiber is \mathbb{P}^1 corresponding to the choice of $[\ell] \in \mathbb{P}(\text{lin}\{\alpha_1, \alpha_2\})$ with $K_3 = \text{lin}\{\alpha_0^3, \alpha_0^2\alpha_1, \alpha_0^2\alpha_2, \alpha_0\alpha_1\ell, \alpha_0\alpha_2\ell\}$. \square

Let

$$Z_1 = \{[I] \in U' \mid \dim_{\mathbb{k}}(I_{\geq 3}: \mathfrak{m})_2 = 1\}$$

and

$$Z_2 = \{[I] \in U' \mid \dim_{\mathbb{k}}(I_{\geq 3}: \mathfrak{m})_2 = 2\}.$$

Lemma 3.87. *The closed subsets $\overline{Z_1}$ and Z_2 are irreducible and 9-dimensional. Moreover, we have $U' = Z_1 \cup Z_2$ set-theoretically.*

Proof. By definition we have $Z_i = \pi^{-1}(W'_i)$ set-theoretically. Moreover, Z_1 is homeomorphic to W'_1 . We claim that the fiber of $Z_2 \rightarrow W'_2$ over every closed point is irreducible and 1-dimensional. Indeed, the fiber over $[K]$ is \mathbb{P}^1 corresponding to the choice of a non-zero element of $(K: \mathfrak{m})_2$.

Therefore, $\overline{Z_1}$ and Z_2 are irreducible and 9-dimensional by Lemmas 3.85, 3.86 and [81, 11.4.C]. \square

Now we can describe the set-theoretic intersection of $\text{Slip}_{5,2}$ with U .

Lemma 3.88. *Set-theoretically we have $U \cap \text{Slip}_{5,2} = Z_1 \cup Z_2$.*

Proof. Containment $U \cap \text{Slip}_{5,2} \subseteq Z_1 \cup Z_2$ follows from Theorem 3.66. Moreover, for every $[I] \in Z_1$, the only ideal $[J] \in \text{Hilb}_S^{h_{5,2}}$ with $J_{\geq 3} = I_{\geq 3}$ is I . Thus, Z_1 is a subset of $\text{Slip}_{5,2}$ by Theorem 3.66.

We show that Z_2 is contained in $\text{Slip}_{5,2}$. Let $I = (\alpha_0\alpha_1, \alpha_0^3, \alpha_0^2\alpha_2, \alpha_0\alpha_2^3 + \alpha_1^4)$. We have $[I] \in \text{Hilb}_S^{h_{5,2}}$. Moreover, the Hilbert function of S/\overline{I} is g . Thus, $[I]$ belongs to $\text{Slip}_{5,2}$ by Lemma 3.81. Consider the initial ideal I' of I with respect to the grevlex order with $\alpha_1 < \alpha_2 < \alpha_0$. The ideal I' is equal to $(\alpha_0\alpha_1, \alpha_0^3, \alpha_0^2\alpha_2, \alpha_0\alpha_2^3, \alpha_1^5)$. The point $[I']$ is in $\text{Slip}_{5,2} \cap Z_2$. Furthermore, we have $\dim_{\mathbb{k}} \text{Hom}_S(I', S/I')_0 = 12 = \dim U + 1$ (see Lemma 3.83). It follows from Theorem 2.75 and Lemma 2.30 that every irreducible component of $\text{Slip}_{5,2} \cap U$ passing through $[I']$ is at least 9-dimensional. This intersection is contained in $Z_1 \cup Z_2$ and this is a union of two irreducible 9-dimensional subsets. Hence it suffices to show that $[I']$ does not belong to $\overline{Z_1}$. Consider the projection $\text{Hilb}_S^{h_{5,2}} \rightarrow \text{Gr}(1, S_2)$. The image of Z_1 is

$$\{\text{lin}\{\ell^2\} \mid \ell \in S_1 \setminus \{0\}\},$$

i.e. it is the image of the second Veronese embedding of $\mathbb{P}S_1$. On the other hand, $\alpha_0\alpha_1 \in I'_2$ is not a power of a linear form. Thus, $[I']$ is not in $\overline{Z_1}$. \square

We summarize the above results in the following proposition which describes $\text{Slip}_{5,2}$.

Proposition 3.89. *Let $S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$ and $[I] \in \text{Hilb}_S^{h_{5,2}}$ be a closed point. The point $[I]$ is in $\text{Slip}_{5,2}$ if and only if $(\overline{I}^2)_3$ is contained in I_3 .*

Proof. If S/\overline{I} has Hilbert function $h_{5,2}$ or g , then $[I]$ is in $\text{Slip}_{5,2}$ by Lemma 3.81. On the other hand, in both this cases $(\overline{I}^2)_3 \subseteq I_3$ holds since $(\overline{I}^2)_3$ is zero.

Assume that S/\overline{I} has Hilbert function $h_{5,1}$. We have $(\overline{I}^2)_3 \subseteq I_3$ if and only if $[I]$ belongs to $Z_1 \cup Z_2$ (see Lemma 3.87). This is equivalent to the condition that $[I]$ is in $\text{Slip}_{5,2}$ by Lemma 3.88. \square

3.7.3 6 points on projective plane

The main result of this subsection is the description of $\text{Slip}_{6,2}$ (see Proposition 3.104). Let $S = \mathbb{k}[\alpha_0, \alpha_1, \alpha_2]$ and let $[I] \in \text{Hilb}_S^{h_{6,2}}$ be a closed point. The Hilbert function of S/\bar{I} is one of the four: $h_{6,2}$, f , g or $h_{6,1}$ where $f: \mathbb{Z} \rightarrow \mathbb{Z}$ is given by

$$f(a) = \begin{cases} 0 & \text{for } a < 0; \\ 2a + 1 & \text{for } a \in \{0, 1, 2\}; \\ 6 & \text{for } a \geq 3 \end{cases}$$

(or, in a brief form, $f = (1, 3, 5, 6, 6, \dots)$) and $g: \mathbb{Z} \rightarrow \mathbb{Z}$ is given by

$$g(a) = \begin{cases} \dim_{\mathbb{k}} S_a & \text{for } a \leq 1; \\ a + 2 & \text{for } a \in \{2, 3\}; \\ 6 & \text{for } a \geq 4 \end{cases}$$

(or, $g = (1, 3, 4, 5, 6, 6, \dots)$).

We start with the points $[I]$ for which the Hilbert function of S/\bar{I} is $h_{6,2}$ or f .

Lemma 3.90. *Let $[I] \in \text{Hilb}_S^{h_{6,2}}$ be a closed point such that S/\bar{I} has Hilbert function $h_{6,2}$ or f . The point $[I]$ is in $\text{Slip}_{6,2}$.*

Proof. If I is saturated, the claim follows from Remark 2.47. Assume that the Hilbert function of S/\bar{I} is f . Observe that we have $f(a) \neq h_{6,2}(a)$ if and only if a equals 2. Thus, the claim follows from Theorem 3.13. \square

Next we consider points $[I]$ corresponding to ideals such that S/\bar{I} has Hilbert function g .

Lemma 3.91. *Let $[I] \in \text{Hilb}_S^{h_{6,2}}$ be a closed point such that S/\bar{I} has Hilbert function g . There is a linear form $\theta \in S_1$ with $\bar{I}_3 \subseteq (\theta)_3$. We have $[I] \in \text{Slip}_{6,2}$ if and only if $(\theta \cdot \bar{I})_3$ is contained in I_3 .*

Proof. The existence of θ as in the statement follows from Theorem 3.75(i). Moreover, by Theorem 3.75(iii) there exists $[J] \in \text{Slip}_{6,2}$ with $J_{\geq 3} = I_{\geq 3}$ if and only if $(\theta \cdot \bar{I})_3$ is a subset of I_3 . We claim that necessarily we have $[J] = [I]$.

There are equalities $h_{6,2}(a) = \dim_{\mathbb{k}} S_a$ for every $a \leq 2$. Therefore, if $[J] \in \text{Hilb}_S^{h_{6,2}}$ satisfies $J_{\geq 3} = I_{\geq 3}$, then there is an equality $J = I$. \square

In order to study the case when S/\bar{I} has Hilbert function $h_{6,1}$, we introduce some notation. Let U_g be a locally closed subset of $\text{Hilb}_S^{h_{6,2}}$, whose closed points correspond to ideals I for which S/\bar{I} has Hilbert function g . Similarly, let U be the closed subset corresponding to Hilbert function $h_{6,1}$. Let $h: \mathbb{Z} \rightarrow \mathbb{Z}$ be given by $h = (1, 3, 6, 10, 6, 6, \dots)$, or more formally,

$$h(a) = \begin{cases} \dim_{\mathbb{k}} S_a & \text{for } a \leq 3; \\ 6 & \text{for } a > 3. \end{cases}$$

We have the natural morphisms $\pi: \text{Hilb}_S^{h_{6,2}} \rightarrow \text{Hilb}_S^h$ and $\pi': \text{Hilb}_S^h \rightarrow \mathcal{Hilb}_6(\mathbb{P}^2)$.

We start with showing that U_g is irreducible and we compute its dimension.

Lemma 3.92. *The locus U_g is irreducible and 13-dimensional.*

Proof. Consider the locally closed subset V_g (with reduced subscheme structure) of $\mathcal{Hilb}_6(\mathbb{P}^2)$ whose closed points correspond to subschemes with Hilbert function g . This locus is irreducible by Proposition 3.11. We claim that V_g has dimension 9. Let $\pi_g: \text{Hilb}_S^g \rightarrow \mathcal{Hilb}_6(\mathbb{P}^2)$ be the natural map. Let $E_g \subseteq \text{Hilb}_S^g$ be the open subset of points corresponding to saturated ideals. It is smooth by Proposition 3.10. Therefore, $\pi_g: E_g \rightarrow \mathcal{Hilb}_6(\mathbb{P}^2)$ factors through V_g . It follows from Lemma 2.29 that V_g is homeomorphic to E_g . Consider the point $[I] \in E_g$ with $I = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^5)$. We have $\dim_{\mathbb{k}} \text{Hom}_S(I, S/I)_0 = 9$. Therefore, by Theorem 2.75 we have $\dim_{\mathbb{k}} \mathbf{T}_{[I]} \text{Hilb}_S^g = 9$. Thus, we get $\dim V_g = \dim E_g = 9$ by Proposition 3.10.

By definition, U_g is the inverse image of V_g under $\pi' \circ \pi$. By [81, 11.4.C], it is enough to show that the fiber over each point is irreducible and 4-dimensional. This follows from Lemma 2.43. \square

Let $U_i = \{[I] \in U \mid \dim_{\mathbb{k}}(I_{\geq 4} : \mathfrak{m})_3 = i\}$. We claim that there is an equality $U = U_4 \cup U_5$. Indeed, let $[I] \in U$ and denote $(I_{\geq 4} : \mathfrak{m})$ by K . We have $I_3 \subseteq K_3 \subsetneq \bar{I}_3$. The latter inclusion is proper since otherwise we obtain $\bar{I}_4 \subseteq I_4$ which contradicts the assumption that $H_{S/I}(4)$ equals 6.

We claim that U_4 and U_5 are irreducible. In order to prove this, we introduce more notation. Let $V \subseteq \mathcal{Hilb}_6(\mathbb{P}^2)$ be the closed subset of points corresponding to the subschemes with Hilbert function $h_{6,1}$. The set U is the set-theoretic inverse image of V under $\pi' \circ \pi$. Let $W \subseteq \text{Hilb}_S^h$ be the set-theoretic inverse image of V under π' .

For an integer $0 \leq i \leq 10$, let $W_i = \{[K] \in W \mid \dim_{\mathbb{k}}(K : \mathfrak{m})_3 = i\}$ and $W_{\geq i} = \bigcup_{j \geq i} W_j$. Observe that $W_{\geq i}$ is closed for every i . Furthermore, $W_5 = W_{\geq 5} \cap \pi(U)$ and $W_4 \cup W_5 = W_{\geq 4} \cap \pi(U)$ are closed in Hilb_S^h .

We show that W_4 and W_5 are irreducible. Moreover, we compute their dimensions. We have natural maps $W_5 \rightarrow V$ and $W_4 \cup W_5 \rightarrow V$ and we want to study their fibers over a closed point $[\text{Proj } S/\bar{I}] \in V$.

Recall the notion of the dual ring from Subsection 2.4.1.

Lemma 3.93. *Let $\bar{I} = (\alpha_0, \theta_6(\alpha_1, \alpha_2))$ for some $\theta_6 \in \mathbb{k}[\alpha_1, \alpha_2]_6 \setminus \{0\}$. The fiber of $\pi': \text{Hilb}_S^h \rightarrow \mathcal{Hilb}_6(\mathbb{P}^2)$ over $[\text{Proj } S/\bar{I}]$ is isomorphic with \mathbb{P}^9 . Moreover, the points of the fiber are in the natural correspondence with points $[F]$ in $\mathbb{P}S_3^*$ where $S^* = \mathbb{k}_{dp}[x_0, x_1, x_2]$ is the dual ring of S .*

Proof. By Lemma 2.43 this fiber is isomorphic to the Grassmannian of codimension 1 subspaces of \bar{I}_4 . Dually, a point of the fiber corresponds to a choice of an element $[G] \in \mathbb{P}(S_4^*/(\bar{I}_4)^\perp)$ where $(\bar{I}_4)^\perp$ is the set of all elements of S_4^* which are annihilated by $\bar{I}_4 = (\alpha_0)_4$. Therefore, we may take $[G] = [x_0F]$ for some $F \in S_3^*$. \square

Given $F \in S_3^*$ we denote by $\text{Cat}_F(1, 2; 3)$ the catalecticant matrix (see [56, §1.1]). This is a 3×6 -matrix of coefficients of $\alpha_i \lrcorner F$ (for $i = 0, 1, 2$) in the basis of S_2^* given by divided power monomials.

In the notation of Lemma 3.93, the condition that a point of the fiber corresponding to $[F] \in \mathbb{P}S_3^*$ is in W_s (for some $s \in \{4, 5\}$) is equivalent to the condition that we have

$$\dim_{\mathbb{k}} \text{lin}\{\alpha_0 \lrcorner x_0 F, \alpha_1 \lrcorner x_0 F, \alpha_2 \lrcorner x_0 F, x_1^{[3]}, x_1^{[2]} x_2, x_1 x_2^{[2]}, x_2^{[3]}\} = 10 - s.$$

Moreover, $\alpha_i \lrcorner (x_0 F) = x_0(\alpha_i \lrcorner F)$ holds for all $i \in \{0, 1, 2\}$. Therefore, the point of the fiber corresponding to $[F] \in \mathbb{P}S_3^*$ is in W_s if and only if the catalecticant matrix $\text{Cat}_F(1, 2; 3)$ has rank $6 - s$, that is either 2 or 1.

Lemma 3.94. *The locus W_5 is irreducible and 10-dimensional.*

Proof. The locus V is irreducible and 8-dimensional by Lemma 3.82. Since W_5 is closed in Hilb_S^h , it is enough to show that points of the fiber of $\pi': \text{Hilb}_S^h \rightarrow \mathcal{Hilb}_6(\mathbb{P}^2)$ belonging to W_5 form an irreducible subset of dimension 2 (see [81, 11.4.C]). In fact, we claim that the locus of these points inside $\mathbb{P}S_3^*$ coincides with $\nu_3(\mathbb{P}S_1^*)$. This follows from [72, Cor. 3.5] since this locus is given by the ideal generated by the 2×2 -minors of the generic catalecticant matrix $\text{Cat}(1, 2; 3)$. \square

The case of W_4 is analogous.

Lemma 3.95. *The locus W_4 is irreducible and 13-dimensional.*

Proof. The locus V is irreducible and 8-dimensional by Lemma 3.82. It is enough to show that the closed subset $W_4 \cup W_5 \subseteq \text{Hilb}_S^h$ is irreducible and 13-dimensional. Thus, by [81, 11.4.C] it suffices to show that the fiber over every closed point is irreducible and 5-dimensional. The fiber is given by the ideal generated by the 3×3 -minors of the generic catalecticant matrix $\text{Cat}(1, 2; 3)$. This coincides set-theoretically with the 2-nd secant variety $\sigma_2(\nu_3(\mathbb{P}S_1^*))$ by [56, Thm. 4.5A]. Furthermore, it is irreducible and 5-dimensional by [56, Prop. 1.23]. \square

Remark 3.96. Observe that if we have $\text{char } \mathbb{k} = 0$, then in fact in the proof of Lemma 3.95 the set-theoretical equality of the fiber with $\sigma_2(\nu_d(\mathbb{P}S_1^*))$ can be strengthened to the equality of their defining ideals. See [73].

Now we show that U_4 and U_5 are irreducible and we compute their dimensions.

Lemma 3.97. *The subsets U_4 and U_5 are irreducible. Moreover, we have $\dim U_4 = 13$ and $\dim U_5 = 14$.*

Proof. Let $i \in \{4, 5\}$, $[I] \in U_i$ and $[K] = \pi([I]) \in W_i$. The ideals I and K differ only in degree 3. Furthermore, we have $\dim_{\mathbb{k}} I_3 = 4$ and $I_3 \subseteq (K : \mathfrak{m})_3$.

Therefore, the natural map $U_4 \rightarrow W_4$ is bijective on closed points. Thus, U_4 is irreducible and 13-dimensional by Lemmas 2.29 and 3.95.

Let $[K] \in W_5$. The fiber of the map $U_5 \rightarrow W_5$ over $[K]$ is irreducible and 4-dimensional. Indeed, it corresponds to the choice of a 4-dimensional subspace of the 5-dimensional linear space $(K : \mathfrak{m})_3$. Thus, W_5 is irreducible and of dimension 14 by Lemma 3.94 and [81, 11.4.C]. \square

Let $U'_i = \{[I] \in U_i \mid (\bar{I}^2)_4 \subseteq I_4\}$ for all $i \in \{4, 5\}$.

Lemma 3.98. *Let $[I] \in U_i$ for $i = 4$ or $i = 5$. There is a point $[J] \in \text{Slip}_{6,2}$ with $I_{\geq 4} = J_{\geq 4}$ if and only if $[I]$ is in U'_i . In particular, we have $U_4 \cap \text{Slip}_{6,2} = U'_4$, set-theoretically.*

Proof. The first part of the lemma follows from Theorem 3.66. If $[I] \in U'_4$ and $[J] \in \text{Hilb}_S^{h_{6,2}}$ satisfy $I_{\geq 4} = J_{\geq 4}$ then we claim that we have $I = J$. Since $I_{\leq 2} = J_{\leq 2}$ is zero it is enough to show that the equality $I_3 = J_3$ holds. However, by the definition of U_4 we get

$$I_3 = (I_{\geq 4} : \mathfrak{m})_3 = (J_{\geq 4} : \mathfrak{m})_3 = J_3.$$

Thus, U'_4 is contained in $\text{Slip}_{6,2}$, set-theoretically. \square

Let $U''_5 = \{[I] \in U'_5 \mid (\bar{I} \cdot \overline{(I_{\leq 4})})_3 \subseteq I_3\}$. We claim that we have $U'_5 \cap \text{Slip}_{6,2} = U''_5$ set-theoretically. We start with describing $\overline{(I_{\leq 4})}$ for all $[I] \in U'_5$.

Lemma 3.99. *Let $[I] \in U'_5$. Up to a linear change of variables, we have $\overline{(I_{\leq 4})} = (\alpha_0^2, \alpha_0\alpha_1)$.*

Proof. Up to a linear change of variables, we may assume that \bar{I} is equal to $(\alpha_0, \theta_6(\alpha_1, \alpha_2))$ for some non-zero $\theta \in \mathbb{k}[\alpha_1, \alpha_2]_6$. Let $[F] \in \mathbb{P}S_3^*$ be the point corresponding to the point $[I_{\geq 4}]$ in the fiber of $\pi': \text{Hilb}_S^h \rightarrow \text{Hilb}_6(\mathbb{P}^2)$ over $[\text{Proj } S/\bar{I}]$ (see Lemma 3.93).

Since $[I]$ belongs to U'_5 we have $\dim_{\mathbb{k}}(I_{\geq 4}: \mathfrak{m})_3 = 5$ and $[F] \in \mathbb{P}(\mathbb{k}_{dp}[x_1, x_2]_3)$. We get

$$\dim_{\mathbb{k}} \text{lin}\{\alpha_1 \lrcorner F, \alpha_2 \lrcorner F\} = 1.$$

Therefore, we may assume by a linear change of variables in $\mathbb{k}[\alpha_1, \alpha_2]$ that F equals x_2^3 .

We have

$$I_4 = \text{Ann}(\text{lin}\{x_0x_2^3, x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^4\})_4.$$

It follows that $(\alpha_0^2, \alpha_0\alpha_1)$ is contained in $\overline{(I_{\leq 4})}$. On the other hand, $\overline{(I_{\leq 4})}$ is a subset of (α_0) . Thus, if we have $(\alpha_0^2, \alpha_0\alpha_1) \subsetneq \overline{(I_{\leq 4})}$ then $\alpha_0\alpha_2^N$ is in $\overline{(I_{\leq 4})}$ for some N . This is impossible since $\alpha_0\alpha_2^N \notin (I_{\leq 4})$ holds for every positive integer N . \square

Let $W'_5 = \{[I] \in W_5 \mid (\bar{I}^2)_4 \subseteq I_4\}$.

Lemma 3.100. *The subset W'_5 is irreducible and 9-dimensional.*

Proof. Consider the natural morphism $W'_5 \rightarrow V$. The target is irreducible and 8-dimensional by Lemma 3.82. By [81, 11.4.C], it is enough to show that the fibers are irreducible and 1-dimensional. Let $T^* = \mathbb{k}_{dp}[x_1, x_2] \subseteq S^* = \mathbb{k}_{dp}[x_0, x_1, x_2]$. In the notation of Lemma 3.93 we now have $[F] \in \mathbb{P}T_3^* \subseteq \mathbb{P}S_3^*$ since we have to choose a codimension one subspace of $(\alpha_0)_4$ containing $(\alpha_0^2)_4$. Therefore, the fiber inside $\mathbb{P}T_3^*$ is set-theoretically given by the vanishing of 2×2 -minors of the generic catalecticant matrix $\text{Cat}(1, 2; 2)$. Thus, it is $\nu_3(\mathbb{P}T_1^*)$ by [72, Cor. 3.5]. \square

Now we show that U''_5 is irreducible and we compute its dimension.

Lemma 3.101. *The subset U''_5 is irreducible and 11-dimensional.*

Proof. Consider the natural map $U''_5 \rightarrow W'_5$. Let $[K] \in W'_5$ be a closed point with $\overline{(K_{\leq 4})} = (\alpha_0^2, \alpha_0\alpha_1)$ (see Lemma 3.99). The fiber over $[K]$ is isomorphic to \mathbb{P}^2 corresponding to a choice of a two-dimensional subspace of $\text{lin}\{\alpha_0\alpha_1^2, \alpha_0\alpha_2^2, \alpha_0\alpha_1\alpha_2\}$. Thus, U''_5 is irreducible and 11-dimensional by Lemma 3.100 and [81, 11.4.C] \square

The key technical step is the following lemma.

Lemma 3.102. *The locus $U'_5 \setminus U''_5$ is disjoint from $\text{Slip}_{6,2}$.*

We defer the proof of this lemma until the end of the subsection. We use it to describe the set-theoretic intersection $U_5 \cap \text{Slip}_{6,2}$.

Lemma 3.103. *We have $U_5 \cap \text{Slip}_{6,2} = U''_5$ set-theoretically.*

Proof. Consider the point $[I] \in \text{Hilb}_S^{h_{6,2}}$ with $I = (\alpha_0^3, \alpha_0^2\alpha_1, \alpha_0\alpha_1^2 + \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \alpha_2^6)$. We have $[I] \in U''_5$. We claim that it also belongs to $\text{Slip}_{6,2}$. Let $J = (\alpha_0\alpha_1\alpha_2 + \alpha_2^3, \alpha_0\alpha_1^2 + \alpha_0^2\alpha_2 + \alpha_1\alpha_2^2, \alpha_0^2\alpha_1 + \alpha_0\alpha_2^2, \alpha_0^3)$. It is a saturated ideal and $[J]$ is in $\text{Hilb}_S^{h_{6,2}}$. It follows that $[J]$ is in $\text{Slip}_{6,2}$ by Lemma 3.90. The initial ideal of J with respect to the weight vector $(3, 2, 1)$ is I .

Now we use Lemma 2.30 to conclude that $U_5 \cap \text{Slip}_{6,2} = U''_5$ holds set-theoretically. Indeed, $U_5 \cap \text{Slip}_{6,2}$ is contained in U''_5 by Lemmas 3.98 and 3.102. Moreover, we have $[I] \in U_5 \cap \text{Slip}_{6,2}$

and $\dim_{\mathbb{k}} \text{Hom}_S(I, S/I)_0 = 15 = \dim U_5 + 1$. It follows from Theorem 2.75 and Lemma 2.30 that the intersection contains an irreducible subset of dimension 11. Therefore, the intersection is U_5'' by Lemma 3.101. \square

The following proposition summarizes the above considerations.

Proposition 3.104. *Let $[I]$ be a point of $\text{Hilb}_S^{h_{6,2}}$. It belongs to $\text{Slip}_{6,2}$ if and only if one of the following holds:*

1. *The ideal I is saturated.*
2. *The algebra S/\bar{I} has Hilbert function $f = (1, 3, 5, 6, 6, \dots)$.*
3. *The algebra S/\bar{I} has Hilbert function $g = (1, 3, 4, 5, 6, 6, \dots)$ and we have $(\theta \cdot \bar{I})_3 \subseteq I_3$ where θ is the common linear divisor of two quadratic generators of \bar{I} .*
4. *The algebra S/\bar{I} has Hilbert function $h_{6,1} = (1, 2, 3, 4, 5, 6, 6, \dots)$ and there are inclusions*

$$((\overline{I_{\leq d+1}}) \cdot \bar{I})_d \subseteq I_d$$

for $d = 3$ and $d = 4$.

Proof. The cases 1–3 follow from Lemmas 3.90 and 3.91.

Observe that if $[I]$ is in U then we have $I_5 = (\bar{I}_1)_5 = \bar{I}_5$ and therefore we get

$$(\bar{I}^2)_4 = ((\overline{I_{\leq 5}}) \cdot \bar{I})_4.$$

Hence in case 4, if $[I]$ belongs to U_5 the claim follows from Lemma 3.103.

Assume that $[I]$ is in U_4 . By Lemma 3.98 the point $[I]$ belongs to $\text{Slip}_{6,2}$ if and only if we have

$$(\bar{I}^2)_4 = ((\overline{I_{\leq 5}}) \cdot \bar{I})_4 \subseteq I_4. \quad (3.105)$$

We need to show that if $[I]$ is in U_4' , then $((\overline{I_{\leq 4}}) \cdot \bar{I})_3$ is a subset of I_3 . Let $f \in ((\overline{I_{\leq 4}}) \cdot \bar{I})_3$ and $i \in \{0, 1, 2\}$. We have $\alpha_i f \in (\bar{I}^2)_4$. From Equation (3.105) we get $\alpha_i f \in I_4$. It follows that we have $f \in (I_{\geq 4} : \mathfrak{m})_3 = I_3$ where the last equality follows from the definition of U_4 . \square

We are left with proving Lemma 3.102. Assume that $[I]$ is a point of $U_5' \setminus U_5''$ with $\bar{I} = (\alpha_0, \theta_6(\alpha_1, \alpha_2))$. By Lemma 3.99 we may assume that we have $(\overline{I_{\leq 4}}) = (\alpha_0^2, \alpha_0 \alpha_1)$. It follows that there is an equality

$$I = (W) + (\alpha_0^2, \alpha_0 \alpha_1)_{\geq 4} + (\alpha_0)_{\geq 5} + (\theta_6) \quad (3.106)$$

where $[W] \in \text{Gr}(4, \text{lin}\{\alpha_0^3, \alpha_0^2 \alpha_1, \alpha_0 \alpha_1^2, \alpha_0^2 \alpha_2, \alpha_0 \alpha_1 \alpha_2\})$ is such that $\text{lin}\{\alpha_0^3, \alpha_0^2 \alpha_1\}$ is not contained in W .

Assume that $[I] \in \text{Slip}_{6,2}$ is as in Equation (3.106) with $\alpha_0^3 \notin W$. By taking the initial ideal with respect to lex order with $\alpha_2 > \alpha_1 > \alpha_0$ we obtain an ideal of the form

$$I' = (\alpha_0^2 \alpha_1, \alpha_0 \alpha_1^2, \alpha_0^2 \alpha_2, \alpha_0 \alpha_1 \alpha_2, \alpha_0^4, \alpha_0 \alpha_2^4, \theta_6'(\alpha_1, \alpha_2)) \quad (3.107)$$

such that $[I']$ belongs to $\text{Slip}_{6,2}$.

On the other hand, if $[I] \in \text{Slip}_{6,2}$ is as in Equation (3.106) with $\alpha_0^3 \in W$ but $\alpha_0^2 \alpha_1 \notin W$, then by taking the initial ideal we get a point $[I']$ in $\text{Slip}_{6,2}$ of the form

$$I' = (\alpha_0^3, \alpha_0 \alpha_1^2, \alpha_0^2 \alpha_2, \alpha_0 \alpha_1 \alpha_2, \alpha_0 \alpha_2^4, \theta'_6(\alpha_1, \alpha_2)). \quad (3.108)$$

We claim that if I' is of the form as in Equation (3.107) or (3.108) then $[I']$ is not in $\text{Slip}_{6,2}$.

Lemma 3.109. *There is no point $[I'] \in \text{Slip}_{6,2}$ with I' as in Equation (3.108).*

Proof. Let $J = I' + \mathfrak{m}^5$. Note that it does not depend on θ'_6 so it is the same for every I' as in Equation (3.108). We have $\dim_{\mathbb{k}} \text{Hom}_S(J, S/J)_0 = 8$. Thus, $[I']$ does not belong to $\text{Slip}_{6,2}$ by Proposition 3.1. \square

We show that there is no point $[I'] \in \text{Slip}_{6,2}$ with I' as in Equation (3.107). First, we introduce some more multigraded Hilbert schemes. Let $\overline{h}_{6,2}: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by

$$\overline{h}_{6,2}(a) = \begin{cases} h_{6,2}(a) & \text{for } a \leq 4; \\ 0 & \text{for } a > 4 \end{cases}$$

or, more briefly, by $\overline{h}_{6,2} = (1, 3, 6, 6, 6, 0, 0, \dots)$. Define $\overline{f}: \mathbb{Z} \rightarrow \mathbb{Z}$ by $\overline{f} = (1, 3, 5, 6, 6, 0, 0, \dots)$ or, more formally, by

$$\overline{f}(a) = \begin{cases} f(a) & \text{for } a \leq 4; \\ 0 & \text{for } a > 4. \end{cases}$$

Finally, let $k: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by

$$k(a) = \begin{cases} \dim_{\mathbb{k}} S_a & \text{for } a \leq 1; \\ a + 3 & \text{for } a \in \{2, 3, 4\}; \\ 0 & \text{for } a > 4 \end{cases}$$

or, in a brief form by $k = (1, 3, 5, 6, 7, 0, 0, \dots)$.

We have natural maps $\text{Hilb}_S^{k, \overline{f}} \rightarrow \text{Hilb}_S^{\overline{f}} \rightarrow \text{Hilb}_S^{\overline{h}_{6,2}}$. Let Z be the set-theoretic image of $\text{Hilb}_S^{k, \overline{f}}$ in $\text{Hilb}_S^{\overline{h}_{6,2}}$. We claim that it is irreducible and 12-dimensional.

Lemma 3.110. *The set Z is irreducible closed and 12-dimensional.*

Proof. Observe that both morphisms $\text{Hilb}_S^{k, \overline{f}} \rightarrow \text{Hilb}_S^{\overline{f}}$ and $\text{Hilb}_S^{\overline{f}} \rightarrow \text{Hilb}_S^{\overline{h}_{6,2}}$ are closed by Theorem 2.36. We claim that they are both injective on closed points. We start with $\text{Hilb}_S^{\overline{f}} \rightarrow \text{Hilb}_S^{\overline{h}_{6,2}}$. Suppose that there are two points $[I] \neq [I'] \in \text{Hilb}_S^{\overline{f}}$ with $I \cap \mathfrak{m}^3 = I' \cap \mathfrak{m}^3$. Let $J = (I_2 \oplus I'_2) \oplus I_{\geq 3}$. The algebra S/J has Hilbert function $(1, 3, 4, 6, 6, 0, 0, \dots)$. This contradicts the Macaulay's bound [10, Thm. 4.2.10].

Now we show that $\text{Hilb}_S^{k, \overline{f}} \rightarrow \text{Hilb}_S^{\overline{f}}$ is injective on closed points. Let $[I' \subseteq I''] \in \text{Hilb}_S^{k, \overline{f}}$. We have $(I'_{\leq 3}) = (\theta_2, \theta_3)$ for some generators $\theta_i \in S_i$. Furthermore, I' has no minimal generator of degree 4 since there is no homogeneous ideal J of S such that S/J has Hilbert function $(1, 3, 5, 6, 8, \dots)$ by Macaulay's bound [10, Thm. 4.2.10]. Therefore, we have $I' = (I''_{\leq 3}) + \mathfrak{m}^5$. As a result, $\text{Hilb}_S^{k, \overline{f}} \rightarrow \text{Hilb}_S^{\overline{f}}$ is injective on closed points.

It follows from the above considerations that it suffices to show that the flag multigraded Hilbert scheme $\text{Hilb}_S^{k, \overline{f}}$ is irreducible and 12-dimensional.

Let $\pi_k: \text{Hilb}_S^{k, \bar{f}} \rightarrow \text{Hilb}_S^k$ be the natural projection. The fiber over a closed point $[K] \in \text{Hilb}_S^k$ is irreducible and 6-dimensional corresponding to the choice of a 9-dimensional subspace of S_4 containing the 8-dimensional subspace K_4 . By [81, 11.4.C] it is enough to show that Hilb_S^k is irreducible and 6-dimensional.

Let X be the pullback

$$\begin{array}{ccc} X & \longrightarrow & \text{Fl}(3, 4, S_2) \\ \downarrow & & \downarrow \\ \text{Gr}(1, S_1) & \longrightarrow & \text{Gr}(3, S_2) \end{array}$$

where the lower horizontal map takes $[\ell]$ to $[\text{lin}\{\alpha_0\ell, \alpha_1\ell, \alpha_2\ell\}]$. It is irreducible and of dimension 4. Moreover, if $[I]$ belongs to Hilb_S^k then the generators of I of degree 2 and 3 have a common linear factor since we have $H_{S/I}(4) = 7 > 6$. Therefore, there is the parametrization $X \times \text{Gr}(1, S_1) \rightarrow \text{Hilb}_S^k$ given by

$$([\ell], [\ell S_1 \subseteq V], [\ell']) \mapsto (\ell\ell') + (V\ell') + \mathfrak{m}^5.$$

Thus, Hilb_S^k is irreducible and of dimension 6. \square

Let $p: \text{Hilb}_S^{h_{6,2}} \rightarrow \text{Hilb}_S^{\overline{h_{6,2}}}$ be the natural map given by $[I] \mapsto [I + \mathfrak{m}^5]$. Let

$$K = (\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0^4) + \mathfrak{m}^5.$$

For every ideal I' as in Equation (3.107), we have $p([I']) = [K]$. Thus, it is enough to show that there is no point $[J] \in \text{Slip}_{6,2}$ with $p([J]) = [K]$.

Lemma 3.111. *If $[K]$ is a point of $p(\text{Slip}_{6,2})$, then there is an irreducible 10-dimensional subset Z' of the set-theoretic intersection $p(\text{Slip}_{6,2}) \cap Z$. Furthermore, $[K]$ belongs to Z' .*

Proof. Let $K' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2) + \mathfrak{m}^5$ and $K'' = K' + (\alpha_0^4 + \alpha_1^4)$. We have $[K' \subseteq K''] \in \text{Hilb}_S^{k, \bar{f}}$. Therefore, $[K'' \cap \mathfrak{m}^3 + \mathfrak{m}^5]$ is in Z . However, the initial ideal of $K'' \cap \mathfrak{m}^3 + \mathfrak{m}^5$ with respect to the lex order with $\alpha_0 > \alpha_1 > \alpha_2$ is K . Thus, $[K]$ belongs to $p(\text{Slip}_{6,2}) \cap Z$. Moreover, we have $\dim_{\mathbb{k}} \mathbf{T}_{[K]} \text{Hilb}_S^{\overline{h_{6,2}}} = \dim_{\mathbb{k}} \text{Hom}_S(K, S/K)_0 = 14$, where the first equality follows from Theorem 2.75. Therefore, the claim follows from Lemmas 2.30 and 3.110 and equalities $\dim p(\text{Slip}_{6,2}) = \dim \text{Slip}_{6,2} = 12$. \square

Suppose that we have $[K] \in p(\text{Slip}_{6,2})$. There exists an irreducible closed subset $Z'' \subseteq \text{Slip}_{6,2}$ with $p(Z'') = Z'$, where Z' is as in Lemma 3.111. Since $p(Z'')$ is contained in Z , it follows that Z'' is disjoint from the set of saturated ideals. Moreover, $p(U_g), p(U_4)$ and $p(U_5)$ are of dimension less than 10 (see Lemmas 3.92 and 3.97). It follows that Z'' is contained in the closure of the locus U_f where

$$U_f = \{[I] \in \text{Hilb}_S^{h_{6,2}} \mid S/\bar{I} \text{ has Hilbert function } f = (1, 3, 5, 6, 6, \dots)\}.$$

Let $q: \text{Hilb}_S^f \rightarrow \text{Hilb}_S^{h_{6,2}}$ be the natural map. It is a closed map by Theorem 2.36 and it is injective on closed points corresponding to saturated ideals. Thus, Z'' is contained in the image of the closure of the locus of all points of Hilb_S^f corresponding to saturated ideals.

It follows that there is an ideal $[I''] \in \text{Hilb}_S^f$ that satisfies $I''_{\leq 4} = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4)_{\leq 4}$ and I'' is a limit of saturated ideals.

By taking an initial ideal, we get that at least one of the following ideals corresponding to points of Hilb_S^f is a limit of saturated ideals:

1. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_1^5)$;
2. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_1^4\alpha_2)$;
3. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_1^3\alpha_2^2)$;
4. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_1^2\alpha_2^3)$;
5. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_1\alpha_2^4)$;
6. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1^6)$;
7. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1^5\alpha_2)$;
8. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1^4\alpha_2^2)$;
9. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1^3\alpha_2^3)$;
10. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1^2\alpha_2^4)$;
11. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1\alpha_2^5)$;
12. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_2^6)$.

We claim that this is impossible. Cases 1–5 can be excluded since then $[I''' \cap \mathfrak{m}^3]$ is in $\text{Slip}_{6,2} \cap U_g$ but $I''' \cap \mathfrak{m}^3$ is not of the form from Lemma 3.91. Furthermore, if I''' is one of the ideals 6–12, then we have $H_{S/(I''')^2}(8) = 17 < 18$. Therefore, by Theorem 3.5, I''' is not in the closure of the locus of all radical ideals. Thus, it is also not in the closure of the locus of all saturated ideals since a general saturated ideal of S such that the quotient algebra has Hilbert function f is radical.

To summarize, we have arrived at a contradiction after assuming that $[K]$ belongs to $p(\text{Slip}_{6,2})$. This shows in particular, that if I' is as in Equation (3.107) then $[I']$ is not in $\text{Slip}_{6,2}$. Together with Lemma 3.109, this finishes the proof of Lemma 3.102 and thus, of Proposition 3.104.

Chapter 4

Criteria for smooth projective toric varieties

In this chapter we work in the category of schemes over the complex numbers. We consider smooth projective toric varieties and the corresponding multigraded Hilbert schemes. The main motivation is to study the case of the product of projective spaces. However, secant varieties of more general toric varieties have also been studied [25], [38].

In Section 4.1 we recall the basic notions of the theory of toric varieties. We mainly follow [28]. In Section 4.2 we consider a morphism with connected fibers $f: X \rightarrow Y$ between smooth projective toric varieties. We present a necessary condition for an ideal I in the Cox ring of X to be in the irreducible component $\text{Slip}_{r,X}$. In Sections 4.3 and 4.4 we present two particular cases of that criterion. In Section 4.3 we assume that X is the blowing up of Y at the closure of a torus orbit. In Section 4.4 we assume that X is a projective toric bundle over Y . In particular, the criterion from this section is applicable to the case of the product of projective spaces. In Section 4.5 we obtain another necessary condition in the case that X is the product of projective spaces. Finally, in Section 4.6 we present two examples of reducible multigraded Hilbert schemes corresponding to two points on a toric surface.

The main technical tool used in this chapter is the possibility to lift a morphism between smooth projective toric varieties to a homomorphism of their Cox rings (see Subsection 4.1.3). This and similar problems have been extensively studied. In particular, in [9] there are general results that could shorten our presentation. This is true, for example for Lemmas 4.4, 4.10, 4.11 and Proposition 4.21. However, since in the generality that we require, most of those results can be presented from scratch, we decided to do so.

4.1 Toric varieties

In Subsections 4.1.1–4.1.3 we recall some basic definitions and results related to toric varieties. This is mainly to fix the notation. Therefore, we omit most of the proofs, referring the reader to [28]. Our notation follows closely the one used there. In Subsection 4.1.3 we recall the main technical tool—lifting a morphism between smooth projective toric varieties to a morphism of their Cox rings.

Subsection 4.1.4 is concerned with morphisms $f: X \rightarrow Y$ between smooth projective toric varieties with $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. This property is assumed in Theorem 4.15 which is one of the main results of this chapter.

In Subsection 4.1.5 we finally give the definition of a multigraded Hilbert scheme of points in general position for a smooth projective toric variety. This is the generality in which it was introduced in [15].

Although the results from this section are stated for smooth projective toric varieties, some of them are still true for more general toric varieties.

4.1.1 Fans and toric varieties

By a *toric variety* we mean a normal variety X over the field of complex numbers such that X contains an algebraic torus $\mathbb{T} \cong (\mathbb{C}^*)^n$ as an open subset and the action of \mathbb{T} on itself extends to an action of \mathbb{T} on X .

Given an algebraic torus $\mathbb{T} \cong (\mathbb{C}^*)^n$, we denote by M the character lattice of \mathbb{T} , i.e. $M = \text{Hom}(\mathbb{T}, \mathbb{C}^*)$ is the set of all algebraic group homomorphisms from \mathbb{T} to the one-dimensional torus. We have $M \cong \mathbb{Z}^n$ and the dual lattice $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ is denoted by N . For every $m \in M$ the corresponding character is $\chi^m: \mathbb{T} \rightarrow \mathbb{C}^*$. We denote by $\langle \cdot, \cdot \rangle$ the natural pairing $M \times N \rightarrow \mathbb{Z}$ and its extension to the \mathbb{R} -vector spaces $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

Toric varieties are obtained by gluing affine toric varieties corresponding to certain combinatorial objects in the \mathbb{R} -vector space $N_{\mathbb{R}}$. We explain in more details how to obtain affine toric varieties. A subset $\sigma \subseteq N_{\mathbb{R}}$ is called a (*rational polyhedral*) *cone* if there is a finite set of elements $\mathbf{u}_1, \dots, \mathbf{u}_k \in N$ with $\sigma = \{\sum_{i=1}^k \lambda_i \mathbf{u}_i \mid \lambda_i \geq 0\}$. Since we do not consider more general cones, we omit the phrase "rational polyhedral". A cone $\sigma \subseteq N_{\mathbb{R}}$ is *strongly convex* if σ does not contain a positive-dimensional vector subspace of $N_{\mathbb{R}}$. Given a cone $\sigma \subseteq N_{\mathbb{R}}$, we can consider the semigroup $S_{\sigma} = \sigma^{\vee} \cap M$ where σ^{\vee} is the dual cone of σ , i.e. $\sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid \langle m, \mathbf{u} \rangle \geq 0 \text{ for all } \mathbf{u} \in \sigma\}$. If σ is strongly convex, then the spectrum of the semigroup algebra $\mathbb{C}[S_{\sigma}]$ is an n -dimensional affine toric variety and is denoted by U_{σ} . As shown in [28, Thm. 1.3.5], all affine toric varieties are of this form.

Given a cone $\sigma \in N_{\mathbb{R}}$, its *face* is the intersection of σ with an affine hyperplane

$$H_m = \{\mathbf{u} \in N_{\mathbb{R}} \mid \langle m, \mathbf{u} \rangle = 0\}$$

in $N_{\mathbb{R}}$ for some $m \in M_{\mathbb{R}}$ satisfying $\langle m, \mathbf{u} \rangle \geq 0$ for every $\mathbf{u} \in \sigma$. A *fan* is a finite collection Σ of strongly convex cones in $N_{\mathbb{R}}$ satisfying conditions:

1. Each face of each cone in Σ is an element of Σ .
2. Every two cones in Σ intersect along a common face.

Given such combinatorial data there is a toric variety X_{Σ} obtained by gluing the affine toric varieties $\{U_{\sigma} \mid \sigma \in \Sigma\}$ (see [28, Thm. 3.1.5]). Moreover, every toric variety with torus \mathbb{T} comes from the above construction for a fan in $N_{\mathbb{R}}$ where N is the lattice dual to the character lattice of the torus (see [28, Cor. 3.1.8]). For a toric variety X we denote by Σ_X a fan such that we have $X_{\Sigma_X} = X$. Note that Σ_X is not uniquely determined by X since we can apply any \mathbb{Z} -linear automorphism of $N_{\mathbb{R}}$ to a fan Σ and obtain the same abstract toric variety. In Sections 4.3 and 4.4 we consider a morphism of toric varieties $X \rightarrow Y$. Starting from an arbitrary choice of a fan Σ_Y corresponding to Y we describe a fan Σ_X corresponding to X that is convenient to use.

4.1.2 Picard groups and Cox rings

Given a smooth projective toric variety X and a corresponding fan Σ_X in $N_{\mathbb{R}}$ the Picard group $\text{Pic}(X)$ of X can be calculated using the combinatorial data of the one-dimensional cones in Σ_X (see [28, Thm. 4.1.3]). Let $\Sigma_X(1)$ be the set of all one-dimensional cones in Σ_X , i.e. cones whose linear spans are one-dimensional real vector subspaces of $N_{\mathbb{R}}$. The torus invariant prime divisors on X are in bijective correspondence with elements of $\Sigma_X(1)$. Given $\rho \in \Sigma_X(1)$ we denote by \mathbf{u}_ρ the ray generator of ρ (i.e. the unique generator of the semigroup $\rho \cap N$) and the corresponding divisor by D_ρ .

Let e_1, \dots, e_n a \mathbb{Z} -basis of M . The group $\text{Pic}(X)$ is generated by classes of $[D_\rho]$ for all $\rho \in \Sigma_X(1)$. Moreover, these generators are subject to relations

$$0 = [\text{div}(\chi^{e_i})] = \sum_{\rho \in \Sigma_X(1)} \langle e_i, \mathbf{u}_\rho \rangle [D_\rho]$$

for all $i = 1, \dots, n$.

Let X be a smooth projective toric variety associated with a fan $\Sigma_X \subseteq N_{\mathbb{R}}$. There is a corresponding polynomial ring $S[X]$ graded by the Picard group $\text{Pic}(X)$. This ring is called the *Cox ring* of X . We have

$$S[X] = \mathbb{C}[\alpha_\rho \mid \rho \in \Sigma_X(1)] \text{ and } \deg(\alpha_\rho) = [D_\rho].$$

By [28, Prop. 5.3.7] we have $S[X]_{[D]} \cong \Gamma(X, \mathcal{O}_X(D))$ for all $[D] \in \text{Pic}(X)$.

Remark 4.1. The construction of a Cox ring can be carried out for more general varieties, see [1]. It does not have to be a polynomial ring. Moreover, unlike for toric varieties, the construction requires some choices so we speak of a Cox ring of X instead of the Cox ring of X .

4.1.3 Irrelevant ideals and the quotient construction

One of the main tools in this chapter is Theorem 4.3 which states that a morphism $f: X \rightarrow Y$ between smooth projective toric varieties can be lifted to a graded homomorphism $\bar{f}^\# : S[Y] \rightarrow S[X]$ of their Cox rings.

We start with recalling the quotient construction of a smooth projective toric variety X presented in [28, Thm. 5.1.11]. Given a cone $\sigma \in \Sigma_X$ we denote by $\sigma(1)$ the set of all 1-dimensional faces of σ . Let $S[X] = \mathbb{C}[\alpha_\rho \mid \rho \in \Sigma_X(1)]$ be the Cox ring of X . For all $\sigma \in \Sigma_X(1)$ we define $\alpha^{\hat{\sigma}}$ to be $\prod_{\rho \in \Sigma_X(1) \setminus \sigma(1)} \alpha_\rho$. The *irrelevant ideal* of X is

$$B(\Sigma_X) = (\alpha^{\hat{\sigma}})_{\sigma \in \Sigma_X} \subseteq S[X].$$

Observe that it is enough to take generators corresponding to maximal cones of Σ_X . We denote the affine space $\text{Spec } S[X]$ by \bar{X} and the open subset $\bar{X} \setminus V(B(\Sigma_X))$ by \hat{X} .

By [28, Prop. 4.2.5] we have $\text{Pic}(X) \cong \mathbb{Z}^k$ for some integer k . Therefore, $H_X = \text{Spec } \mathbb{C}[\text{Pic}(X)]$ is a torus. Since $S[X]$ is $\text{Pic}(X)$ -graded, there is a natural action of the torus H_X on \bar{X} . The variety X is the geometric quotient by the induced action of H_X on \hat{X} . We denote the open immersion $\hat{X} \rightarrow \bar{X}$ by i_X and the quotient $\hat{X} \rightarrow X$ by π_X .

Definition 4.2. Suppose that $f: X \rightarrow Y$ is a morphism between smooth projective toric varieties and let $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$ be the pullback map. Suppose that there exists a \mathbb{C} -algebra

homomorphism $\bar{f}^\# : S[Y] \rightarrow S[X]$ such that the following conditions hold:

1. $\bar{f}^\#(S[Y]_{[D]})$ is contained in $S[X]_{f^*([D])}$ for every $[D] \in \text{Pic}(Y)$;
2. the corresponding morphism $\bar{f} : \bar{X} \rightarrow \bar{Y}$ restricts to a morphism $\hat{f} : \hat{X} \rightarrow \hat{Y}$;
3. we have $\pi_Y \circ \hat{f} = f \circ \pi_X$.

We call $\bar{f}^\#$ a *lift* of f .

If $\bar{f}^\#$ is a lift of a morphism $f : X \rightarrow Y$ between smooth projective toric varieties, we have a commutative diagram

$$\begin{array}{ccc}
 \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\
 i_X \uparrow & & i_Y \uparrow \\
 \hat{X} & \xrightarrow{\hat{f}} & \hat{Y} \\
 \downarrow \pi_X & & \downarrow \pi_Y \\
 X & \xrightarrow{f} & Y.
 \end{array}$$

Observe that once conditions 1–2 from Definition 4.2 are satisfied, there is a unique morphism $f' : X \rightarrow Y$ with $f' \circ \pi_X = \pi_Y \circ \hat{f}$. Indeed, $\pi_Y \circ \hat{f}$ is constant on H_X -orbits and π_X is a categorical quotient (see [28, Thm 5.1.11]). Condition 3 says, that we have $f' = f$.

The possibility of lifting a morphism $f : X \rightarrow Y$ to a homomorphism $\bar{f}^\# : S[Y] \rightarrow S[X]$ has been studied in various settings. The version suitable for our needs is considered in [26]. The case of rational maps of toric varieties using multi-valued maps of Cox rings is studied in [9]. Analogous quotient construction holds for the so called Mori dream spaces. These are varieties admitting a Cox ring that is a finitely generated \mathbb{C} -algebra. Lifting of rational maps of Mori dream spaces is discussed in [18] and the case of a regular map can be found in [52].

Now we can state the key existence theorem.

Theorem 4.3 ([26, Thm. 3.2]). *Let $f : X \rightarrow Y$ be a morphism between smooth projective toric varieties. There exists a lift $\bar{f}^\#$ of f .*

Let X and Y be smooth projective toric varieties corresponding to fans $\Sigma_X \subseteq (N_X)_{\mathbb{R}}$ and $\Sigma_Y \subseteq (N_Y)_{\mathbb{R}}$, respectively. Assume that $f : X \rightarrow Y$ is a *toric* morphism, i.e. it maps the torus \mathbb{T}_X of X into the torus \mathbb{T}_Y of Y and the restricted map $\mathbb{T}_X \rightarrow \mathbb{T}_Y$ is a group homomorphism. Such morphisms correspond to \mathbb{Z} -linear maps $\phi : N_X \rightarrow N_Y$ such that for every cone $\sigma \in \Sigma_X$, there is a cone $\sigma' \in \Sigma_Y$ satisfying $\phi_{\mathbb{R}}(\sigma) \subseteq \sigma'$ (see [28, Thm. 3.3.4]). Here $\phi_{\mathbb{R}}$ denotes $\phi \otimes_{\mathbb{Z}} \text{id}_{\mathbb{R}}$. We say that ϕ is *compatible with the fans Σ_X and Σ_Y* .

In the following lemma we study the condition under which a homomorphism of graded rings $\bar{f}^\# : S[Y] \rightarrow S[X]$ is a lift of the given toric morphism $f : X \rightarrow Y$.

Lemma 4.4. *Let $f : X \rightarrow Y$ be a toric morphism between smooth projective toric varieties. Let $S[X] = \mathbb{C}[\alpha_\rho \mid \rho \in \Sigma_X(1)]$ and $S[Y] = \mathbb{C}[\beta_\rho \mid \rho \in \Sigma_Y(1)]$ be the Cox rings of X and Y , respectively. Let $\phi : N_X \rightarrow N_Y$ be the map corresponding to f .*

Assume that we are given a homomorphism of rings $\bar{f}^\# : S[Y] \rightarrow S[X]$ satisfying conditions 1 and 2 from Definition 4.2. The homomorphism $\bar{f}^\#$ is a lift of f , if and only if we have

$$\prod_{\rho \in \Sigma_Y(1)} (\bar{f}^\#(\beta_\rho))^{\langle m, \mathbf{u}_\rho \rangle} = \prod_{\rho \in \Sigma_X(1)} \alpha_\rho^{\langle \psi(m), \mathbf{u}_\rho \rangle} \quad (4.5)$$

for every $m \in M_Y$, where $\psi: M_Y \rightarrow M_X$ is the dual map of $\phi: N_X \rightarrow N_Y$.

Proof. Let $f': X \rightarrow Y$ be the morphism induced by \bar{f} . It is enough to show that f and f' define the same morphism $U_\sigma \rightarrow U_{\sigma'}$ of affine toric varieties for every pair of cones $\sigma \in \Sigma_X$ and $\sigma' \in \Sigma_Y$ satisfying $\phi_{\mathbb{R}}(\sigma) \subseteq \sigma'$.

Recall that we have $\beta^{\hat{\sigma}'} = \prod_{\rho \in \Sigma_Y(1) \setminus \sigma'(1)} \beta_\rho$ and $\alpha^{\hat{\sigma}} = \prod_{\rho \in \Sigma_X(1) \setminus \sigma(1)} \alpha_\rho$. There is an isomorphism $\mathbb{C}[(\sigma')^\vee \cap M_Y] \cong (S[Y]_{\beta^{\hat{\sigma}'}})_0$ given by $\chi^m \mapsto \prod_{\rho \in \Sigma_Y(1)} \beta_\rho^{\langle m, \mathbf{u}_\rho \rangle}$ (see the proof of [28, 5.1.11]) and a similar one $\mathbb{C}[\sigma^\vee \cap M_X] \cong (S[X]_{\alpha^{\hat{\sigma}}})_0$.

The map $U_\sigma \rightarrow U_{\sigma'}$ induced by f corresponds to the homomorphism $\mathbb{C}[(\sigma')^\vee \cap M_Y] \rightarrow \mathbb{C}[\sigma^\vee \cap M_X]$ given by $\chi^m \mapsto \chi^{\psi(m)}$. On the other hand, the map $U_\sigma \rightarrow U_{\sigma'}$ induced by f' corresponds to the map $(\bar{f}^{\#}_{\beta^{\hat{\sigma}'}})_0: (S[Y]_{\beta^{\hat{\sigma}'}})_0 \rightarrow (S[X]_{\alpha^{\hat{\sigma}}})_0$. Therefore, f and f' induce the same map $U_\sigma \rightarrow U_{\sigma'}$ if and only if Equation (4.5) holds. Indeed, this is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{C}[(\sigma')^\vee \cap M_Y] & \xrightarrow{\chi^m \mapsto \chi^{\psi(m)}} & \mathbb{C}[\sigma^\vee \cap M_X] \\ \chi^m \mapsto \prod_{\rho \in \Sigma_Y(1)} \beta_\rho^{\langle m, \mathbf{u}_\rho \rangle} \downarrow & & \downarrow \chi^m \mapsto \prod_{\rho \in \Sigma_X(1)} \alpha_\rho^{\langle m, \mathbf{u}_\rho \rangle} \\ (S[Y]_{\beta^{\hat{\sigma}'}})_0 & \xrightarrow{(\bar{f}^{\#}_{\beta^{\hat{\sigma}'}})_0} & (S[X]_{\alpha^{\hat{\sigma}}})_0. \end{array}$$

□

The fact that a morphism $\bar{f}: \bar{X} \rightarrow \bar{Y}$ restricts to a morphism $\hat{f}: \hat{X} \rightarrow \hat{Y}$ has the following algebraic consequence.

Lemma 4.6. *Let $f: X \rightarrow Y$ be a morphism between smooth projective toric varieties. Assume that $\bar{f}^{\#}: S[Y] \rightarrow S[X]$ is a homomorphism of \mathbb{C} -algebras satisfying condition 2 from Definition 4.2. If $I \subseteq S[X]$ is a homogeneous ideal which is saturated with respect to $B(\Sigma_X)$, then $(\bar{f}^{\#})^{-1}(I)$ is saturated with respect to $B(\Sigma_Y)$.*

Proof. By assumption that $\bar{f}: \bar{X} \rightarrow \bar{Y}$ restricts to a morphism $\hat{f}: \hat{X} \rightarrow \hat{Y}$ we conclude that there is an inclusion

$$\bar{f}^{-1}\left(V(B(\Sigma_Y))\right) \subseteq V(B(\Sigma_X)). \quad (4.7)$$

By [6, Prop. 3 §6.2] we have $\bar{f}^{-1}\left(V(B(\Sigma_Y))\right) = V\left(\bar{f}^{\#}(B(\Sigma_Y))\right)$. Therefore, from Equation (4.7) we get

$$B(\Sigma_X) \subseteq \sqrt{\bar{f}^{\#}(B(\Sigma_Y)) \cdot S[X]}.$$

Since $S[X]$ is a Noetherian ring, there is a positive integer k with

$$B(\Sigma_X)^k \subseteq \bar{f}^{\#}(B(\Sigma_Y)) \cdot S[X]. \quad (4.8)$$

Let $J = (\bar{f}^{\#})^{-1}(I)$. Take an element F of $(J: B(\Sigma_Y))$. We need to show that F is in J , or equivalently, that $\bar{f}^{\#}(F)$ belongs to I . Since I is saturated with respect to $B(\Sigma_X)$, it is enough to show that $\bar{f}^{\#}(F)$ is an element of $(I: B(\Sigma_X)^k)$. We have

$$\bar{f}^{\#}(F) \cdot B(\Sigma_X)^k \stackrel{(4.8)}{\subseteq} \bar{f}^{\#}(F) \cdot \bar{f}^{\#}(B(\Sigma_Y)) \cdot S[X] \subseteq \bar{f}^{\#}(F \cdot B(\Sigma_Y)) \cdot S[X] \subseteq \bar{f}^{\#}(J) \cdot S[X] \subseteq I,$$

where the penultimate containment follows from the choice of F and the ultimate one is by the definition of J . □

4.1.4 Morphism with connected fibers

In this subsection X and Y are smooth projective toric varieties. We consider a morphism $f: X \rightarrow Y$ such that the natural map $f: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is an isomorphism. In particular, f is dominant and therefore surjective. Furthermore, f has connected fibers [48, Cor. III.11.3]. Conversely, since the base field \mathbb{C} has characteristic zero, for every surjective morphism $f: X \rightarrow Y$ whose fibers are connected we have an isomorphism $f_*\mathcal{O}_X \cong \mathcal{O}_Y$ [30, §1.13].

In what follows we do not assume that f is a toric morphism. However, if f happens to be a toric morphism, then the condition $f_*\mathcal{O}_X \cong \mathcal{O}_Y$ has an equivalent combinatorial reformulation (see [29, Prop. 2.1]).

Lemma 4.9. *Let $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$ be the pullback map. The equality*

$$\dim_{\mathbb{C}} S[X]_{f^*([D])} = \dim_{\mathbb{C}} S[Y]_{[D]}$$

holds for every $[D] \in \text{Pic}(Y)$.

Proof. We have

$$\begin{aligned} S[X]_{f^*([D])} &\cong H^0(X, f^*\mathcal{O}_Y(D)) \cong H^0(Y, f_*(f^*\mathcal{O}_Y(D))) \\ &\cong H^0(Y, f_*\mathcal{O}_X \otimes \mathcal{O}_Y(D)) \cong H^0(Y, \mathcal{O}_Y(D)) \cong S[Y]_{[D]}. \end{aligned}$$

The middle equality follows from projection formula [48, Ex. II.5.1]. The first and the last equality follow from [28, Prop. 5.3.7]. \square

We lift f to a homomorphism $\bar{f}^\#: S[Y] \rightarrow S[X]$ as in Definition 4.2. The following lemma is used in the proof of Theorem 4.15—one of the main results of this chapter.

Lemma 4.10. *We have:*

- (i) *the pullback map $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$ is injective;*
- (ii) *the map $\bar{f}^\#$ induces an isomorphism of the \mathbb{C} -vector spaces $S[Y]_{[D]} \rightarrow S[X]_{f^*([D])}$ for every $[D] \in \text{Pic} Y$.*

Proof.

- (i) Since $f_*\mathcal{O}_X \rightarrow \mathcal{O}_Y$ is an isomorphism, it follows from projection formula (see [48, Ex. II.5.1]) that the map $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$ is injective.
- (ii) The pullback $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$ is injective by part (i). Hence the corresponding map of algebraic tori $H_X = \text{Spec } \mathbb{C}[\text{Pic}(X)] \rightarrow \text{Spec } \mathbb{C}[\text{Pic}(Y)] = H_Y$ is dominant. Thus, it is surjective by [28, Prop. 1.1.1].

Since f is a projective morphism with $f_*\mathcal{O}_X \cong \mathcal{O}_Y$, it is surjective. We claim that \bar{f} is dominant. It is enough to show that $\hat{f}: \hat{X} \rightarrow \hat{Y}$ is surjective. Let $\hat{y} \in \hat{Y}$. Since f and π_X are surjective, there is a point \hat{x} with $f \circ \pi_X(\hat{x}) = \pi_Y(\hat{y})$. Thus, there is an element $t \in H_Y$ satisfying $t \cdot (\hat{f}(\hat{x})) = \hat{y}$. Using the fact that the map of tori is surjective and \hat{f} is equivariant, we conclude that there is $t' \in H_X$ with $\hat{f}(t' \cdot \hat{x}) = \hat{y}$.

This shows \bar{f} is dominant and hence $\bar{f}^\#$ is injective. In particular, it induces injections $S[Y]_{[D]} \rightarrow S[X]_{f^*([D])}$ for every $[D] \in \text{Pic}(Y)$. These maps are surjective by Lemma 4.9. \square

Given a finite set of points $\{p_1, \dots, p_r\} \in X$, we denote by $I(\{p_1, \dots, p_r\})$ the unique $B(\Sigma_X)$ -saturated homogeneous ideal of $S[X]$ defining this set of points as a reduced subscheme of X .

Lemma 4.11. *We have $(\bar{f}^\#)^{-1}(I(\{p_1, \dots, p_r\})) = I(\{f(p_1), \dots, f(p_r)\})$.*

Proof. Let $R \subseteq X$ be the (reduced) subscheme $\{p_1, \dots, p_r\}$. Let $i: R \rightarrow X$ be the closed immersion. Let R' be the scheme-theoretic image of R . Since R is reduced and f is closed, R' is the (reduced) subscheme $\{f(p_1), \dots, f(p_r)\}$.

The scheme theoretic image R' of R is defined by the ideal sheaf $\ker(\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X \rightarrow f_*i_*\mathcal{O}_R)$. By assumption $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is an isomorphism. Therefore, the ideal sheaf of R' is $\ker(f_*\mathcal{O}_X \rightarrow f_*i_*\mathcal{O}_R)$. Moreover, f_* is left-exact, so the ideal sheaf of R' is the pushforward under f of the ideal sheaf of R .

The ideal $I(\{p_1, \dots, p_r\})$ is saturated with respect to $B(\Sigma_X)$. Hence, the subscheme of Y corresponding to $(\bar{f}^\#)^{-1}(I(\{p_1, \dots, p_r\})) \subseteq S[Y]$ is R' by [66, Thm. 3.5] and Lemma 4.10. By Lemma 4.6, the ideal $(\bar{f}^\#)^{-1}(I(\{p_1, \dots, p_r\}))$ is saturated with respect to $B(\Sigma_Y)$. Thus, we have

$$(\bar{f}^\#)^{-1}(I(\{p_1, \dots, p_r\})) = I(\{f(p_1), \dots, f(p_r)\}),$$

as claimed. □

4.1.5 Multigraded Hilbert schemes

Let Y be a smooth projective toric variety with Cox ring $S[Y]$. A natural generalization of the function $h_{r, \mathbb{P}^n}: \mathbb{Z} \rightarrow \mathbb{Z}$ studied in Chapters 2 and 3 is the function $h_{r, Y}: \text{Pic}(Y) \rightarrow \mathbb{Z}$ defined by

$$h_{r, Y}([D]) = \min\{\dim_{\mathbb{C}} H^0(Y, \mathcal{O}_Y(D)), r\} = \min\{\dim_{\mathbb{C}} S[Y]_{[D]}, r\},$$

where the latter equality follows from [28, Prop. 5.3.7]. Observe that in the case of the projective space we have implicitly used the identifications $\text{Pic}(\mathbb{P}^n) = \mathbb{Z}[H]$ where H is a hyperplane divisor and $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(dH)) \cong \mathbb{C}[\alpha_0, \dots, \alpha_n]_d$.

Let $\text{Hilb}_{S[Y]}^{h_{r, Y}}$ be the corresponding multigraded Hilbert scheme (see Subsection 2.2.2). We denote by $\text{Sip}_{r, Y}$ the subset of $\text{Hilb}_{S[Y]}^{h_{r, Y}}$ whose closed points correspond to $B(\Sigma_Y)$ -saturated, homogeneous ideals of r -tuples of disjoint points in Y . Let $\text{Slip}_{r, Y}$ be the closure of $\text{Sip}_{r, Y}$ in $\text{Hilb}_{S[Y]}^{h_{r, Y}}$. By [15, Prop. 3.13] it is an irreducible component of $\text{Hilb}_{S[Y]}^{h_{r, Y}}$.

We construct a natural morphism from r -tuples of distinct points of Y in general position to $\text{Hilb}_{S[Y]}^{h_{r, Y}}$. Let $Y_{dis}^r = \{(p_1, \dots, p_r) \mid p_i \neq p_j \text{ for all } i \neq j\}$ be the set of r -tuples of distinct points of Y . This is an open subset of Y^r so it has a natural scheme structure.

Recall that, given a point $(p_1, \dots, p_r) \in Y_{dis}^r$, we denote by $I(\{p_1, \dots, p_r\})$ the unique $B(\Sigma_Y)$ -saturated homogeneous ideal defining this set of points as a reduced subscheme of Y . Let

$$Y_{gen}^r = \{(p_1, \dots, p_r) \in Y_{dis}^r \mid S[Y]/I(\{p_1, \dots, p_r\}) \text{ has Hilbert function } h_{r, Y}\}.$$

We use the following key observation from [14].

Theorem 4.12. *The set Y_{gen}^r is a dense open subset of Y_{dis}^r . In particular, it has a natural scheme structure.*

Proof. The following sketch of proof is based on [14]. The subset Y_{gen}^r is dense in Y_{dis}^r by [15, Lem. 3.9]. In order to show that it is open we introduce some notation. Let $\mathcal{U}_{dis} \subseteq Y_{dis}^r \times Y$ be

the (reduced) closed subscheme $\coprod_{i=1}^r Z_i$ where we have

$$Z_i = \{(p_1, \dots, p_r), q \mid p_i = q\}.$$

Let $\pi_{dis}: \mathcal{U}_{dis} \rightarrow Y_{dis}^r$ be the projection. The family \mathcal{U}_{dis} is flat over Y_{dis}^r , since each Z_i is mapped isomorphically to Y_{dis}^r . By construction, the fiber of π_{dis} over a closed point (p_1, \dots, p_r) of Y_{dis}^r is the (reduced) subscheme $\{p_1, \dots, p_r\}$ of Y .

It suffices to show that for every affine open subset $U = \text{Spec } A$ of Y_{dis}^r the set $U \cap Y_{gen}^r$ is open. Therefore, we can replace Y_{dis}^r by an affine open subset $U = \text{Spec } A$. Let \mathcal{Z} denote $\mathcal{U}_{dis}|_{\pi_{dis}^{-1}(U)}$. Let $\mathcal{I}_{\mathcal{Z}}$ denote the ideal sheaf of \mathcal{Z} and for a point $u \in U$ let $\mathcal{I}_{\mathcal{Z}_u}$ denote the ideal sheaf of the fiber $\mathcal{Z}_u \subseteq Y_{\kappa(u)}$. Pick a degree $[D] \in \text{Pic}(Y)$ of the grading group. Since \mathcal{Z} is flat over U , from the exact sequence

$$0 \rightarrow \mathcal{I}_{\mathcal{Z}}(D) \rightarrow \mathcal{O}_{Y_A}(D) \rightarrow \mathcal{O}_{\mathcal{Z}}(D) \rightarrow 0$$

we get the exact sequence

$$0 \rightarrow \mathcal{I}_{\mathcal{Z}_u}(D) \rightarrow \mathcal{O}_{Y_{\kappa(u)}}(D) \rightarrow \mathcal{O}_{\mathcal{Z}_u}(D) \rightarrow 0.$$

Therefore, there is the exact sequence

$$0 \rightarrow H^0(Y_{\kappa(u)}, \mathcal{I}_{\mathcal{Z}_u}(D)) \rightarrow H^0(Y, \mathcal{O}_Y(D)) \otimes_{\mathbb{C}} \kappa(u) \rightarrow H^0(Y_{\kappa(u)}, \mathcal{O}_{\mathcal{Z}_u}(D)) \rightarrow \dots$$

We claim that the set

$$U_{[D]} := \{u \in U \mid \dim_{\kappa(u)} \frac{H^0(Y, \mathcal{O}_Y(D)) \otimes_{\mathbb{C}} \kappa(u)}{H^0(Y_{\kappa(u)}, \mathcal{I}_{\mathcal{Z}_u}(D))} < h_{r,Y}([D])\}$$

is a closed subset of U . The point $u \in U$ belongs to $U_{[D]}$ if and only if we have

$$\dim_{\kappa(u)} H^0(Y_{\kappa(u)}, \mathcal{I}_{\mathcal{Z}_u}(D)) > \dim_{\mathbb{C}} H^0(Y, \mathcal{O}_Y(D)) - h_{r,Y}([D]).$$

Therefore, $U_{[D]}$ is closed by [48, Thm. III.12.8].

Observe that there is an equality $Y_{gen}^r \cap U = U \setminus \bigcup_{[D] \in \text{Pic}(Y)} U_{[D]}$ but the union is over a countable indexing set. Therefore, the observation that each $U_{[D]}$ is closed is not sufficient to conclude that $Y_{gen}^r \cap U$ is open in U .

Assume that Y is the projective n -space and let H be a divisor such that the corresponding sheaf $\mathcal{O}_Y(H)$ is isomorphic with $\mathcal{O}_Y(1)$. By Gotzmann's regularity theorem [10, Thm. 4.3.2] and [32, Thm. 4.2] we get $U_{[D]} = \emptyset$ unless $[D]$ belongs to the set $\{[0], [H], [2H], \dots, [(r-1)H]\}$. Thus, in this case the union $\bigcup_{[D] \in \text{Pic}(Y)} U_{[D]}$ is actually a finite union. For more general toric varieties it is harder to conclude that $\bigcup_{[D] \in \text{Pic}(Y)} U_{[D]}$ is closed (see [14]). \square

The rest of this subsection is devoted to showing that $(p_1, \dots, p_r) \mapsto [I(\{p_1, \dots, p_r\})]$ defines a morphism $Y_{gen}^r \rightarrow \text{Hilb}_{S[Y]}^{h_{r,Y}}$. Let \mathcal{U}_{dis} be as in the proof of Theorem 4.12. We denote by \mathcal{U} the restriction of \mathcal{U}_{dis} to $Y_{gen}^r \times Y$. Let $\pi: Y_{gen}^r \times Y \rightarrow Y_{gen}^r$ be the natural projection. Consider the exact sequence of sheaves of $\mathcal{O}_{Y_{gen}^r \times Y}$ -modules

$$0 \rightarrow \bigoplus_{[D] \in \text{Pic}(Y)} \mathcal{I}_{\mathcal{U}}(D) \rightarrow \bigoplus_{[D] \in \text{Pic}(Y)} \mathcal{O}_{Y_{gen}^r \times Y}(D) \xrightarrow{\eta} \bigoplus_{[D] \in \text{Pic}(Y)} \mathcal{O}_{\mathcal{U}}(D) \rightarrow 0. \quad (4.13)$$

Let $\mathcal{A} = \text{im}(\pi_*\eta)$. We verify the following claims:

1. we have $\pi_*\left(\bigoplus_{[D] \in \text{Pic}(Y)} \mathcal{O}_{Y_{gen}^r \times Y}(D)\right) \cong \mathcal{O}_{Y_{gen}^r} \otimes_{\mathbb{C}} S[Y]$;
2. \mathcal{A} is a sheaf of $\mathcal{O}_{Y_{gen}^r} \otimes_{\mathbb{C}} S[Y]$ -algebras;
3. $\mathcal{A}_{[D]}$ is a locally free sheaf of $\mathcal{O}_{Y_{gen}^r}$ -modules of rank $h_{r,Y}([D])$ for every $[D] \in \text{Pic}(Y)$.

The first claim follows from [48, Prop. III.9.3] since there is an isomorphism $\Gamma(Y, \mathcal{O}_Y(D)) \cong S[Y]_{[D]}$. The second claim follows from the fact that in exact sequence (4.13) the $\mathcal{O}_{Y_{gen}^r \times Y}$ -submodule $\bigoplus_{[D] \in \text{Pic}(Y)} \mathcal{I}_{\mathcal{U}}(D)$ of the sheaf of $\mathcal{O}_{Y_{gen}^r \times Y}$ -algebras $\bigoplus_{[D] \in \text{Pic}(Y)} \mathcal{O}_{Y_{gen}^r \times Y}(D)$ is in fact a sheaf of ideals. Therefore, from left-exactness of the pushforward, it follows that the kernel of $\pi_*(\eta)$ is a sheaf of ideals of the sheaf $\mathcal{O}_{Y_{gen}^r} \otimes_{\mathbb{C}} S[Y]$.

Finally, we address the third claim. We consider two cases:

- (a) $h_{r,Y}([D]) = r$;
- (b) $h_{r,Y}([D]) < r$.

Observe that by definition of \mathcal{U} , for every $y \in Y_{gen}^r$ we have

$$\dim_{\mathbb{C}} H^0((Y_{gen}^r \times Y)_y, (\mathcal{I}_{\mathcal{U}}(D))_y) = \dim_{\mathbb{C}} S[Y]_{[D]} - h_{r,Y}([D]).$$

Similarly, we have $\dim_{\mathbb{C}} H^0((Y_{gen}^r \times Y)_y, (\mathcal{O}_{\mathcal{U}}(D))_y) = r$ for every $y \in Y_{gen}^r$. Moreover, both $\mathcal{O}_{\mathcal{U}}(D)$ and $\mathcal{I}_{\mathcal{U}}(D)$ are flat over Y_{gen}^r . Therefore, by [48, Cor. III.12.9] the sheaves of $\mathcal{O}_{Y_{gen}^r}$ -modules $\pi_*(\mathcal{I}_{\mathcal{U}}(D))$ and $\pi_*(\mathcal{O}_{\mathcal{U}}(D))$ are locally free of rank $\dim_{\mathbb{C}} S[Y]_{[D]} - h_{r,Y}([D])$ and r , respectively. In particular, this establishes claim 3 in the case (b) since then we have $\mathcal{A}_{[D]} \cong \mathcal{O}_{Y_{gen}^r} \otimes_{\mathbb{C}} S[Y]_{[D]}$.

Thus, it is enough to show that if $h_{r,Y}([D])$ equals r then $\pi_*(\eta)$ induces a surjection $\mathcal{O}_{Y_{gen}^r} \otimes_{\mathbb{C}} S[Y]_{[D]} \rightarrow \pi_*(\mathcal{O}_{\mathcal{U}}(D))$. This can be checked on stalks over closed points, and by Nakayama's lemma it is even enough to check this on fibers. Let $y \in Y_{gen}^r$ correspond to the subscheme $Z \subseteq Y$ and let I_Z denote its $B(\Sigma_Y)$ -saturated ideal. Using [48, Cor. III.12.9], it is enough to show that natural map

$$S[Y]_{[D]} \rightarrow \Gamma(Y, \mathcal{O}_Z(D))$$

is surjective for every $[D] \in \text{Pic}(Y)$ with $h_{r,Y}([D]) = r$. However, the kernel of this map is $(I_Z)_{[D]}$ and we have

$$r = \dim_{\mathbb{C}} \Gamma(Y, \mathcal{O}_Z(D)) = \dim_{\mathbb{C}} S[Y]_{[D]} - \dim_{\mathbb{C}} (I_Z)_{[D]}$$

by the choice of $[D]$. This finishes the proof of claims 1–3.

Lemma 4.14. *There is a natural morphism $\psi_{r,Y}: Y_{gen}^r \rightarrow \text{Hilb}_{S[Y]}^{h_{r,Y}}$ which on closed points maps $\{p_1, \dots, p_r\}$ to $I(\{p_1, \dots, p_r\})$.*

Proof. By properties 1–3 above, \mathcal{A} defines an admissible family over Y_{gen}^r for the Hilbert function $h_{r,Y}$. Thus, there is a morphism $Y_{gen}^r \rightarrow \text{Hilb}_{S[Y]}^{h_{r,Y}}$. By construction, on closed points, it maps $\{p_1, \dots, p_r\}$, to $I(\{p_1, \dots, p_r\})$. \square

4.2 Criterion based on a morphism of toric varieties

Let $f: X \rightarrow Y$ be a morphism between smooth projective toric varieties with $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Let r be a positive integer. In this section we present a necessary condition for a closed point

$[I] \in \text{Hilb}_{S[X]}^{h_{r,X}}$ to be in the irreducible component $\text{Slip}_{r,X}$. The most interesting case is when we have $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{k+1}}$ and $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ for some positive integers k, n_1, \dots, n_{k+1} and f is the natural projection.

Theorem 4.15. *Let $f: X \rightarrow Y$ be a morphism between smooth projective toric varieties with $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Let r be a positive integer and $[I] \in \text{Hilb}_{S[X]}^{h_{r,X}}$ be a closed point. Let $\bar{f}^\# : S[Y] \rightarrow S[X]$ be a lift of f as in Definition 4.2.*

- (i) $\bar{f}^\#$ induces a morphism $\pi: \text{Hilb}_{S[X]}^{h_{r,X}} \rightarrow \text{Hilb}_{S[Y]}^{h_{r,Y}}$ given on closed points by $[I] \mapsto [(\bar{f}^\#)^{-1}(I)]$;
- (ii) The morphism $\pi: \text{Hilb}_{S[X]}^{h_{r,X}} \rightarrow \text{Hilb}_{S[Y]}^{h_{r,Y}}$ from part (i) induces a surjection $\text{Slip}_{r,X} \rightarrow \text{Slip}_{r,Y}$.

Proof. The existence of a lift $\bar{f}^\#$ follows from Theorem 4.3.

- (i) Using Lemma 4.10 we may identify (as $\text{Pic}(Y)$ -graded rings) the ring $S[Y]$ with the subring $\bigoplus_{[D] \in \text{Pic} Y} S[X]_{f^*(D)}$ of $S[X]$. Under this identification, we have

$$(\bar{f}^\#)^{-1}(I) = I|_{f^*(\text{Pic}(Y))} := \bigoplus_{[D] \in \text{Pic}(Y)} I_{f^*(D)}.$$

It follows that if $S[X]/I$ has Hilbert function $h_{r,X}$, then $S[Y]/(\bar{f}^\#)^{-1}(I)$ has Hilbert function $h_{r,Y}$. Thus, we have a natural transformation of functors of points

$$\underline{\text{Hilb}}_{S[X]}^{h_{r,X}} \rightarrow \underline{\text{Hilb}}_{S[Y]}^{h_{r,Y}}$$

given on a \mathbb{C} -algebra R by

$$\underline{\text{Hilb}}_{S[X]}^{h_{r,X}}(R) \ni I \mapsto I|_{f^*(\text{Pic}(Y))} \in \underline{\text{Hilb}}_{S[Y]}^{h_{r,Y}}(R).$$

Hence we have the corresponding morphism of schemes $\pi: \text{Hilb}_{S[X]}^{h_{r,X}} \rightarrow \text{Hilb}_{S[Y]}^{h_{r,Y}}$.

- (ii) We first show that $\pi(\text{Slip}_{r,X})$ is contained in $\text{Slip}_{r,Y}$ set-theoretically. If I is radical then $(\bar{f}^\#)^{-1}(I)$ is also radical. Moreover, if I is saturated with respect to $B(\Sigma_X)$, then $(\bar{f}^\#)^{-1}(I)$ is saturated with respect to $B(\Sigma_Y)$ by Lemma 4.6. It follows that we have $\pi(\text{Slip}_{r,X}) \subseteq \text{Slip}_{r,Y}$ set-theoretically. Therefore, $\pi(\text{Slip}_{r,X})$ is a subset of $\text{Slip}_{r,Y}$.

Now we show that in fact $\pi: \text{Slip}_{r,X} \rightarrow \text{Slip}_{r,Y}$ is surjective. Recall the definition of Y_{gen}^r from Subsection 4.1.5. Consider the product morphism $f^r: X^r \rightarrow Y^r$. By Chevalley's theorem [41, Thm. 10.20] the image of X_{gen}^r in Y^r is constructible. Moreover, f^r is projective and surjective and X_{gen}^r is dense in X^r by Theorem 4.12. It follows that $f^r(X_{gen}^r)$ is dense in Y^r . Thus, there is an open subset $U \subseteq Y^r$ contained in $f^r(X_{gen}^r)$ (see [48, Ex. II.3.18]). Let $V = U \cap Y_{gen}^r$ and $W = (f^r)^{-1}(V) \cap X_{gen}^r$. We have a diagram

$$\begin{array}{ccc} \text{Hilb}_{S[X]}^{h_{r,X}} & \xrightarrow{\pi} & \text{Hilb}_{S[Y]}^{h_{r,Y}} \\ (\psi_{r,X})|_W \uparrow & & (\psi_{r,Y})|_V \uparrow \\ W & \xrightarrow{f^r|_W} & V \end{array}$$

where the maps $\psi_{r,X}$ and $\psi_{r,Y}$ are as in Lemma 4.14. We claim that this diagram is commutative. Let (p_1, \dots, p_r) be a point of W . We have $\psi_{r,Y} \circ f^r(p_1, \dots, p_r) = [I(\{f(p_1), \dots, f(p_r)\})]$ and

$$\pi \circ \psi_{r,X}(p_1, \dots, p_r) = [(\overline{f^\#})^{-1}(I(\{p_1, \dots, p_r\}))] = [I(\{f(p_1), \dots, f(p_r)\})].$$

Here the last equality follows from Lemma 4.11. Thus, the diagram commutes.

By construction, $f^r(W)$ equals V , so it is dense in Y_{gen}^r . Since π is projective, it follows that we have

$$\text{Slip}_{r,Y} = \overline{\psi_{r,Y} \circ f^r(W)} = \overline{\pi \circ \psi_{r,X}(W)} = \overline{\pi(\psi_{r,X}(W))} = \pi(\text{Slip}_{r,X})$$

set-theoretically. □

We obtain the following corollary.

Corollary 4.16. *In the notation of Theorem 4.15, assume that $[J] \in \text{Slip}_{r,Y}$ is such that there exists a unique closed point $[I] \in \text{Hilb}_{S[X]}^{h_{r,X}}$ with $(\overline{f^\#})^{-1}(I) = J$. The point $[I]$ is in $\text{Slip}_{r,X}$.*

Remark 4.17. The usual Hilbert scheme $\mathcal{Hilb}_r(X)$ is usually not functorial in X . That is, let $f: X \rightarrow Y$ be a regular non-constant morphism of algebraic varieties. A general r -tuple of distinct points of X is mapped to an r -tuple of points of Y and this assignment induces a rational map of the smoothable components $\pi: \mathcal{Hilb}_r^{sm}(X) \dashrightarrow \mathcal{Hilb}_r^{sm}(Y)$. However, this map needs not to extend to a regular morphism $\mathcal{Hilb}_r(X) \rightarrow \mathcal{Hilb}_r(Y)$, or even $\mathcal{Hilb}_r^{sm}(X) \rightarrow \mathcal{Hilb}_r^{sm}(Y)$.

Let $r = 2$ and $f: X \rightarrow Y$ be the blowup of \mathbb{P}^2 at a point (thus we are in the situation of Theorem 4.15). There are degree 2 finite subschemes contained in the exceptional divisor, which is contracted to a point. It is straightforward to verify that there is no continuous map that extends $\pi: \mathcal{Hilb}_2^{sm}(X) \dashrightarrow \mathcal{Hilb}_2^{sm}(\mathbb{P}^2)$ to the points representing such subschemes.

In contrast, Theorem 4.15 shows that the multigraded Hilbert scheme $\text{Hilb}_{S[X]}^{h_{r,X}}$ and the analogue of its smoothable component $\text{Slip}_{r,X}$ behave nicely (functorially), at least under some special morphisms. In some sense, the induced map $\text{Slip}_{r,X} \rightarrow \text{Slip}_{r,Y}$ is a natural resolution of the rational map $\mathcal{Hilb}_r^{sm}(X) \dashrightarrow \mathcal{Hilb}_r^{sm}(Y)$.

4.3 Blowup of the closure of a torus orbit

In this section we study a special case of Theorem 4.15—the blowup of a smooth projective toric variety at the closure of a torus orbit.

Let Y be a smooth projective n -dimensional toric variety associated with a fan $\Sigma_Y \subseteq N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $f: X \rightarrow Y$ be the blowup of Y at the closure of the torus orbit $V(\tau) = \overline{O(\tau)}$ corresponding to a cone $\tau \in \Sigma_Y$ (see [28, Thm. 3.2.6]). In that case, as follows from [28, pp. 132–133], the variety X is the toric variety associated with the fan $\Sigma_Y^*(\tau) \subseteq N \otimes_{\mathbb{Z}} \mathbb{R}$, whose construction we recall below. Moreover, the blowup $f: X \rightarrow Y$ corresponds to the identity map on the lattice N . Observe that the special case, when τ is n -dimensional (or, equivalently, when $V(\tau)$ is a torus invariant point of Y) is [28, Prop. 3.3.15].

Now we recall the construction of the fan $\Sigma_Y^*(\tau)$. Given a cone $\sigma \in \Sigma_Y$, we denote by $\sigma(1)$ the set of all edges of σ . Let $\mathbf{u}_\tau = \sum_{\rho \in \tau(1)} \mathbf{u}_\rho$ be the sum of the ray generators of edges of τ .

Let $\sigma \in \Sigma_Y$ be a cone containing τ and consider the set

$$(\Sigma_Y)_\sigma^*(\tau) = \{\text{Cone}(A) \mid A \subseteq \{\mathbf{u}_\tau\} \cup \bigcup_{\rho \in \sigma(1)} \{\mathbf{u}_\rho\} \text{ such that } \tau(1) \not\subseteq A\}.$$

We have

$$\Sigma_Y^*(\tau) = \{\sigma \in \Sigma_Y \mid \tau \not\subseteq \sigma\} \cup \bigcup_{\tau \subseteq \sigma} (\Sigma_Y)_\sigma^*(\tau).$$

Let $\Sigma_Y(1) = \{\rho_1, \dots, \rho_n, \rho'_1, \dots, \rho'_t\}$ and $\tau = \text{Cone}(\rho_i \mid i = 1, \dots, s)$ for some $1 \leq s \leq n$. Let e_1, \dots, e_n be the standard basis of M and let $\mathbf{e}_1^*, \dots, \mathbf{e}_n^*$ be the dual basis. Since Y is smooth, we may assume that the ray generator \mathbf{u}_{ρ_i} of ρ_i for all $i = 1, \dots, n$ is \mathbf{e}_i^* . We can express the ray generators of ρ'_j in terms of this basis

$$\mathbf{u}_{\rho'_j} = \sum_{i=1}^n a_{ij} \mathbf{e}_i^* \quad (4.18)$$

for some $a_{ij} \in \mathbb{Z}$. We have $\Sigma_X(1) = \Sigma_Y(1) \cup \{\rho_\tau\}$, where ρ_τ equals $\text{Cone}(\mathbf{u}_\tau)$.

By [28, Thm. 4.1.3], the Picard group $\text{Pic}(Y)$ is generated by the classes

$$[D_{\rho_1}], \dots, [D_{\rho_n}], [D_{\rho'_1}], \dots, [D_{\rho'_t}]$$

of prime torus invariant divisors modulo the relations

$$0 = [\text{div}(\chi^{e_i})] = \sum_{j=1}^n \langle e_i, \mathbf{u}_{\rho_j} \rangle [D_{\rho_j}] + \sum_{j=1}^t \langle e_i, \mathbf{u}_{\rho'_j} \rangle [D_{\rho'_j}] \text{ for all } i = 1, \dots, n. \quad (4.19)$$

It follows from Equations (4.18) and (4.19) that we have

$$\text{Pic}(Y) = \bigoplus_{i=1}^t \mathbb{Z}[D_{\rho'_i}]$$

and

$$[D_{\rho_i}] = \sum_{j=1}^t -a_{ij} [D_{\rho'_j}]$$

for all $i = 1, \dots, n$. We use this description to identify $\text{Pic}(Y)$ with \mathbb{Z}^t .

Similarly, we obtain

$$\text{Pic}(X) = \bigoplus_{i=1}^t \mathbb{Z}[D_{\rho'_i}] \oplus [D_{\rho_\tau}],$$

$$[D_{\rho_i}] = - \sum_{j=1}^t a_{ij} [D_{\rho'_j}] - [D_{\rho_\tau}]$$

for all $i = 1, \dots, s$ and

$$[D_{\rho_i}] = - \sum_{j=1}^t a_{ij} [D_{\rho'_j}]$$

for all $i = s + 1, \dots, n$.

It follows that the Cox ring of Y is $S[Y] = \mathbb{C}[\beta_1, \dots, \beta_n, \beta'_1, \dots, \beta'_t]$ with $\deg(\beta_i) = [D_{\rho_i}] = -\sum_{j=1}^t a_{ij} \mathbf{e}_j$ for all $i = 1, \dots, n$ and $\deg(\beta'_i) = \mathbf{e}_i$ where $\mathbf{e}_1, \dots, \mathbf{e}_t$ is the standard basis of $\mathbb{Z}^t \cong \text{Pic}(Y)$.

Similarly, the Cox ring of X is $S[X] = \mathbb{C}[\alpha_1, \dots, \alpha_n, \alpha'_1, \dots, \alpha'_t, \alpha'']$ with

$$\begin{aligned} \deg(\alpha_i) &= [D_{\rho_i}] = -\sum_{j=1}^t a_{ij} \mathbf{f}_j - \mathbf{f}_{t+1} \text{ for all } i = 1, \dots, s, \\ \deg(\alpha_i) &= [D_{\rho_i}] = -\sum_{j=1}^t a_{ij} \mathbf{f}_j \text{ for all } i = s+1, \dots, n, \\ \deg(\alpha'_i) &= [D_{\rho'_i}] = \mathbf{f}_i \text{ for all } i = 1, \dots, t \\ \deg(\alpha'') &= [D_{\rho''}] = \mathbf{f}_{t+1} \end{aligned}$$

where $\mathbf{f}_1, \dots, \mathbf{f}_{t+1}$ is the standard basis of $\mathbb{Z}^{t+1} \cong \text{Pic}(X)$.

We lift the map $f: X \rightarrow Y$ to a map of Cox rings $\bar{f}^\# : S[Y] \rightarrow S[X]$ as in Definition 4.2. We start with describing the pullback map $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$.

Lemma 4.20. *The pullback map $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$ is given by $\mathbf{e}_i \mapsto \mathbf{f}_i$ for all $i = 1, \dots, t$.*

Proof. For all $i = 1, \dots, t$, let $\varphi_{D_{\rho'_i}}: N \rightarrow \mathbb{R}$ be the support function of the torus invariant Cartier divisor $D_{\rho'_i}$ on Y (see [28, Def. 4.2.11]). By [28, Prop. 6.2.7], $f^*([D_{\rho'_i}])$ is the class of the torus invariant Cartier divisor on X corresponding to the same support function. By definition we have

$$\varphi_{D_{\rho'_i}}(\mathbf{u}_\rho) = -\delta_{\rho\rho'_i} \text{ for all } \rho \in \Sigma_Y(1).$$

In particular, $\varphi_{D_{\rho'_i}}(\mathbf{u}_\tau)$ is zero since $\varphi_{D_{\rho'_i}}$ is zero on each ray generator of the cone τ . Thus, we get $f^*(\mathbf{e}_i) = \mathbf{f}_i$ for all $i = 1, \dots, t$. \square

Now we describe a lift of $f: X \rightarrow Y$ to a map $\bar{f}^\# : S[Y] \rightarrow S[X]$.

Proposition 4.21. *The \mathbb{C} -algebra homomorphism $\bar{f}^\# : S[Y] \rightarrow S[X]$ given by*

$$\begin{aligned} \beta_i &\mapsto \alpha_i \cdot \alpha'' \text{ for all } i = 1, \dots, s \\ \beta_i &\mapsto \alpha_i \text{ for all } i = s+1, \dots, n \\ \beta'_i &\mapsto \alpha'_i \text{ for all } i = 1, \dots, t \end{aligned}$$

is a lift of $f: X \rightarrow Y$ as in Definition 4.2. In particular, if r is a positive integer and $[I]$ belongs to $\text{Slip}_{r,X}$, then $[(\bar{f}^\#)^{-1}(I)]$ is in $\text{Slip}_{r,Y}$.

Proof. By Lemma 4.20, the homomorphism $\bar{f}^\#$ is a map of graded rings with respect to the homomorphism $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$ of grading groups, i.e.

$$\bar{f}^\#(S[Y]_{[D]}) \subseteq S[X]_{f^*([D])}$$

holds for every $[D] \in \text{Pic}(Y)$.

It follows that $\bar{f}^\#$ defines an equivariant map $\bar{f}: \bar{X} = \text{Spec } S[X] \rightarrow \text{Spec } S[Y] = \bar{Y}$. Let $B(\Sigma_X) \subseteq S[X]$ and $B(\Sigma_Y) \subseteq S[Y]$ be the irrelevant ideals. We claim that \bar{f} restricts to a map

$$\hat{f}: \bar{X} \setminus V(B(\Sigma_X)) = \hat{X} \rightarrow \hat{Y} = \bar{Y} \setminus V(B(\Sigma_Y)).$$

Recall that $\Sigma_Y(n)$ is the set of all maximal cones of the fan Σ_Y . For all $\sigma \in \Sigma_Y(n)$ let $\beta^{\widehat{\sigma}}$ be the product of variables in $S[Y]$ corresponding to rays in $\Sigma_Y(1)$ which are not rays in $\sigma(1)$. We have

$$B(\Sigma_Y) = (\beta^{\widehat{\sigma}})_{\sigma \in \Sigma_Y(n)}$$

by [28, p. 207].

We consider two types of maximal cones in $\Sigma_Y(n)$. Namely, there is an equality $\Sigma_Y(n) = \Sigma'_Y(n) \cup \Sigma''_Y(n)$ with

$$\Sigma'_Y(n) = \{\sigma \in \Sigma_Y(n) \mid \tau \not\subseteq \sigma\}$$

and

$$\Sigma''_Y(n) = \{\sigma \in \Sigma_Y(n) \mid \tau \subseteq \sigma\}.$$

For a cone $\sigma \in \Sigma''_Y(n)$ and $i \in \{1, \dots, s\}$ we define $\sigma_i = \text{Cone}\left(\left(\bigcup_{\rho \in \sigma(1), \rho \neq \rho_i} \{\mathbf{u}_\rho\}\right) \cup \{\mathbf{u}_\tau\}\right)$. We have $\Sigma_X(n) = \Sigma'_Y(n) \cup \bigcup_{\sigma \in \Sigma''_Y(n)} \bigcup_{i=1}^s \{\sigma_i\}$. For a cone $\sigma \in \Sigma_X(n)$, let $\alpha^{\widehat{\sigma}}$ be the product of variables of $S[X]$ corresponding to rays from $\Sigma_X(1) \setminus \sigma(1)$. There is an equality

$$B(\Sigma_X) = (\alpha^{\widehat{\sigma}})_{\sigma \in \Sigma_X(n)}.$$

The map $\bar{f}: \mathbb{A}^{n+t+1} \rightarrow \mathbb{A}^{n+t}$ is given by

$$p := (a_1, \dots, a_n, a'_1, \dots, a'_t, a'') \mapsto (a_1 a'', \dots, a_s a'', a_{s+1}, \dots, a_n, a'_1, \dots, a'_t).$$

Assume that $\bar{f}(p)$ is in $V(B(\Sigma_Y))$. We show that p belongs to $V(B(\Sigma_X))$.

Let $\sigma \in \Sigma'_Y(n)$. We have

$$0 = \beta^{\widehat{\sigma}}(\bar{f}(p)) = (a'')^k \cdot (\beta^{\widehat{\sigma}}(a_1, \dots, a_n, a'_1, \dots, a'_t)) = (a'')^{k-1} \cdot (\alpha^{\widehat{\sigma}}(p))$$

where $s \geq k \geq 1$ is the number of rays in $\tau(1) \setminus \sigma(1)$. It follows that $\alpha^{\widehat{\sigma}}(p)$ is zero.

For every $\sigma \in \Sigma''_Y(n)$ and $i \in \{1, \dots, s\}$ there are equalities

$$\alpha^{\widehat{\sigma}_i}(p) = a_i \cdot (\beta^{\widehat{\sigma}}(\bar{f}(p))) = 0.$$

We have shown that p is in $V(B(\Sigma_X))$.

By Lemma 4.4, in order to verify that $\bar{f}^\#$ is a lift of f , it suffices to show that

$$\prod_{i=1}^s (\alpha_i \alpha'')^{\langle m, \mathbf{u}_{\rho_i} \rangle} \cdot \prod_{i=s+1}^n \alpha_i^{\langle m, \mathbf{u}_{\rho_i} \rangle} \cdot \prod_{i=1}^t \alpha'_i{}^{\langle m, \mathbf{u}_{\rho'_i} \rangle} = \prod_{i=1}^s \alpha_i^{\langle m, \mathbf{u}_{\rho_i} \rangle} \cdot \prod_{i=s+1}^n \alpha_i^{\langle m, \mathbf{u}_{\rho_i} \rangle} \cdot \prod_{i=1}^t \alpha'_i{}^{\langle m, \mathbf{u}_{\rho'_i} \rangle} \cdot \alpha''^{\langle m, \mathbf{u}_\tau \rangle}$$

holds for every $m \in M$. This holds since we have $\mathbf{u}_\tau = \sum_{i=1}^s \mathbf{u}_{\rho_i}$.

The last part of the proposition follows from Theorem 4.15. \square

4.4 Toric bundle

In this section we study another special case of Theorem 4.15—where X is a decomposable toric vector bundle.

Let Y be a smooth projective toric variety defined by a fan $\Sigma_Y \subseteq N_{\mathbb{R}}$. Let n be a positive integer and consider torus invariant divisors $D_i = \sum_{\rho \in \Sigma_Y(1)} a_{i\rho} D_\rho$ for all $i = 0, \dots, n$, where

D_ρ 's are prime torus invariant divisors of Y corresponding to rays of Σ_Y and $a_{i\rho}$'s are integers. Let $\mathcal{E} = \mathcal{O}_Y(D_0) \oplus \cdots \oplus \mathcal{O}_Y(D_n)$ and let $X = \mathbb{P}(\mathcal{E})$. The variety X is a smooth projective toric variety (see [28, Prop. 7.3.3.]). We construct a lift $\bar{f}^\# : S[Y] \rightarrow S[X]$ of the natural projection $f : X \rightarrow Y$. A special case of interest is when we have $Y = \mathbb{P}^a \times \mathbb{P}^b$, $X = \mathbb{P}^a \times \mathbb{P}^b \times \mathbb{P}^c$ and f is the projection.

We start with describing the Cox ring of X . Let f_1, \dots, f_m be a basis of the lattice M and let $\mathbf{f}_1^*, \dots, \mathbf{f}_m^*$ be its dual basis. Consider \mathbb{Z}^n with the standard \mathbb{Z} -basis $\mathbf{e}_1^*, \dots, \mathbf{e}_n^*$ and let $\mathbf{e}_0^* = -\mathbf{e}_1^* - \cdots - \mathbf{e}_n^*$. Given a ray generator $\mathbf{u}_\rho \in N_{\mathbb{R}}$ with $\rho \in \Sigma_Y(1)$ we define

$$\mathbf{v}_\rho = \mathbf{u}_\rho + (a_{1\rho} - a_{0\rho})\mathbf{e}_1^* + \cdots + (a_{n\rho} - a_{0\rho})\mathbf{e}_n^* \in N_{\mathbb{R}} \times \mathbb{R}^n.$$

The cones of the fan $\Sigma_X \subseteq N_{\mathbb{R}} \times \mathbb{R}^n$ of X are of the form

$$\sigma_i = \text{Cone}(\mathbf{v}_\rho \mid \rho \in \sigma(1)) + \text{Cone}(\mathbf{e}_0^*, \dots, \widehat{\mathbf{e}}_i^*, \dots, \mathbf{e}_n^*)$$

together with their faces, where we have $\sigma \in \Sigma_Y$, $i \in \{0, \dots, n\}$ and $\widehat{\mathbf{e}}_i^*$ means that \mathbf{e}_i^* is omitted ([28, Prop. 7.3.3.]). Thus, the ray generators of the fan of X are

$$\{\mathbf{v}_\rho \mid \rho \in \Sigma_Y(1)\} \cup \{\mathbf{e}_0^*, \dots, \mathbf{e}_n^*\}.$$

In particular, by [28, Thm. 4.1.3] the Picard group of X is generated by the classes of torus invariant divisors F_ρ for all $\rho \in \Sigma_Y(1)$ corresponding to \mathbf{v}_ρ and the classes of E_0, \dots, E_n corresponding to $\mathbf{e}_0^*, \dots, \mathbf{e}_n^*$. Moreover, these generators are subject to the relations

$$0 = [\text{div}(\chi^{f_i})] = \sum_{\rho \in \Sigma_Y(1)} \langle f_i, \mathbf{u}_\rho \rangle [F_\rho] \text{ for all } i = 1, \dots, m$$

and

$$0 = [\text{div}(\chi^{e_i})] = [E_i] - [E_0] + \sum_{\rho \in \Sigma_Y(1)} (a_{i\rho} - a_{0\rho}) [F_\rho] \text{ for all } i = 1, \dots, n.$$

Therefore, we have an isomorphism $\text{Pic}(X) \cong \text{Pic}(Y) \times \mathbb{Z}$ given by

$$\begin{aligned} [F_\rho] &\mapsto ([D_\rho], 0) && \text{for all } \rho \in \Sigma_Y(1) \\ [E_i] &\mapsto \left(-\sum_{\rho \in \Sigma_Y(1)} (a_{i\rho} - a_{0\rho}) [D_\rho], 1\right) && \text{for all } i = 0, \dots, n. \end{aligned}$$

In particular, $[E_0] \in \text{Pic}(X)$ corresponds to $(0, 1) \in \text{Pic}(Y) \times \mathbb{Z}$. From these considerations, it follows that the Cox rings of Y and X are

$$S[Y] = \mathbb{C}\{\{\beta_\rho \mid \rho \in \Sigma_Y(1)\}\} \text{ and } S[X] = \mathbb{C}\{\{\alpha_\rho \mid \rho \in \Sigma_Y(1)\}, \alpha'_0, \dots, \alpha'_n\}$$

with

$$\begin{aligned} \deg(\beta_\rho) &= [D_\rho] \in \text{Pic}(Y) \text{ for all } \rho \in \Sigma_Y(1), \\ \deg(\alpha_\rho) &= [F_\rho] = ([D_\rho], 0) \in \text{Pic}(Y) \times \mathbb{Z} \text{ for all } \rho \in \Sigma_Y(1) \text{ and} \\ \deg(\alpha'_i) &= [E_i] \text{ for all } i = 0, \dots, n. \end{aligned}$$

Let $\phi : N \times \mathbb{Z}^n \rightarrow N$ be the natural surjection of lattices. If τ is a face of $\sigma_i \in \Sigma_X$ for some

$\sigma \in \Sigma_Y$ and $i \in \{0, \dots, n\}$, then $\phi_{\mathbb{R}}(\tau)$ is a face of σ . Therefore, the map ϕ is compatible with the fans of X and Y . Thus, it induces a toric morphism $f: X \rightarrow Y$ [28, Thm. 3.3.4]. We want to lift this morphism to a homomorphism of Cox rings as in Definition 4.2. First we describe the pullback map $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$.

Lemma 4.22. *The pullback map $f^*: \text{Pic}(Y) \rightarrow \text{Pic}(X)$ maps $[D_\rho]$ to $[F_\rho]$ for every $\rho \in \Sigma_Y(1)$.*

Proof. Let $\varphi_{D_\rho}: N_{\mathbb{R}} \rightarrow \mathbb{R}$ be the support function corresponding to D_ρ , i.e. it is linear on each cone of Σ_Y and we have $\varphi_{D_\rho}(\mathbf{u}_{\rho'}) = -\delta_{\rho\rho'}$ (see [28, Thm. 4.2.12]). By [28, Thm. 6.2.7] $f^*([D_\rho])$ is the class of the torus invariant Cartier divisor with support function

$$\psi: N_{\mathbb{R}} \times \mathbb{R}^n \xrightarrow{\phi_{\mathbb{R}}} N_{\mathbb{R}} \xrightarrow{\varphi_{D_\rho}} \mathbb{R}.$$

We have $\psi(\mathbf{v}_{\rho'}) = \varphi_{D_\rho}(\mathbf{u}_{\rho'}) = -\delta_{\rho\rho'}$ and $\psi(\mathbf{e}_i^*) = 0$. Thus, ψ is the support function of F_ρ . \square

Now we describe a lift of f to a map $\bar{f}^\#: S[Y] \rightarrow S[X]$.

Proposition 4.23. *The \mathbb{C} -algebra homomorphism $\bar{f}^\#: S[Y] \rightarrow S[X]$ given by $\beta_\rho \mapsto \alpha_\rho$ for all $\rho \in \Sigma_Y(1)$ is a lift of f as in Definition 4.2. In particular, if $[I] \in \text{Slip}_{r,X}$ is a closed point, then $[(\bar{f}^\#)^{-1}(I)]$ is in $\text{Slip}_{r,Y}$.*

Proof. By Lemma 4.22, the homomorphism $\bar{f}^\#$ is a homomorphism of graded rings, i.e.

$$\bar{f}^\#(S[Y]_{[D]}) \subseteq S[X]_{f^*([D])}$$

holds for every $[D] \in \text{Pic}(Y)$. Therefore, it induces an equivariant map $\bar{f}: \bar{X} \rightarrow \bar{Y}$. We claim that it restricts to a morphism $\hat{f}: \hat{X} \rightarrow \hat{Y}$. The map \bar{f} is defined by

$$(a_1, \dots, a_s, a'_0, \dots, a'_n) \mapsto (a_1, \dots, a_s),$$

where the first $s = \#\Sigma_Y(1)$ coordinates of the affine space \bar{X} correspond to α_ρ 's. By $\Sigma_Y(m)$ we denote the set of all m -dimensional cones of Σ_Y . Given $\sigma \in \Sigma_Y$, let $\beta^{\hat{\sigma}} = \prod_{\rho \in \Sigma_Y(1) \setminus \sigma(1)} \beta_\rho$. Recall that $B(\Sigma_Y)$ is equal to $(\beta^{\hat{\sigma}} \mid \sigma \in \Sigma_Y(m))$. Similarly, we have $B(\Sigma_X) = (\alpha^{\hat{\sigma}_i} \mid \sigma \in \Sigma_Y(m), i \in \{0, \dots, n\})$. Assume that (a_1, \dots, a_s) belongs to $V(B(\Sigma_Y))$. Let $\sigma \in \Sigma_Y(m)$ and $i \in \{0, \dots, n\}$. The equality $\beta^{\hat{\sigma}}(a_1, \dots, a_s) = 0$ implies that we have

$$0 = a'_i \cdot \beta^{\hat{\sigma}}(a_1, \dots, a_s) = \alpha^{\hat{\sigma}_i}(a_1, \dots, a_s, a'_0, \dots, a'_n).$$

Thus, $(a_1, \dots, a_s, a'_0, \dots, a'_n)$ is in $V(B(\Sigma_X))$.

We have shown, that $\bar{f}: \bar{X} \rightarrow \bar{Y}$ restricts to a map $\hat{f}: \hat{X} \rightarrow \hat{Y}$. Therefore, there is an induced morphism $f': X \rightarrow Y$. We claim that we have $f = f'$. By Lemma 4.4, it is enough to show that

$$\prod_{\rho \in \Sigma_Y(1)} \alpha_\rho^{\langle m, \mathbf{u}_\rho \rangle} = \prod_{\rho \in \Sigma_Y(1)} \alpha_\rho^{\langle \psi(m), \mathbf{v}_\rho \rangle} \cdot \prod_{i=0}^n \alpha_i^{\langle \psi(m), \mathbf{e}_i^* \rangle}$$

holds for every $m \in M$, where $\psi: M \rightarrow M \times \mathbb{Z}^n$ is dual to $N \times \mathbb{Z}^n \rightarrow N$ (i.e., it is the natural inclusion). The claimed equality follows from the definition of \mathbf{v}_ρ .

The last part of the proposition is implied by Theorem 4.15. \square

4.5 Product of projective spaces

Let X be the product of projective spaces $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ for some positive integers $k \geq 2, n_1, \dots, n_k$. Proposition 4.23 gives a necessary condition for $[I] \in \text{Hilb}_{S[X]}^{h_{r,X}}$ to be in $\text{Slip}_{r,X}$. In this section, we present another condition that must be fulfilled for $[I]$ to be in the irreducible component $\text{Slip}_{r,X}$.

The Cox ring of X is of the form

$$S[X] = \mathbb{C}[\alpha_{1,0}, \dots, \alpha_{1,n_1}, \alpha_{2,0}, \dots, \alpha_{2,n_2}, \dots, \alpha_{k,0}, \dots, \alpha_{k,n_k}].$$

It has a grading in \mathbb{Z}^k . In the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_k$ of \mathbb{Z}^k we have $\deg(\alpha_{i,j}) = \mathbf{e}_i$ for all $i \in \{1, \dots, k\}$ and all $j \in \{0, \dots, n_i\}$. The Cox ring of the i -th factor of $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is the polynomial ring $\mathbb{C}[\alpha_{i,0}, \dots, \alpha_{i,n_i}]$ with the standard \mathbb{Z} -grading, i.e. $\deg(\alpha_{i,j})$ equals 1 for all $j \in \{0, \dots, n_i\}$. The irrelevant ideal $B(\Sigma_i)$ of \mathbb{P}^{n_i} is $B(\Sigma_i) = (\alpha_{i,0}, \dots, \alpha_{i,n_i})$. The irrelevant ideal of X is $B(\Sigma_X) = B(\Sigma_1) \cdots B(\Sigma_k) = (S[X]_{(1,1,\dots,1)})$.

We use the following lemma about Hilbert functions of quotient algebras of homogeneous ideals in $S[X]$.

Lemma 4.24. *Let $I \neq (1)$ be a \mathbb{Z}^k -graded ideal in $S[X]$. If I is saturated with respect to the irrelevant ideal $B(\Sigma_X)$, then for all $i \in \{1, \dots, k\}$ there is a homogeneous element $\gamma_i \in S[X]_{\mathbf{e}_i}$ such that γ_i is a nonzerodivisor on $S[X]/I$. Therefore:*

(i) *For all $\mathbf{u} \in \mathbb{Z}^k$ and for all $i \in \{1, 2, \dots, k\}$ we have $H_{S[X]/I}(\mathbf{u}) \leq H_{S[X]/I}(\mathbf{u} + \mathbf{e}_i)$.*

(ii) *Let $\mathbf{u} \in \mathbb{Z}_{\geq 0}^k$ and assume that there is $i \in \{1, 2, \dots, k\}$ with*

$$H_{S[X]/I}(\mathbf{u}) = H_{S[X]/I}(\mathbf{u} + \mathbf{e}_i).$$

We have $H_{S[X]/I}(\mathbf{u} + \mathbf{e}_i) = H_{S[X]/I}(\mathbf{u} + 2\mathbf{e}_i)$.

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the associated primes of $S[X]/I$. These are \mathbb{Z}^k -graded ideals. We claim that for each i , the \mathbb{C} -vector subspace $\bigcup_{j=1}^s (\mathfrak{p}_j)_{\mathbf{e}_i} \subseteq S[X]_{\mathbf{e}_i}$ is a proper subspace. Indeed, otherwise, there is $j \in \{1, \dots, s\}$ with $(\mathfrak{p}_j)_{\mathbf{e}_i} = S[X]_{\mathbf{e}_i}$. Therefore, $B(\Sigma_X)$ is contained in \mathfrak{p}_j . We obtain $(I : B(\Sigma_X)) \neq I$ which contradicts the assumption that I is saturated with respect to $B(\Sigma_X)$. Having established the claim, we proceed to the proof of the lemma.

(i) By the above claim, for every $i \in \{1, \dots, k\}$ there is a homogeneous nonzerodivisor γ_i on $S[X]/I$ of degree \mathbf{e}_i . It follows that the map $(S[X]/I)_{\mathbf{u}} \rightarrow (S[X]/I)_{\mathbf{u} + \mathbf{e}_i}$ given by multiplication by γ_i is injective.

(ii) Let $[\Theta] \in (S[X]/I)_{\mathbf{u} + 2\mathbf{e}_i}$. There are $\Theta_0, \dots, \Theta_{n_i} \in S[X]_{\mathbf{u} + \mathbf{e}_i}$ with

$$[\Theta] = [\alpha_{i,0}\Theta_0 + \cdots + \alpha_{i,n_i}\Theta_{n_i}].$$

Using the notation of the proof of part (i), multiplication by γ_i gives an isomorphism $(S[X]/I)_{\mathbf{u}} \rightarrow (S[X]/I)_{\mathbf{u} + \mathbf{e}_i}$. Therefore, there are $\Gamma_0, \dots, \Gamma_{n_i} \in S[X]_{\mathbf{u}}$ with $[\Theta_j] = [\gamma_i \Gamma_j]$ for all $j \in \{0, \dots, n_i\}$. It follows that we have $[\Theta] = \gamma_i([\alpha_{i,0}\Gamma_0 + \cdots + \alpha_{i,n_i}\Gamma_{n_i}])$. Thus, the injective map $(S[X]/I)_{\mathbf{u} + \mathbf{e}_i} \rightarrow (S[X]/I)_{\mathbf{u} + 2\mathbf{e}_i}$ given by multiplication by γ_i is in fact bijective. \square

We present a necessary condition for $[I] \in \text{Hilb}_{S[X]}^{h_{r,X}}$ to be in the irreducible component $\text{Slip}_{r,X}$.

Theorem 4.25. *Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ for some positive integers $k \geq 2, n_1, \dots, n_k$. For all $i \in \{1, \dots, k\}$ let $B(\Sigma_i) \subseteq S[X]$ be the extension of the irrelevant ideal of \mathbb{P}^{n_i} under the natural inclusion $S[\mathbb{P}^{n_i}] \rightarrow S[X]$. If $[I]$ is a point of $\text{Slip}_{r,X}$ for some positive integer r , then*

$$\dim_{\mathbb{C}} \text{Hom}_{S[X]} \left(I + B(\Sigma_i)^2, S[X]/(I + B(\Sigma_i)^2) \right)_0 \geq r(n_1 + \cdots + n_k)$$

holds for all $i \in \{1, \dots, k\}$.

Proof. Assume that $[I]$ belongs to $\text{Hilb}_{S[X]}^{h_{r,X}}$ where $I \subseteq S[X]$ is an ideal saturated with respect to $B(\Sigma_X)$. Fix an integer $i \in \{1, \dots, k\}$ and let

$$\mathcal{A}_i = \{ \mathbf{u} = (u_1, \dots, u_k) \in \mathbb{Z}^k \mid u_j \geq 0 \text{ for all } j \in \{1, \dots, k\} \text{ and } u_i \in \{0, 1\} \}.$$

Let J be the ideal of $S[X]$ generated by

$$\bigoplus_{\mathbf{u} \in \mathcal{A}_i} I_{\mathbf{u}}.$$

We claim that we have $(J : B(\Sigma_X)^\infty) = I$. We first show how to conclude the proof using the claim. Let g be the Hilbert function of $S[X]/(I + B(\Sigma_i)^2)$ and consider the natural map

$$\chi: \text{Hilb}_{S[X]}^{h_{r,X}} \rightarrow \text{Hilb}_{S[X]}^g$$

given on closed points by $\chi([I]) = [I + B(\Sigma_i)^2]$. It follows from the claim that χ is injective on points corresponding to ideals that are saturated with respect to $B(\Sigma_X)$. Indeed, the inverse map is $[I'] \mapsto [(\bigoplus_{\mathbf{u} \in \mathcal{A}_i} I'_{\mathbf{u}}) : B(\Sigma_X)^\infty]$. Since a general closed point of $\text{Slip}_{r,X}$ corresponds to an ideal of $S[X]$ that is saturated with respect to $B(\Sigma_X)$, it follows that the image of $\text{Slip}_{r,X}$ under χ is of dimension $\dim \text{Slip}_{r,X} = r(n_1 + \cdots + n_k)$. Therefore, if $[I]$ belongs to $\text{Slip}_{r,X}$ then the tangent space to $\text{Hilb}_{S[X]}^g$ at $\chi([I])$ is of dimension at least $r(n_1 + \cdots + n_k)$. Application of Theorem 2.75 finishes the proof of the theorem.

We are left with proving the claim. Let $K = (J : B(\Sigma_X)^\infty)$. Since there are inclusions $J \subseteq K \subseteq I$, it follows that $K_{\mathbf{u}} = J_{\mathbf{u}} = I_{\mathbf{u}}$ holds for all $\mathbf{u} \in \mathcal{A}_i$. Let $\mathbf{u} = (u_1, \dots, u_k) \in \mathcal{A}_i$ with $u_j \geq r$ for all $j \neq i$ and $u_i = 0$. Since we have

$$H_{S[X]/K}(\mathbf{u}) = H_{S[X]/K}(\mathbf{u} + \mathbf{e}_i) = r,$$

it follows from Lemma 4.24(ii) that $H_{S[X]/K}(\mathbf{u})$ equals r for all $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{Z}^k$ with $u_j \geq r$ for all $j \in \{1, 2, \dots, k\}$. Therefore, we get $H_{S[X]/K}(\mathbf{u}) \leq r$ for all $\mathbf{u} \in \mathbb{Z}^k$ by Lemma 4.24(i). Since K is contained in I and $S[X]/I$ has Hilbert function $h_{r,X}$, it follows that the Hilbert function of $S[X]/K$ is also $h_{r,X}$. Thus, we have $K = I$ as claimed. \square

4.6 Small examples of reducible multigraded Hilbert schemes

In this section we show that $\text{Hilb}_{S[X]}^{h_{2,X}}$ need not be irreducible for a smooth projective toric surface X . We present two examples: Hirzebruch surface $\mathcal{H}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ and $\mathbb{P}^1 \times \mathbb{P}^1$. These examples illustrate that the necessary condition described in Theorem 4.15 is in general not

sufficient even for small values of r and $\dim X$. In fact, as explained below, this condition is trivially satisfied in these two cases.

Both special versions of Theorem 4.15 studied in Sections 4.3 and 4.4 apply to multigraded Hilbert scheme $\text{Hilb}_{S[\mathcal{H}_1]}^{h_2, \mathcal{H}_1}$ since \mathcal{H}_1 is also the blowup of \mathbb{P}^2 at a torus invariant point. However, we present also the example of $\text{Hilb}_{S[\mathbb{P}^1 \times \mathbb{P}^1]}^{h_2, \mathbb{P}^1 \times \mathbb{P}^1}$ to demonstrate that Theorem 4.25 gives some insight into $\text{Slip}_{2, \mathbb{P}^1 \times \mathbb{P}^1}$ even though Theorem 4.15 is of no use in this case.

Proposition 4.26. *Let $S[\mathbb{P}^1 \times \mathbb{P}^1] = \mathbb{C}[\alpha_0, \alpha_1, \beta_0, \beta_1]$ be the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ with $\deg(\alpha_0) = \deg(\alpha_1) = (1, 0)$ and $\deg(\beta_0) = \deg(\beta_1) = (0, 1)$. The scheme $\text{Hilb}_{S[\mathbb{P}^1 \times \mathbb{P}^1]}^{h_2, \mathbb{P}^1 \times \mathbb{P}^1}$ is not irreducible. In fact, we have $(\alpha_0\alpha_1, \alpha_0\beta_0, \alpha_1\beta_0, \beta_0\beta_1) \in \text{Hilb}_{S[\mathbb{P}^1 \times \mathbb{P}^1]}^{h_2, \mathbb{P}^1 \times \mathbb{P}^1} \setminus \text{Slip}_{2, \mathbb{P}^1 \times \mathbb{P}^1}$.*

Proof. Let $I = (\alpha_0\alpha_1, \alpha_0\beta_0, \alpha_1\beta_0, \beta_0\beta_1)$. The point $[I]$ belongs to $\text{Hilb}_{S[\mathbb{P}^1 \times \mathbb{P}^1]}^{h_2, \mathbb{P}^1 \times \mathbb{P}^1}$. We claim that it is not in $\text{Slip}_{2, \mathbb{P}^1 \times \mathbb{P}^1}$. Let $\mathfrak{a} = (\alpha_0, \alpha_1)$ and $J = I + \mathfrak{a}^2$. The \mathbb{C} -vector space $\text{Hom}_{S[\mathbb{P}^1 \times \mathbb{P}^1]}(J, S[\mathbb{P}^1 \times \mathbb{P}^1]/J)_0$ has dimension 2. Thus, we get $[I] \notin \text{Slip}_{2, \mathbb{P}^1 \times \mathbb{P}^1}$ by Theorem 4.25. \square

The case of the Hirzebruch surface \mathcal{H}_1 is more involved since we lack a criterion analogous to Theorem 4.25.

Proposition 4.27. *Let $\mathcal{H}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ be the Hirzebruch surface. The multigraded Hilbert scheme $\text{Hilb}_{S[\mathcal{H}_1]}^{h_2, \mathcal{H}_1}$ is not irreducible.*

Proof. We start with calculating the Cox ring of \mathcal{H}_1 . Let $S[\mathbb{P}^2] = \mathbb{C}[\beta_0, \beta_1, \beta_2]$ with $\deg(\beta_0) = \deg(\beta_1) = \deg(\beta_2) = 1$. The Hirzebruch surface \mathcal{H}_1 can be constructed as the blowup of \mathbb{P}^2 at the torus invariant point $[0 : 1 : 0]$.

The Cox ring $S[\mathcal{H}_1]$ of \mathcal{H}_1 is $\mathbb{C}[\alpha_0, \alpha_1, \alpha_2, \alpha_3]$ with $\deg(\alpha_0) = \deg(\alpha_2) = (1, -1)$, $\deg(\alpha_1) = (1, 0)$ and $\deg(\alpha_3) = (0, 1)$ (see Section 4.3). Moreover, by Proposition 4.21 the graded homomorphism of graded rings $S[\mathbb{P}^2] \rightarrow S[\mathcal{H}_1]$ given by $\beta_0 \mapsto \alpha_0\alpha_3$, $\beta_1 \mapsto \alpha_1$, $\beta_2 \mapsto \alpha_2\alpha_3$ is a lift of the natural map $\mathcal{H}_1 \rightarrow \mathbb{P}^2$. We identify $S[\mathbb{P}^2]$ with its image in $S[\mathcal{H}_1]$.

Let W be the locus of those points $[I]$ of $\text{Hilb}_{S[\mathbb{P}^2]}^{h_2, \mathbb{P}^2}$ for which the unique linear generator of I is of the form $a\alpha_0\alpha_3 + b\alpha_2\alpha_3$ for some $a, b \in \mathbb{C}$ (or geometrically, the locus of all points defining subschemes contained in a line passing through the center of the blowup). We claim that W is irreducible and 3-dimensional. Indeed, it is a $\text{Hilb}_{S[\mathbb{P}^1]}^{h_2, \mathbb{P}^1}$ -bundle over the projective line $\mathbb{P}(\text{lin}\{\alpha_0\alpha_3, \alpha_2\alpha_3\})$. Since $\text{Hilb}_{S[\mathbb{P}^1]}^{h_2, \mathbb{P}^1}$ is isomorphic to \mathbb{P}^2 (see Proposition 3.38), our claim follows.

Let $\pi: \text{Hilb}_{S[\mathcal{H}_1]}^{h_2, \mathcal{H}_1} \rightarrow \text{Hilb}_{S[\mathbb{P}^2]}^{h_2, \mathbb{P}^2}$ be the natural map from Theorem 4.15. We claim that the set-theoretic inverse image V of W is of dimension at least 4. Let $[I] \in W$ be a closed point. We may assume that we have

$$I = (\alpha_0\alpha_3, A\alpha_2^2\alpha_3^2 + B\alpha_1\alpha_2\alpha_3 + C\alpha_1^2)$$

for some $A, B, C \in \mathbb{C}$, not all zero. It is enough to show that the fiber over $[I]$ is of positive dimension. Let $[a : b] \in \mathbb{P}^1$. We claim that

$$[J] = [(\alpha_0\alpha_3, A\alpha_2^2\alpha_3^2 + B\alpha_1\alpha_2\alpha_3 + C\alpha_1^2, \alpha_0(a\alpha_0 + b\alpha_2), \alpha_0\alpha_1)]$$

is a point of that fiber. We need to check two things:

1. we have $J \cap S[\mathbb{P}^2] = I$;

2. $S[\mathcal{H}_1]/J$ has Hilbert function h_{2,\mathcal{H}_1} .

We start with 1. Clearly, I is contained in $J \cap S[\mathbb{P}^2]$. Suppose that f is an element of $(\alpha_0(a\alpha_0 + b\alpha_2), \alpha_0\alpha_1)_{(d,0)}$ for some positive integer d . We show that f belongs to $(\alpha_0\alpha_3)$. Observe that we have $\deg(\alpha_0(a\alpha_0 + b\alpha_2)) = (2, -2)$ and $\deg(\alpha_0\alpha_1) = (2, -1)$. Since $\deg(f)$ equals $(d, 0)$, we get that α_3 divides f . This shows that 1 is fulfilled.

Now we show that 2 holds. The ideal J is contained in $K = (\alpha_0, A\alpha_2^2\alpha_3^2 + B\alpha_1\alpha_2\alpha_3 + C\alpha_1^2)$. Moreover, if $J_{(a,b)}$ and $K_{(a,b)}$ are not equal to each other then we have $b = -a < 0$ and $\dim_{\mathbb{C}}(S[\mathcal{H}_1]/K)_{(a,-a)} = \dim_{\mathbb{C}}(S[\mathcal{H}_1]/J)_{(a,-a)} - 1$. Thus, it is enough to establish the following equality:

$$\dim_{\mathbb{C}}(S[\mathcal{H}_1]/K)_{(a,b)} = \begin{cases} h_{2,\mathcal{H}_1}(a,b) - 1 = 1 & \text{if } b = -a < 0; \\ h_{2,\mathcal{H}_1}(a,b) & \text{otherwise.} \end{cases}$$

We can rewrite this as follows:

$$\dim_{\mathbb{C}}(S[\mathcal{H}_1]/K)_{(a,b)} = \begin{cases} 0 & \text{if } a + b < 0 \text{ or } a < 0; \\ 1 & \text{if } a + b = 0 \text{ and } a \geq 0; \\ 1 & \text{if } a = 0 \text{ and } b \geq 0; \\ 2 & \text{if } a + b > 0 \text{ and } a > 0. \end{cases} \quad (4.28)$$

Let $R = \mathbb{C}[\alpha_1, \alpha_2, \alpha_3] \subseteq S[\mathcal{H}_1]$. We have

$$\dim_{\mathbb{C}} R_{(a,b)} = \begin{cases} 0 & \text{if } a + b < 0 \text{ or } a < 0; \\ a + b + 1 - \max\{0, b\} & \text{otherwise.} \end{cases} \quad (4.29)$$

Indeed, the case $a + b < 0$ or $a < 0$ is clear since there are equalities $\deg(\alpha_1) = (1, 0)$, $\deg(\alpha_2) = (1, -1)$ and $\deg(\alpha_3) = (0, 1)$. On the other hand, if we have $a + b \geq 0$ and $a \geq 0$ then $R_{(a,b)}$ is spanned by

$$\{\alpha_1^{a+b-c}\alpha_2^{c-b}\alpha_3^c \mid \max\{0, b\} \leq c \leq a + b\}.$$

There are equalities

$$\dim_{\mathbb{C}}(S[\mathcal{H}_1]/K)_{(a,b)} = \dim_{\mathbb{C}}(R/(A\alpha_2^2\alpha_3^2 + B\alpha_1\alpha_2\alpha_3 + C\alpha_1^2))_{(a,b)} = \dim_{\mathbb{C}} R_{(a,b)} - \dim_{\mathbb{C}} R_{(a-2,b)}. \quad (4.30)$$

Equations (4.29) and (4.30) imply Equation (4.28) and thus, finish the proof that V is of dimension at least 4.

If $\text{Hilb}_{S[\mathcal{H}_1]}^{h_{2,\mathcal{H}_1}}$ is irreducible, then it is of dimension $\dim \text{Slip}_{2,\mathcal{H}_1} = 4$. It follows that we have $V = \text{Hilb}_{S[\mathcal{H}_1]}^{h_{2,\mathcal{H}_1}}$ set-theoretically. This contradicts Theorem 4.15 since W is not equal to $\text{Hilb}_{S[\mathbb{P}^2]}^{h_{2,\mathbb{P}^2}} = \text{Slip}_{2,\mathbb{P}^2}$. \square

We conclude this chapter with a remark.

Remark 4.31. In this section we considered two examples of toric morphisms $f: X \rightarrow Y$ between smooth projective toric varieties with $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Hence Theorem 4.15 applies to these cases. However, we have $Y = \mathbb{P}^1$ or \mathbb{P}^2 (depending on example) and $r = 2$. Therefore, $\text{Hilb}_{S[Y]}^{h_{2,Y}}$ equals $\text{Slip}_{2,Y}$ (see Proposition 3.39). Thus, the necessary condition from Theorem 4.15 is trivially satisfied. Nevertheless, we have $\text{Slip}_{2,X} \neq \text{Hilb}_{S[X]}^{h_{2,X}}$ in both cases.

Chapter 5

Applications of border apolarity to secant varieties

In this chapter we present some applications of the border apolarity lemma (Proposition 2.92) for studying secant varieties. Section 5.1 deals with existence of homogeneous wild polynomials, i.e. polynomials whose border rank is strictly smaller than the smoothable rank. In Subsection 5.1.1 we show that there are no wild degree d polynomials in three variables of border rank at most $d + 2$. In Subsection 5.1.2 we prove that there is no wild quartic in four variables of border rank at most 6. In Subsection 5.1.3 we give an example of a wild quintic in four variables of border rank 7. In Subsection 5.1.4 we show that the known example of a wild cubic of border rank 5 in five variables (see [12, Thm. 1.3]) is a unique such example up to a change of variables. Results from Section 5.1 depend on the criteria developed in Chapter 3. Subsections 5.1.1, 5.1.2 and 5.1.4 are based on [67].

Sections 5.2, 5.3 and 5.4 are based on [40]. This paper uses Proposition 2.92 in a simple form. Namely, we do not use any criteria for distinguishing $\text{Slip}_{r,n}$ from $\text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,n}}$. Therefore, these results are not directly related to the rest of the thesis. Consequently, we present them in special versions, where the proofs are simpler. In Section 5.2 we calculate cactus and border cactus rank of a homogeneous subspace of a divided power ring that is divisible by a large power of a linear form. This is then used in Section 5.3 to describe the points in $\kappa_{14}(\nu_d(\mathbb{P}^6)) \setminus \sigma_{14}(\nu_d(\mathbb{P}^6))$ for all $d \geq 7$. Results from Section 5.3 have their analogues for $\kappa_{8,3}(\nu_d(\mathbb{P}^4)) \setminus \sigma_{8,3}(\nu_d(\mathbb{P}^4))$. In Section 5.4 we state the main theorem in that direction.

5.1 Wild polynomials

In this section we assume that the base field \mathbb{k} is the field of complex numbers \mathbb{C} since we cite results from papers in which this is assumed. Let n be a positive integer and $S = \mathbb{C}[\alpha_0, \dots, \alpha_n]$ be the polynomial ring with the standard \mathbb{Z} -grading. We consider the dual polynomial ring $S^* = \mathbb{C}[x_0, \dots, x_n]$ with the structure of an S -module on S^* given by partial differentiation. We denote this action by \lrcorner . Given a homogeneous polynomial $F \in S^*$ we denote by $\text{Ann}(F)$ the ideal $\{\theta \in S \mid \theta \lrcorner F = 0\}$.

We use the following consequence of the border apolarity lemma (Proposition 2.92).

Corollary 5.1. *Let d be a positive integer and $F \in S_d^*$. Assume that $\text{br}(F) \leq r < \text{cr}(F)$ holds for some integer r . There exists an ideal $I \subseteq \text{Ann}(F)$ with $[I] \in \text{Slip}_{r,n}$ and $\bar{I}_d \neq I_d$.*

Proof. By Proposition 2.92, there is an ideal $[I] \in \text{Slip}_{r,n}$ with $I \subseteq \text{Ann}(F)$. If we have $\bar{I}_d = I_d$, then there is an inclusion $\bar{I} \subseteq \text{Ann}(F)$ by [11, Prop. 3.4]. Thus, $\text{cr}(F) \leq r$ follows from Proposition 2.91. This is a contradiction. \square

We always have $\text{br}(F) \leq \text{sr}(F)$ (see [56, Lem. 5.17]). Recall that we say that F is wild, if the inequality is strict. Wild polynomials are more difficult to control using standard, existing methods. Therefore, new methods need to be developed in order to study them effectively. For example, see [3, Prop. 11] and its applications, [19, Rmk. 1.5] and [40].

We study wild homogeneous polynomials F satisfying $\text{br}(F) \leq \text{deg}(F) + 2$. If the inequality $\text{br}(F) \leq \text{deg}(F) + 1$ holds then F is not wild. This is established in [11, Prop. 2.5] based on a result in [3]. Therefore, we assume that we have $\text{br}(F) = \text{deg}(F) + 2$.

We use the following observation.

Lemma 5.2. *Let $r \geq 2$ be an integer and $[I] \in \text{Hilb}_{S[\mathbb{P}^n]}^{h_{r,n}}$ be a closed point. If we have $\bar{I}_{r-2} \neq I_{r-2}$ then S/\bar{I} has Hilbert function $h_{r,1}$.*

Proof. Let g be the Hilbert function of S/\bar{I} . The Hilbert polynomial of S/\bar{I} is r . Therefore, by Lemma 2.9(ii) we get $g(r-2) \leq r-1$. Lemma 2.9(iii) implies that we have $g(0) = 1 < g(1) < g(2) < \dots < g(r-2) \leq r-1$. It follows that $g(a) = a+1$ holds for every $a \in \{0, \dots, r-2\}$. Using Lemma 2.9 again we obtain the equalities $g(a) = r$ for all $a \geq r-1$. Thus, g equals $h_{r,1}$. \square

5.1.1 Polynomials in three variables of small border rank

In this subsection we prove that there is no wild homogeneous polynomial F in three variables of border rank at most $\text{deg}(F) + 2$.

We start with the following observation.

Lemma 5.3. *Let d be a positive integer and let $e = \lceil \frac{d+1}{2} \rceil$. Let $H_{d-1} \in \mathbb{C}[x_1, x_2]_{d-1}$ and $H_d \in \mathbb{C}[x_1, x_2]_d$ be homogeneous polynomials. There exists an element $\alpha_0 \xi_{e-1} + \xi_e \in \text{Ann}(x_0 H_{d-1} + H_d)$ with $\xi_{e-1} \in \mathbb{C}[\alpha_1, \alpha_2]_{e-1}$, $\xi_e \in \mathbb{C}[\alpha_1, \alpha_2]_e$ and $\xi_e \neq 0$.*

Proof. Let $T^* = \mathbb{C}[x_1, x_2]$ and $T = \mathbb{C}[\alpha_1, \alpha_2]$. We consider the restriction of the action of $S = \mathbb{C}[\alpha_0, \alpha_1, \alpha_2]$ on $S^* = \mathbb{C}[x_0, x_1, x_2]$ to an action of T on T^* . If H is an element of T^* , we write $\text{Ann}_T(H)$ if we compute the annihilator ideal with respect to the T action.

Let $F = x_0 H_{d-1} + H_d$. We have $(\alpha_0 \xi_{e-1} + \xi_e) \lrcorner F = x_0 (\xi_e \lrcorner H_{d-1}) + (\xi_{e-1} \lrcorner H_{d-1} + \xi_e \lrcorner H_d)$. Therefore, we need to choose $\xi_e \in \text{Ann}_T(H_{d-1})$. If there exists a non-zero $\xi_e \in \text{Ann}_T(H_{d-1}) \cap \text{Ann}_T(H_d)$ we can set $\xi_{e-1} = 0$ and we are done.

Otherwise, let h be the Hilbert function of $T/\text{Ann}_T(H_{d-1})$. The \mathbb{C} -vector space

$$\text{lin}\{\xi_e \lrcorner H_d \mid \xi_e \in \text{Ann}_T(H_{d-1})_e\}$$

has dimension $e+1-h(e)$. On the other hand the vector space

$$\text{lin}\{\xi_{e-1} \lrcorner H_{d-1} \mid \xi_{e-1} \in T_{e-1}\}$$

is of dimension $h(e-1)$. It is enough to show that these two vector subspaces of T_{d-e}^* have a non-zero intersection. It suffices to establish the inequality

$$e+1-h(e)+h(e-1) \geq d-e+2. \quad (5.4)$$

By the definition of e we have $d + 1 \leq 2e$. We claim that $h(e - 1) - h(e)$ is non-negative. If $\text{Ann}_T(H_{d-1})_{e-1}$ is non-zero, then the inequality $h(e - 1) - h(e) \geq 0$ follows from Lemma 2.12. On the other hand, if we have $\text{Ann}_T(H_{d-1})_{e-1} = 0$ and $h(e) > h(e - 1)$, then $\text{Ann}_T(H_{d-1})_e$ is zero. Since $T/\text{Ann}_T(H_{d-1})$ is Gorenstein, we get $h(d - 1 - e) = h(e) = e + 1$. This gives a contradiction with $d + 1 \leq 2e$. These remarks imply Equation (5.4). \square

Now we present the main result of this subsection.

Proposition 5.5. *Let $S = \mathbb{C}[\alpha_0, \alpha_1, \alpha_2]$ be a polynomial ring with dual ring $S^* = \mathbb{C}[x_0, x_1, x_2]$. Let $F \in S_d^*$ be a non-zero polynomial for some $d \in \mathbb{Z}_{>0}$. If the border rank of F is at most $d + 2$, then we have $\text{cr}(F) = \text{sr}(F) = \text{br}(F)$.*

Proof. By [11, Prop. 2.5] we may assume that $\text{br}(F)$ equals $d + 2$. Furthermore, $\text{sr}(F) = \text{cr}(F)$ holds since $\mathcal{Hilb}_r(\mathbb{P}^2)$ is irreducible for every positive integer r . If we have $\text{cr}(F) \leq d + 2$, it follows from

$$d + 2 = \text{br}(F) \leq \text{sr}(F) = \text{cr}(F) \leq d + 2$$

that there are equalities $\text{cr}(F) = \text{sr}(F) = \text{br}(F)$. Assume that we have $\text{cr}(F) > d + 2$. From Corollary 5.1 we obtain that there is an ideal $I \subseteq \text{Ann}(F)$ with $[I] \in \text{Slip}_{d+2,2}$ and $I_d \neq \bar{I}_d$. From Lemma 5.2 we get that S/\bar{I} has Hilbert function $h_{d+2,1}$. We may assume that \bar{I} equals $(\alpha_0, F_{d+2}(\alpha_1, \alpha_2))$. It follows from Theorem 3.66 that $(\alpha_0^2) \cdot (\alpha_0, \alpha_1, \alpha_2)^{d-2}$ is contained in $I \subseteq \text{Ann}(F)$. Thus, α_0^2 is in $\text{Ann}(F)$ and consequently, F is of the form $F = x_0 H_{d-1} + H_d$ with $H_{d-1} \in \mathbb{C}[x_1, x_2]_{d-1}$ and $H_d \in \mathbb{C}[x_1, x_2]_d$.

Let $e = \lceil \frac{d+1}{2} \rceil$. By Lemma 5.3, there is an element $\eta_e = \alpha_0 \xi_{e-1} + \xi_e \in \text{Ann}(F)$ with $\xi_{e-1} \in \mathbb{C}[\alpha_1, \alpha_2]_{e-1}$ and non-zero $\xi_e \in \mathbb{C}[\alpha_1, \alpha_2]_e$. Let $J = (\alpha_0^2, \eta_e)$. We have $J \subseteq \text{Ann}(F)$ and S/J has Hilbert polynomial $2e \leq d + 2$. Moreover, J is saturated. Indeed, let $>$ be the lex order with $\alpha_2 > \alpha_1 > \alpha_0$ and let J' be the initial ideal of J with respect to the order $>$. We have $J' = (\alpha_0^2, M)$ where M belongs to $\{\alpha_1^e, \alpha_1^{e-1} \alpha_2, \dots, \alpha_2^e\}$. In particular, J' is saturated. Thus, so is J by Lemma 2.7. It follows from Proposition 2.91 that we have $\text{cr}(F) \leq d + 2$. \square

Remark 5.6. In [3, p. 37] in the paragraph above Remark 13, there is an example that suggests that there could exist a wild polynomial in $\sigma_8(\nu_6(\mathbb{P}^2))$. Proposition 5.5 shows that there is no such polynomial.

In the context of Proposition 5.5, there is the following natural question.

Problem 5.7. Does there exist a homogeneous polynomial $F \in \mathbb{C}[x_0, x_1, x_2]$ with $\text{br}(F) \neq \text{sr}(F)$? If it does, what is the smallest possible degree of such a polynomial?

It follows from Proposition 5.5 that if there exists a wild polynomial $F \in \mathbb{C}[x_0, x_1, x_2]_d$, then we have $d \geq 6$ since otherwise $\sigma_{d+2}(\nu_d(\mathbb{P}^2))$ is equal to $\mathbb{P}^{\binom{2+d}{d}-1}$ by the Alexander-Hirschowitz theorem [7, Thm. 1.2].

5.1.2 Quartics in four variables of small border rank

There are no wild cubics in four variables [12, Thm. 1.3]. In this subsection we prove that there are no wild homogeneous quartics in four variables of border rank at most 6.

Proposition 5.8. *Let $S = \mathbb{C}[\alpha_0, \dots, \alpha_3]$ be a polynomial ring with dual ring $S^* = \mathbb{C}[x_0, \dots, x_3]$. Let $F \in S_4^*$ be non-zero. If the border rank of F is at most 6, then we have $\text{cr}(F) = \text{sr}(F) = \text{br}(F)$.*

Proof. By [11, Prop. 2.5] we may assume that $\text{br}(F)$ equals 6. Since $\mathcal{Hilb}_6(\mathbb{P}^3)$ is irreducible (see [20, Thm. 1.1]), it is enough to show that we have $\text{cr}(F) \leq 6$. Assume that it does not hold.

By Corollary 5.1 there is an ideal $[I] \in \text{Slip}_{6,3}$ with $I \subseteq \text{Ann}(F)$ and $I_4 \neq \bar{I}_4$. It follows from Lemma 5.2 that S/\bar{I} has Hilbert function $h_{6,1}$. We may assume that \bar{I}_1 is equal to $(\alpha_0, \alpha_1)_1$. By Theorem 3.66 we get $(\alpha_0, \alpha_1)^2 \subseteq \text{Ann}(F)$. Thus, we may restrict our attention to the case that we have $F = x_0C_1 + x_1C_2 + D$ with $C_1, C_2 \in \mathbb{C}[x_2, x_3]_3$ and $D \in \mathbb{C}[x_2, x_3]_4$.

There is a polynomial $\theta \in \mathbb{C}[\alpha_2, \alpha_3]_3$ satisfying $\theta \lrcorner C_1 = \theta \lrcorner C_2 = 0$. By a linear change of variables in $\mathbb{C}[\alpha_2, \alpha_3]$ we may assume that θ is one of the following:

1. $\theta = \alpha_2^3$;
2. $\theta = \alpha_2^2\alpha_3$;
3. $\theta = \alpha_2\alpha_3(\alpha_2 - \alpha_3)$.

We study this case by case. We further simplify F by a linear change of variables and in each case we find a homogeneous ideal $J \subseteq \text{Ann}(F)$ whose initial ideal with respect to the lex order with $\alpha_2 > \alpha_3 > \alpha_1 > \alpha_0$ is saturated and the Hilbert polynomial of the corresponding quotient algebra is 6. Thus, $\text{cr}(F) \leq 6$ is a consequence of Lemma 2.7 and Proposition 2.91. In each case the given set of generators of J is a Gröbner basis. We may assume that $\text{Ann}(F)_1$ is zero by Proposition 5.5 and [12, §3.1].

We start with case 1. Up to a linear change of variables in S^* we have one of the cases:

- 1.A $F = x_0(x_2^2x_3 + ax_3^3) + x_1x_2x_3^2 + Q$ with $a \in \mathbb{C}$ and $Q \in \mathbb{C}[x_2, x_3]_4$;
- 1.B $F = x_0(x_2^2x_3 + ax_2x_3^2) + x_1x_3^3 + Q$ with $a \in \mathbb{C}$ and $Q \in \mathbb{C}[x_2, x_3]_4$;
- 1.C $F = x_0x_2x_3^2 + x_1x_3^3 + Q$ with $Q \in \mathbb{C}[x_2, x_3]_4$.

Let $\alpha_2^3 \lrcorner Q = Ax_2 + Bx_3$. Corresponding to the above cases, the following ideals contained in $\text{Ann}(F)$ show that the cactus rank of F is at most 6:

- 1.A $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_2 - \alpha_1\alpha_3, \alpha_1\alpha_2^2, \alpha_2^3 - \frac{A}{2}\alpha_0\alpha_2\alpha_3 - \frac{B}{2}\alpha_1\alpha_2\alpha_3)$;
- 1.B $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_1\alpha_2, \alpha_0\alpha_2^2 - \frac{1}{3}\alpha_1\alpha_2^3, \alpha_2^3 - \frac{A}{2}\alpha_0\alpha_2\alpha_3 + \frac{aA-B}{6}\alpha_1\alpha_2^3)$;
- 1.C $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_1\alpha_2, \alpha_0\alpha_2^2, \alpha_2^3 - \frac{A}{2}\alpha_0\alpha_2^3 - \frac{B}{6}\alpha_1\alpha_2^3)$.

Now we consider case 2, namely we assume that $\alpha_2^2\alpha_3 \lrcorner C_1 = \alpha_2^2\alpha_3 \lrcorner C_2$ is zero. Up to a linear change of variables in S^* and excluding possibilities already considered in case 1, we have one of the cases:

- 2.A $F = x_0(x_2^3 + ax_3^3) + x_1(x_2x_3^2 + bx_3^3) + Q$ with $a, b \in \mathbb{C}$ and $Q \in \mathbb{C}[x_2, x_3]_4$;
- 2.B $F = x_0(x_2^3 + ax_2x_3^2) + x_1x_3^3 + Q$ with $a \in \mathbb{C}$ and $Q \in \mathbb{C}[x_2, x_3]_4$.

Let $\alpha_2^2\alpha_3 \lrcorner Q = Ax_2 + Bx_3$. Corresponding to the above cases, the following ideals contained in $\text{Ann}(F)$ show that the cactus rank of G is at most 6:

- 2.A If a is non-zero, take $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, a\alpha_1\alpha_2 - \frac{1}{3}\alpha_0\alpha_3, \alpha_0\alpha_2\alpha_3, \alpha_2^2\alpha_3 - \frac{A}{6}\alpha_0\alpha_2^2 - \frac{B}{2}\alpha_1\alpha_2\alpha_3)$.
If a is zero, take $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_3, \alpha_1\alpha_2^2, \alpha_2^2\alpha_3 - \frac{A}{6}\alpha_0\alpha_2^2 - \frac{B}{2}\alpha_1\alpha_2\alpha_3)$;
- 2.B Take $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_1\alpha_2, \alpha_0\alpha_2\alpha_3 - \frac{a}{3}\alpha_1\alpha_2^3, \alpha_2^2\alpha_3 - \frac{A}{6}\alpha_0\alpha_2^2 - \frac{B}{6}\alpha_1\alpha_2^3)$.

Finally we consider case 3, that is we assume that $\alpha_2\alpha_3(\alpha_2 - \alpha_3)\lrcorner C_1 = \alpha_2\alpha_3(\alpha_2 - \alpha_3)\lrcorner C_2$ is zero. Up to a linear change of variables in S^* and excluding possibilities considered in case 1, we have one of the cases:

3.A $F = x_0(x_2^3 + ax_3^3) + x_1(x_2^2x_3 + x_2x_3^2 + bx_3^3) + Q$ with $a, b \in \mathbb{C}$ and $Q \in \mathbb{C}[x_2, x_3]_4$;

3.B $F = x_0(x_2^3 + ax_2^2x_3 + ax_2x_3^2) + x_1x_3^3 + Q$ with $a \in \mathbb{C}$ and $Q \in \mathbb{C}[x_2, x_3]_4$.

Let $(\alpha_2^2\alpha_3 - \alpha_2\alpha_3^2)\lrcorner Q = Ax_2 + Bx_3$. Corresponding to the above cases, the following ideals show that the cactus rank of G is at most 6:

3.A If a is non-zero, take $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, a\alpha_1\alpha_2 + \frac{a}{3}\alpha_0\alpha_2 - a\alpha_1\alpha_3 + (b - \frac{1}{3})\alpha_0\alpha_3, \alpha_0\alpha_2\alpha_3, \alpha_2^2\alpha_3 - \alpha_2\alpha_3^2 - \frac{A}{6}\alpha_0\alpha_2^2 - \frac{B}{2}\alpha_1\alpha_2^2)$.

If a is zero, take $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_3, \alpha_1\alpha_2^2 + \frac{1}{3}\alpha_0\alpha_2^2 - \alpha_1\alpha_2\alpha_3, \alpha_2^2\alpha_3 - \alpha_2\alpha_3^2 - \frac{A}{6}\alpha_0\alpha_2^2 - \frac{B}{2}\alpha_1\alpha_2^2)$;

3.B If a is zero, then we are in case 2.B. Therefore, assume that a is non-zero. Take

$$J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_1\alpha_2, \alpha_0\alpha_2\alpha_3 - \alpha_0\alpha_3^2 - \frac{a}{3}\alpha_1\alpha_3^2, \alpha_2^2\alpha_3 - \alpha_2\alpha_3^2 - \frac{A}{6}\alpha_0\alpha_2^2 + \frac{aA - 3B}{18}\alpha_1\alpha_3^2).$$

□

5.1.3 Wild quintic in four variables of border rank 7

In Proposition 5.8 we showed that there are no wild quartics in four variables of border rank 6. In this subsection, we give an example of a wild quintic in four variables of border rank 7.

Proposition 5.9. *Let $S = \mathbb{C}[\alpha_0, \alpha_1, \alpha_2, \alpha_3]$ be polynomial ring with graded dual ring $S^* = \mathbb{C}[x_0, x_1, x_2, x_3]$. Let $F = x_0x_2^4 + x_0x_2^3x_3 + x_1x_2^2x_3^2 + x_1x_3^4$. We have $\text{br}(F) = 7$ and $\text{cr}(F) > 7$. Thus, F is wild.*

Proof. The Hilbert scheme $\mathcal{Hilb}_7(\mathbb{P}^3)$ is irreducible by [20, Thm. 1.1]. Therefore, there is an equality $\text{sr}(F) = \text{cr}(F)$ so it is enough to show that we have $\text{br}(F) = 7 < \text{cr}(F)$. Furthermore, by [11, Prop. 2.5] it suffices to show that the inequalities $\text{br}(F) \leq 7 < \text{cr}(F)$ hold.

We have $(\alpha_0, \alpha_1)^2 \subseteq \text{Ann}(F)$. If J is equal to $(\text{Ann}(F)_{\leq 3}) + (\alpha_2^7)$, then $[J]$ is in $\text{Hilb}_S^{h_{7,3}}$ and it follows from Theorem 3.66 that there is an ideal $[J'] \in \text{Slip}_{7,3}$ with $(J')_{\geq 5} = J_{\geq 5}$. In particular, we have $J'_5 = J_5 \subseteq \text{Ann}(F)_5$ so J' is contained in $\text{Ann}(F)$. The inequality $\text{br}(F) \leq 7$ follows from Proposition 2.92.

Now we show that we have $\text{cr}(F) > 7$. Otherwise, by Proposition 2.91 there exists a homogeneous, saturated ideal $K \subseteq \text{Ann}(F)$ such that S/K has Hilbert polynomial 7. Since $H_{S/\text{Ann}(F)}(a) = h_{7,3}(a)$ holds for all $a \leq 3$, we have $K_{\leq 3} = \text{Ann}(F)_{\leq 3}$. We get $(\alpha_0, \alpha_1) = \overline{(\text{Ann}(F)_{\leq 3})} \subseteq K$. This is a contradiction since $K_1 = \text{Ann}(F)_1$ is zero.

The claims that $[J]$ is in $\text{Hilb}_S^{h_{7,3}}$ and that we have $(\alpha_0, \alpha_1) = \overline{(\text{Ann}(F)_{\leq 3})}$ were checked in Macaulay2 [43] using the following code:

```
S=QQ[x_0..x_3];
F=x_0*x_2^4+x_0*x_2^3*x_3+x_1*x_2^2*x_3^2+x_1*x_3^4;
Ann=inverseSystem(F);
AnnUpTo3 = ideal (select ((entries mingens Ann)#0, (i) -> (degree i)#0 <= 3));
J=AnnUpTo3 + ideal(x_2^7);
```

hilbertSeries(J, Order=>12)
 saturate(AnnUpTo3).

□

5.1.4 Cubics in five variables of minimal border rank

In this subsection we let $S = \mathbb{C}[\alpha_0, \dots, \alpha_4]$ be a polynomial ring and $S^* = \mathbb{C}[x_0, \dots, x_4]$ be the dual ring. Let $C \in S^*$ be a homogeneous polynomial of degree 3. We say that C is concise if $\text{Ann}(C)_1$ is zero. It is known that there exists a wild concise cubic in S^* of border rank 5 (see [12, Thm. 1.3]) and that a concise cubic in S^* of border rank 5 is wild if and only if its Hessian is zero (see [53, Thm. 4.9]).

Using Theorem 3.66 we obtain in a simple way that up to a linear change of variables, the cubic given in [12, Thm. 1.3] is the unique wild cubic in S^* of border rank 5.

Proposition 5.10. *Let $S = \mathbb{C}[\alpha_0, \dots, \alpha_4]$ be a polynomial ring with graded dual ring $S^* = \mathbb{C}[x_0, \dots, x_4]$. Up to a linear change of variables, the cubic*

$$x_0x_3^2 - x_1(x_3 + x_4)^2 + x_2x_4^2 \quad (5.11)$$

is the unique wild cubic in S^ of border rank 5.*

Proof. Let $C \in S$ be a wild cubic of border rank 5. By [12, Thm. 1.3] we may assume that C is concise. By Proposition 2.92 there is an ideal $I \subseteq \text{Ann}(C)$ with $[I] \in \text{Slip}_{5,4}$. If the Hilbert function of S/\bar{I} is not $h_{5,1}$, then we have $I_3 = \bar{I}_3$ by Lemma 5.2. Thus, $\text{cr}(C) \leq 5$ holds by Corollary 5.1. Consequently, C is not wild since there is an equality $\text{cr}(C) = \text{sr}(C)$ (see [20, Thm. 1.1]). Therefore, we may assume that \bar{I} is equal to $(\alpha_0, \alpha_1, \alpha_2, F(\alpha_3, \alpha_4))$ for some $F \in \mathbb{C}[\alpha_3, \alpha_4]_5$. Since $[I]$ belongs to $\text{Slip}_{5,4}$ it follows from Theorem 3.66 that $(\alpha_0, \alpha_1, \alpha_2)^2 \cdot (\alpha_0, \alpha_1, \dots, \alpha_4)$ is a subset of $\text{Ann}(C)$ and thus, $(\alpha_0, \alpha_1, \alpha_2)^2$ is contained in $\text{Ann}(C)$. Hence we have

$$C = x_0Q_0 + x_1Q_1 + x_2Q_2 + C' \text{ where } Q_0, Q_1, Q_2 \in \mathbb{C}[x_3, x_4]_2 \text{ and } C' \in \mathbb{C}[x_3, x_4]_3.$$

Moreover, $Q_0, Q_1, Q_2 \in \mathbb{C}[x_3, x_4]_2$ are linearly independent since C is concise. Therefore, after a linear change of variables we may reduce C to the form given in Equation (5.11). □

Remark 5.12. The annihilator ideal $\text{Ann}(C)$ of a concise cubic C has a minimal generator of degree 3 (see [15, Thm. 5.4] for a vast generalization). Therefore, the form given in Equation (5.11) should be compared to the form given in [16, Thm. 4.5].

5.2 (Border) cactus rank of a homogeneous subspace divisible by a large power of a linear form

In this section we compute the cactus rank and the border cactus rank of a homogeneous subspace of a divided power ring divisible by a large power of a linear form. This result is based on [40, Thm. 4.2 and 4.3]. However, here we strengthen the assumptions to omit technical difficulties.

Let \mathbb{k} be an algebraically closed field, let n be a positive integer and consider polynomial rings $S = \mathbb{k}[\alpha_1, \dots, \alpha_n] \subseteq \mathbb{k}[\alpha_0, \dots, \alpha_n] = T$ with graded dual rings $S^* = \mathbb{k}_{dp}[x_1, \dots, x_n] \subseteq$

$\mathbb{k}_{dp}[x_0, \dots, x_n] = T^*$. Let d_1 be a positive integer and $W \subseteq S_{\leq d_1}^*$ be a linear subspace. For a non-negative integer d_2 we define

$$W^{hom, d_2} = \{f^{hom, d_2} \mid f \in W\} \subseteq T_{d_1+d_2}^*.$$

where we have

$$f^{hom, d_2} = \sum_{i=0}^{\deg(f)} F_i x_0^{[d_2+d_1-i]} \in T_{d_1+d_2}^*$$

for every $f = F_{\deg f} + \dots + F_0 \in W$ with $F_i \in S_i^*$. We show that there are equalities

$$\text{cr}(W^{hom, d_2}) = \text{bcr}(W^{hom, d_2}) = \dim_{\mathbb{k}} S / \text{Ann}(W)$$

if the inequality $d_2 \geq d_1$ holds.

We start with a lemma which is based on [4, Lem. 2].

Lemma 5.13 ([40, Lem. 3.6 and 3.8]). *We have:*

- (i) $\text{Ann}(W)^{hom} \subseteq \text{Ann}(W^{hom, d_2})$, and
- (ii) $(\text{Ann}(W)^{hom})_{\leq d_2} = \text{Ann}(W^{hom, d_2})_{\leq d_2}$.

Proof. The proof of the lemma is based on the following calculation. Let

$$\Gamma = \alpha_0^d \Theta_0 + \alpha_0^{d-1} \Theta_1 + \dots + \Theta_d,$$

with $\Theta_i \in S_i$ and let $f \in W$. We can rewrite $\Gamma \lrcorner f^{hom, d_2}$ as follows

$$\begin{aligned} \Gamma \lrcorner f^{hom, d_2} &= \sum_{e=0}^{d_1} \sum_{j=0}^{\min(d_1-e, d)} (\alpha_0^{d-j} \Theta_j) \lrcorner (x_0^{[d_1+d_2-(e+j)]} F_{e+j}) \\ &= \sum_{e=0}^{d_1} \sum_{j=0}^{\min(d_1-e, d)} (\alpha_0^{d-j} \lrcorner x_0^{[d_1+d_2-(e+j)]}) (\Theta_j \lrcorner F_{e+j}) \\ &= \sum_{e=0}^{\min(d_1, d_1+d_2-d)} \sum_{j=0}^{\min(d_1-e, d)} x_0^{[d_1+d_2-d-e]} (\Theta_j \lrcorner F_{e+j}) \\ &= \sum_{e=0}^{\min(d_1, d_1+d_2-d)} x_0^{[d_1+d_2-d-e]} \sum_{j=0}^{\min(d_1-e, d)} \Theta_j \lrcorner F_{e+j}. \end{aligned} \tag{5.14}$$

- (i) Let $\theta = \Theta_0 + \dots + \Theta_d \in \text{Ann}(W)$, where Θ_i is homogeneous of degree i . We show that $\theta^{hom} = \alpha_0^d \Theta_0 + \alpha_0^{d-1} \Theta_1 + \dots + \Theta_d$ belongs to $\text{Ann}(W^{hom, d_2})$. We have

$$\text{Ann}(W^{hom, d_2}) = \bigcap_{f \in W} \text{Ann}(f^{hom, d_2}). \tag{5.15}$$

Thus, it suffices to show that θ^{hom} is in $\text{Ann}(f^{hom, d_2})$ for every $f \in W$. If f belongs to W , then θ is in $\text{Ann}(f)$. We set $\Gamma = \theta^{hom}$ in Equation (5.14). For every $e = 0, \dots, \min(d_1, d_1 + d_2 - d)$ the sum $\sum_{j=0}^{\min(d_1-e, d)} \Theta_j \lrcorner F_{e+j}$ is zero since $\theta \lrcorner f$ is zero. We get $\theta^{hom} \lrcorner f^{hom, d_2} = 0$, as claimed.

(ii) The inclusion $\text{Ann}(W)^{\text{hom}} \subseteq \text{Ann}(W^{\text{hom}, d_2})$ follows from part (i). We claim that we have

$$(\Theta|_{\alpha_0=1}) \in \text{Ann}(f) \text{ for every } f \in W \text{ and } \Theta \in \text{Ann}(f^{\text{hom}, d_2})_{\leq d_2}. \quad (5.16)$$

Before proving the claim, we show how to conclude the proof of part (ii). Let $\Theta \in \text{Ann}(W^{\text{hom}, d_2})_{\leq d_2}$. It follows from Equation (5.15) that Θ is in $\text{Ann}(f^{\text{hom}, d_2})$ for every $f \in W$. Thus, by Equation (5.16) we get

$$\Theta|_{\alpha_0=1} \in \bigcap_{f \in W} \text{Ann}(f) = \text{Ann}(W).$$

As a result Θ belongs to $(\text{Ann}(W)^{\text{hom}})_{\leq d_2}$.

We are left with proving the claimed Equation (5.16). Pick $f \in W$. Assume that the inequality $d \leq d_2$ holds and let $\Theta = \alpha_0^d \Theta_0 + \alpha_0^{d-1} \Theta_1 + \cdots + \Theta_d$ with $\Theta_i \in S_i$ be such that $\Theta \lrcorner f^{\text{hom}, d_2}$ is zero. We claim that $(\Theta|_{\alpha_0=1}) \lrcorner f$ is zero. By Equation (5.14) (with $\Gamma = \Theta$) we have

$$0 = \sum_{e=0}^{d_1} x_0^{[d_1+d_2-d-e]} \sum_{j=0}^{\min(d_1-e, d)} \Theta_j \lrcorner F_{e+j}.$$

Since the exponents at x_0 are pairwise different, we get

$$\sum_{j=0}^{\min(d_1-e, d)} \Theta_j \lrcorner F_{e+j} = 0 \text{ for every } d_1 \geq e \geq 0.$$

This implies that $(\Theta|_{\alpha_0=1}) \lrcorner f$ is zero. \square

We use the following result, which bounds the degree from which $T/\text{Ann}(W)^{\text{hom}}$ agrees with its Hilbert polynomial.

Lemma 5.17 ([40, Cor. 3.3]). *Let $W \subseteq S_{\leq d_1}^*$ be a linear subspace. The equality*

$$H(T/\text{Ann}(W)^{\text{hom}}, e) = \dim_{\mathbb{k}} S/\text{Ann}(W)$$

holds for every $e \geq d_1$.

Now we present the main result of this section. The version for polynomials instead of arbitrary subspaces can be strengthened, see [40, Thm. 4.3]. Recall the notion of a standard Hilbert function from Definition 2.93.

Theorem 5.18 ([40, Thm. 4.2]). *Let $W \subseteq S_{\leq d_1}^*$ be a linear subspace and $r = \dim_{\mathbb{k}} S/\text{Ann}(W)$. Let d_2 be a non-negative integer. We have the following:*

- (i) *The cactus rank of W^{hom, d_2} is at most r .*
- (ii) *If we have $d_2 \geq d_1$, then there is no homogeneous ideal $J \subseteq \text{Ann}(W^{\text{hom}, d_2})$ such that T/J has an $(r-1, n+1)$ -standard Hilbert function. In particular, the border cactus rank $\text{bcr}(W^{\text{hom}, d_2})$ of W^{hom, d_2} equals r .*
- (iii) *If we have $d_2 \geq d_1 + 1$ and $J \subseteq \text{Ann}(W^{\text{hom}, d_2})$ is a homogeneous ideal such that T/J has an $(r, n+1)$ -standard Hilbert function, then the equality $\bar{J} = \text{Ann}(W)^{\text{hom}}$ holds.*

Proof.

- (i) We have $\text{Ann}(W)^{\text{hom}} \subseteq \text{Ann}(W^{\text{hom}, d_2})$ by Lemma 5.13(i). The Hilbert polynomial of $T/\text{Ann}(W)^{\text{hom}}$ is $\dim_{\mathbb{k}} S/\text{Ann}(W) = r$. Moreover, the ideal $\text{Ann}(W)^{\text{hom}}$ is saturated. Hence the claim follows from Proposition 2.91.
- (ii) We have $H(T/\text{Ann}(W)^{\text{hom}}, d_1) = r$ by Lemma 5.17. Therefore, by Lemma 5.13(ii) we have

$$H(T/\text{Ann}(W^{\text{hom}, d_2}), d_1) = r.$$

Thus, there exists no ideal $J \subseteq \text{Ann}(W^{\text{hom}, d_2})$ such that T/J has an $(r-1, n+1)$ -standard Hilbert function. By Proposition 2.94 we get $\text{bcr}(W^{\text{hom}, d_2}) \geq r$, which together with part (i) implies that $\text{bcr}(W^{\text{hom}, d_2})$ equals r .

- (iii) Assume that $J \subseteq \text{Ann}(W^{\text{hom}, d_2})$ is such that T/J has an $(r, n+1)$ -standard Hilbert function. By Lemmas 5.13(ii) and 5.17 we get

$$H(T/\text{Ann}(W^{\text{hom}, d_2}), d_2) = H(T/\text{Ann}(W)^{\text{hom}}, d_2) = r.$$

In particular, we have $J_{d_2} = (\text{Ann}(W)^{\text{hom}})_{d_2}$. Since $\text{Ann}(W)^{\text{hom}}$ is generated in degrees smaller or equal $d_1 + 1 \leq d_2$, it follows that $J_d \supseteq (\text{Ann}(W)^{\text{hom}})_d$ holds for every $d \geq d_2$.

Ideals J and $\text{Ann}(W)^{\text{hom}}$ have the same Hilbert polynomial. Hence we get

$$\bar{J} = \overline{(\text{Ann}(W)^{\text{hom}})} = \text{Ann}(W)^{\text{hom}}. \quad \square$$

5.3 Distinguishing secant from cactus varieties

In this section we work over the field of complex numbers. We show that for all $d \geq 7$, the cactus variety $\kappa_{14}(\nu_d(\mathbb{P}^6))$ has two irreducible components: $\eta_{14}(\nu_d(\mathbb{P}^6))$ and the secant variety $\sigma_{14}(\nu_d(\mathbb{P}^6))$. Moreover, we describe the points of $\eta_{14}(\nu_d(\mathbb{P}^6))$ and present an algorithm that verifies whether a point in $\kappa_{14}(\nu_d(\mathbb{P}^6))$ is in $\sigma_{14}(\nu_d(\mathbb{P}^6))$. These results are special cases of [40, Thm. 1.4 and Thm. 1.6] which address the case of $\kappa_{14}(\nu_d(\mathbb{P}^n))$ for all $n \geq 6$ and all $d \geq 5$ ([40, Thm. 1.4]) or $d \geq 6$ ([40, Thm. 1.6]). Our presentation follows [40] with minor changes and some simplifications due to additional assumptions.

For $X = \mathbb{A}^n$ or \mathbb{P}^n , we denote by $\mathcal{Hilb}_r^{\text{Gor}}(X)$ the open subset of the Hilbert scheme $\mathcal{Hilb}_r(X)$ consisting of points corresponding to Gorenstein subschemes. Let $\mathcal{Hilb}_r^{\text{Gor}, \text{sm}}(X)$ denote its smoothable component. It is a key observation, that the cactus variety $\kappa_r(\nu_d(\mathbb{P}^n))$ is actually determined by the Gorenstein locus of the Hilbert scheme. More precisely, we have

$$\kappa_r(\nu_d(\mathbb{P}^n)) = \overline{\bigcup \{ \langle \nu_d(R) \rangle \mid [R] \in \mathcal{Hilb}_r^{\text{Gor}}(\mathbb{P}^n) \}} \quad (5.19)$$

by [11, Prop. 2.2]. Therefore, if $\mathcal{Hilb}_r^{\text{Gor}}(\mathbb{P}^n)$ is irreducible then $\kappa_r(\nu_d(\mathbb{P}^n))$ is equal to $\sigma_r(\nu_d(\mathbb{P}^n))$. Note that a description of the cactus variety, similar to the one given by Equation (5.19), works over an arbitrary field (see [17, Cor. 6.20]).

If we have either $r \leq 13$ or $r = 14$ and $n < 6$, the scheme $\mathcal{Hilb}_r^{\text{Gor}}(\mathbb{A}^n)$ is irreducible by [21, Thm. A]. Therefore, in that case, $\kappa_r(\nu_d(\mathbb{P}^n))$ is equal to $\sigma_r(\nu_d(\mathbb{P}^n))$. Thus, the cactus variety $\kappa_{14}(\nu_d(\mathbb{P}^6))$ that we study in this section is the simplest example of a reducible cactus variety.

We assume that we have $d \geq 7$. However, the presented results hold for all $d \geq 6$, and some of them, even for $d = 5$, with more technical proofs. See [40] for this as well as the case $n > 6$.

We start with defining for all $d \geq 3$ an irreducible, closed subset $\eta_{14}(\nu_d(\mathbb{P}^6))$. Consider the following rational map φ , which assigns to a scheme R its projective linear span $\langle \nu_d(R) \rangle$

$$\varphi : \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6) \dashrightarrow \text{Gr}(14, \text{Sym}^d \mathbb{C}^7).$$

Let $U \subseteq \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6)$ be a dense open subset on which φ is regular. Consider the projectivized universal bundle $\mathbb{P}\mathcal{S}$ over $\text{Gr}(14, \text{Sym}^d \mathbb{C}^7)$, given as a set by

$$\mathbb{P}\mathcal{S} = \{([P], [p]) \in \text{Gr}(14, \text{Sym}^d \mathbb{C}^7) \times \mathbb{P}(\text{Sym}^d \mathbb{C}^7) \mid p \in P\},$$

together with the inclusion $i : \mathbb{P}\mathcal{S} \hookrightarrow \text{Gr}(14, \text{Sym}^d \mathbb{C}^7) \times \mathbb{P}(\text{Sym}^d \mathbb{C}^7)$. We pull the commutative diagram

$$\begin{array}{ccc} \mathbb{P}\mathcal{S} & \xleftarrow{i} & \text{Gr}(14, \text{Sym}^d \mathbb{C}^7) \times \mathbb{P}(\text{Sym}^d \mathbb{C}^7) \\ & \searrow \pi & \swarrow \text{pr}_1 \\ & \text{Gr}(14, \text{Sym}^d \mathbb{C}^7) & \end{array}$$

back along φ to U , getting the commutative diagram

$$\begin{array}{ccc} \varphi^*(\mathbb{P}\mathcal{S}) & \xleftarrow{\varphi^*i} & U \times \mathbb{P}(\text{Sym}^d \mathbb{C}^7) \\ & \searrow \varphi^*\pi & \swarrow \text{pr}_1 \\ & U & \end{array}$$

Let Y be the closure of $\varphi^*(\mathbb{P}\mathcal{S})$ inside $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6) \times \mathbb{P}(\text{Sym}^d \mathbb{C}^7)$. The scheme Y has two irreducible components, Y_1 and Y_2 , corresponding to two irreducible components of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6)$, the schemes $\mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{P}^6)$ and \mathcal{H}_{1661} , respectively. For the description of irreducible components of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6)$ see [21].

We have

$$\begin{aligned} \sigma_{14}(\nu_d(\mathbb{P}^6)) &= \text{pr}_2(Y_1), \text{ and we define} \\ \eta_{14}(\nu_d(\mathbb{P}^6)) &= \text{pr}_2(Y_2). \end{aligned}$$

By construction, if $\kappa_{14}(\nu_d(\mathbb{P}^6))$ is reducible, then $\eta_{14}(\nu_d(\mathbb{P}^6))$ and $\sigma_{14}(\nu_d(\mathbb{P}^6))$ are its irreducible components.

In the next lemma, we bound the dimension of $\eta_{14}(\nu_d(\mathbb{P}^6))$.

Lemma 5.20 ([40, Prop. 5.5]). *The irreducible set $\eta_{14}(\nu_d(\mathbb{P}^6))$ has dimension at most 89.*

Proof. There is the following commutative diagram

$$\begin{array}{ccccc} \mathbb{P}(\text{Sym}^d \mathbb{C}^7) \supseteq \sigma \cup \eta & \longleftarrow & Y_1 \cup Y_2 & \dashrightarrow & \mathbb{P}\mathcal{S} \\ & & \downarrow \chi & & \downarrow \\ \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6) & \longleftarrow & \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{P}^6) \cup \mathcal{H}_{1661} & \dashrightarrow & \text{Gr}(14, \text{Sym}^d \mathbb{C}^7) \end{array}$$

where σ and η denote $\sigma_{14}(\nu_d(\mathbb{P}^6))$ and $\eta_{14}(\nu_d(\mathbb{P}^6))$ respectively, and $\chi : Y_1 \cup Y_2 \rightarrow \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6)$ is the projection. We have $\dim \eta_{14}(\nu_d(\mathbb{P}^6)) \leq \dim(Y_2) = m + 13$, where m equals $\dim \mathcal{H}_{1661}$

and 13 is the dimension of the general fiber of the map $\chi|_{Y_2} : Y_2 \rightarrow \mathcal{H}_{1661}$. It follows from [59, Thm. 1.1], that m is equal to 76 and therefore, we get $\dim \eta_{14}(\nu_d(\mathbb{P}^6)) \leq 89$. \square

One of the reasons why the case of \mathbb{P}^6 is simpler than the case of \mathbb{P}^n for $n > 6$ is the following lemma which follows from [59].

Lemma 5.21. *Let $S = \mathbb{C}[\alpha_1, \dots, \alpha_6]$ be a polynomial ring and let $S^* = \mathbb{C}[x_1, \dots, x_6]$ be its graded dual ring. For a polynomial $f = F_3 + F_2 + F_1 + F_0 \in S_{\leq 3}^*$ with $F_i \in S_i^*$ consider the conditions:*

- (i) *Apolar(f) has (local) Hilbert function $(1, 6, 6, 1)$;*
- (ii) *Apolar(F_3) has Hilbert function $(1, 6, 6, 1)$;*
- (iii) *[Spec Apolar(f)] is a point of $\mathcal{Hilb}_{14}^{Gor}(\mathbb{A}^6) \setminus \mathcal{Hilb}_{14}^{Gor,sm}(\mathbb{A}^6)$;*
- (iv) *[Spec Apolar(F_3)] is a point of $\mathcal{Hilb}_{14}^{Gor}(\mathbb{A}^6) \setminus \mathcal{Hilb}_{14}^{Gor,sm}(\mathbb{A}^6)$.*

Conditions (i) and (ii) are equivalent. Conditions (iii) and (iv) are equivalent.

Proof. The equivalence of (i) and (ii) follows from [59, Rmk. 2.2] and the other equivalence is a consequence of [59, Thm. 1.1]. \square

In Lemma 5.22 we identify for all $d \geq 7$ some points in $\kappa_{14}(\nu_d(\mathbb{P}^6)) \setminus \sigma_{14}(\nu_d(\mathbb{P}^6))$. Later, we show that the closure of the locus of these points is the irreducible component $\eta_{14}(\nu_d(\mathbb{P}^6))$.

Lemma 5.22 ([40, Prop. 5.6]). *Let $T = \mathbb{C}[\alpha_0, \dots, \alpha_6]$ be a polynomial ring with graded dual ring $T^* = \mathbb{C}[x_0, \dots, x_6]$. Let (y_0, y_1, \dots, y_6) be a \mathbb{C} -basis of T_1^* . Assume that G is equal to $y_0^{d-3}P$ for some natural number $d \geq 7$ and $P \in T_3^*$. Define $f := P|_{y_0=1} = F_3 + F_2 + F_1 + F_0 \in R^* := \mathbb{C}[y_1, \dots, y_6]$. If f satisfies the following conditions:*

- (a) *Apolar(f) has (local) Hilbert function $(1, 6, 6, 1)$,*
- (b) *we have [Spec Apolar(f)] $\notin \mathcal{Hilb}_{14}^{Gor,sm}(\mathbb{A}^6)$,*

then $[G]$ is a point of $\kappa_{14}(\nu_d(\mathbb{P}^6)) \setminus \sigma_{14}(\nu_d(\mathbb{P}^6))$.

Proof. Let $F'_i = (d-i)!F_i$ for all $i \in \{0, 1, 2, 3\}$ and let $f' = F'_3 + F'_2 + F'_1 + F'_0$. We have

$$G = \sum_{i=0}^3 y_0^{[d-i]} F'_i. \quad (5.23)$$

By Lemma 5.21, conditions (a) and (b) hold with f' instead of f . There is an equality $\dim_{\mathbb{C}}(R/\text{Ann}(f')) = 14$ by condition (a). Therefore, from Theorem 5.18(i) and Equation (5.23) we get $\text{cr}(G) \leq 14$.

From Proposition 2.92, if $[G]$ belongs to $\sigma_{14}(\nu_d(\mathbb{P}^6))$ then there exists $J \subseteq \text{Ann}(G)$ with $[J] \in \text{Slip}_{14, \mathbb{P}T_1^*} \subseteq \text{Hilb}_T^{h_{14,6}}$. Thus, $[\text{Proj}(T/\bar{J})]$ is in $\mathcal{Hilb}_{14}^{sm}(\mathbb{P}^6)$. From Theorem 5.18(iii) it follows that we have $\bar{J} = \text{Ann}(f')^{\text{hom}}$, so we get

$$[\text{Spec}(R/\text{Ann}(f'))] \in \mathcal{Hilb}_{14}^{Gor,sm}(\mathbb{A}^6).$$

This contradicts condition (b). \square

The following lemma is an analogue of [40, Lem. 5.3].

Lemma 5.24. *Let $S = \mathbb{C}[\alpha_1, \dots, \alpha_6]$ be a polynomial ring and $S^* = \mathbb{C}[x_1, \dots, x_6]$ be the graded dual ring. Define subsets*

$$\widehat{A} = \{f \in S^*_{\leq 3} \mid S/\text{Ann}(f) \text{ has local Hilbert function } (1, 6, 6, 1)\}$$

and

$$\widehat{B} = \{f \in \widehat{A} \mid [\text{Spec } S/\text{Ann}(f)] \notin \mathcal{Hilb}_{14}^{\text{Gor}, \text{sm}}(\mathbb{A}^6)\}.$$

The subset \widehat{A} is irreducible and 84-dimensional. Furthermore, \widehat{B} is dense in \widehat{A} .

Proof. It follows from Lemma 5.21 that for a given $f = F_3 + F_2 + F_1 + F_0 \in S^*_{\leq 3}$ we have $f \in \widehat{A}$ (respectively, $f \in \widehat{B}$) if and only if F_3 belongs to \widehat{A} (respectively, F_3 is in \widehat{B}). Moreover, \widehat{A} is open in $S^*_{\leq 3}$ so there are equalities $\dim \widehat{A} = \dim S^*_{\leq 3} = 84$.

We have a well defined morphism

$$\pi: \widehat{A} \rightarrow \mathcal{Hilb}_{14}^{\text{Gor}}(\mathbb{A}^6) \subseteq \mathcal{Hilb}_{14}(\mathbb{A}^6)$$

that maps f to $[\text{Spec } S/\text{Ann}(f)]$ (see [40, Thm. 7.1] which is based on [59, Prop. 2.12]). By definition, \widehat{B} is equal to $\pi^{-1}(\mathcal{H}_{1661} \setminus \mathcal{Hilb}_{14}^{\text{Gor}, \text{sm}}(\mathbb{A}^6))$, where \mathcal{H}_{1661} is the irreducible component of $\mathcal{Hilb}_{14}^{\text{Gor}}(\mathbb{A}^6)$ other than the smoothable component. It follows that \widehat{B} is open, and hence dense in \widehat{A} . \square

Now we present the main result of this section.

Theorem 5.25 ([40, Thm. 1.1]). *Let $d \geq 7$ be an integer and $T^* = \mathbb{C}[x_0, x_1, \dots, x_6]$. The cactus variety $\kappa_{14}(\nu_d(\mathbb{P}T_1^*))$ has two irreducible components: the secant variety $\sigma_{14}(\nu_d(\mathbb{P}T_1^*))$ and the other one, denoted by $\eta_{14}(\nu_d(\mathbb{P}T_1^*))$. Consider the map $\psi: \mathbb{P}T_1^* \times \mathbb{P}T_3^* \rightarrow \mathbb{P}T_d^*$ given by $([y_0], [P]) \mapsto [y_0^{d-3}P]$. Its image is $\eta_{14}(\nu_d(\mathbb{P}T_1^*))$.*

Proof. Let

$$U_0 = \{[a_0x_0 + \dots + a_6x_6] \in \mathbb{P}T_1^* \mid a_i \in \mathbb{C} \text{ and } a_0 \neq 0\}.$$

Let $S^* = \mathbb{C}[x_1, \dots, x_6]$. Given $[y_0] \in U_0$ with $y_0 = a_0x_0 + \dots + a_6x_6$ and $P \in T_d^*$, we can consider $P|_{y_0=1}$ as an element of S^* . Note that it is independent of the choice of representative y_0 of the class $[y_0]$. Indeed, it is obtained by substituting $x_0 = 1 - \sum_{i=1}^6 \frac{a_i}{a_0}x_i$. Recall the definition of the set \widehat{B} from Lemma 5.24. We use the following subset of $\mathbb{P}T_1^* \times \mathbb{P}T_3^*$:

$$D = \{[y_0], [P] \in \mathbb{P}T_1^* \times \mathbb{P}T_3^* \mid [y_0] \in U_0 \text{ and } P|_{y_0=1} \in \widehat{B}\}.$$

Observe that the condition $P|_{y_0=1} \in \widehat{B}$ is independent of the choice of representatives y_0 and P of $[y_0]$ and $[P]$. By construction and Lemma 5.24, D is irreducible and of dimension $6 + \dim \widehat{B} - 1 = 89$. We have $\dim(\mathbb{P}T_1^* \times \mathbb{P}T_3^*) = 6 + 83 = 89$. Thus, \overline{D} is equal to $\mathbb{P}T_1^* \times \mathbb{P}T_3^*$. The morphism ψ is closed. We get

$$\psi(\mathbb{P}T_1^* \times \mathbb{P}T_3^*) = \overline{\psi(D)}.$$

By Lemma 5.22 the set-theoretic image $\psi(D)$ is contained in $\eta_{14}(\nu_d(\mathbb{P}T_1^*))$. Therefore, $\psi(\mathbb{P}T_1^* \times \mathbb{P}T_3^*)$ is a subset of $\eta_{14}(\nu_d(\mathbb{P}T_1^*))$. Observe that ψ has finite fibers. It follows that $\psi(\mathbb{P}T_1^* \times \mathbb{P}T_3^*)$ is an 89-dimensional, irreducible closed subset of $\eta_{14}(\nu_d(\mathbb{P}T_1^*))$. The latter variety

is irreducible and of dimension at most 89 (see Lemma 5.20). Thus, we have $\psi(\mathbb{P}T_1^* \times \mathbb{P}T_3^*) = \eta_{14}(\nu_d(\mathbb{P}T_1^*))$. \square

Having described the irreducible component of $\kappa_{14}(\nu_d(\mathbb{P}^6))$ other than $\sigma_{14}(\nu_d(\mathbb{P}^6))$ we can algorithmically check whether a given point $[G] \in \kappa_{14}(\nu_d(\mathbb{P}^6))$ belongs to $\sigma_{14}(\nu_d(\mathbb{P}^6))$.

We start with the following lemma which characterizes the points of $\kappa_{14}(\nu_d(\mathbb{P}^6)) \setminus \sigma_{14}(\nu_d(\mathbb{P}^6))$.

Lemma 5.26. *Let $T = \mathbb{C}[\alpha_0, \dots, \alpha_6]$ be a polynomial ring and $T^* = \mathbb{C}[x_0, \dots, x_6]$ be the graded dual ring. Let $d \geq 7$. The point $[G] \in \kappa_{14}(\nu_d(\mathbb{P}^6))$ does not belong to $\sigma_{14}(\nu_d(\mathbb{P}^6))$ if and only if there exist $y_0 \in T_1^*$ and $P \in T_3^*$ such that G is equal to $y_0^{d-3}P$ and for any completion of y_0 to a basis (y_0, \dots, y_6) of T_1^* (with dual basis equal to $(\beta_0, \dots, \beta_6)$) we have:*

(a) $\text{Apolar}(P|_{y_0=1})$ has Hilbert function $(1, 6, 6, 1)$,

(b) $[\text{Spec Apolar}(P|_{y_0=1})] \notin \mathcal{Hilb}_{14}^{\text{Gor,sm}}(\mathbb{A}^6)$.

Proof. If $y_0 \in T_1^*$ and $P \in T_3^*$ are such that G equals $y_0^{d-3}P$ and there exists a completion of y_0 to a basis (y_0, \dots, y_6) of T_1^* for which conditions (a) and (b) hold, we get $[G] \notin \sigma_{14}(\nu_d(\mathbb{P}^6))$ by Lemma 5.22.

Assume that we have $[G] \notin \sigma_{14}(\nu_d(\mathbb{P}^6))$. By Theorem 5.25 there exists a linear form $y_0 \in T_1^*$ with $y_0^{d-3}|G$. We claim that G is not divisible by y_0^{d-2} . Indeed, otherwise, from Theorem 5.18(ii) we get $\text{bcr}(G) \leq 8$, so $[G]$ belongs to $\kappa_8(\nu_d(\mathbb{P}^6)) = \sigma_8(\nu_d(\mathbb{P}^6)) \subseteq \sigma_{14}(\nu_d(\mathbb{P}^6))$. We showed that G is equal to $y_0^{d-3}P$ for some $P \in T_3^*$. Extend y_0 to a basis y_0, y_1, \dots, y_6 . Let $f = P|_{y_0=1} \in \mathbb{C}[y_1, \dots, y_6]$. Suppose f equals $F_3 + F_2 + F_1 + F_0$.

Now we prove that conditions (a) and (b) hold. Let $f' = F'_3 + F'_2 + F'_1 + F'_0 \in \mathbb{C}[y_1, \dots, y_6]$ with $F'_i = (d-i)!F_i$. By Lemma 5.21, it is enough to show that conditions (a) and (b) hold for f' instead of $f = P|_{y_0=1}$. We have

$$G = \sum_{i=0}^3 y_0^{[d-i]} F'_i.$$

By Lemma 5.13 (i) there is an inclusion

$$\text{Ann}(f')^{\text{hom}} \subseteq \text{Ann}(G).$$

If we have $\dim_{\mathbb{C}}(\text{Apolar}(f')) \leq 13$, then $\text{cr}(G) \leq 13$ holds by Proposition 2.91, since $\text{Ann}(f')^{\text{hom}}$ is saturated and $T/\text{Ann}(f')^{\text{hom}}$ has Hilbert polynomial $\dim_{\mathbb{C}}(\text{Apolar}(f'))$. We have $[G] \in \kappa_{13}(\nu_d(\mathbb{P}^6)) = \sigma_{13}(\nu_d(\mathbb{P}^6)) \subseteq \sigma_{14}(\nu_d(\mathbb{P}^6))$, a contradiction.

From Theorem 5.18(ii) we obtain $\dim_{\mathbb{C}}(\text{Apolar}(f')) \leq 14$, and thus, $\dim_{\mathbb{C}}(\text{Apolar}(f')) = 14$. We claim that we have $[\text{Proj } T/\text{Ann}(f')^{\text{hom}}] \notin \mathcal{Hilb}_{14}^{\text{sm}}(\mathbb{P}^6)$. Indeed, otherwise, there is a point $[J] \in \text{Slip}_{14,6}$ with

$$\bar{J} = \text{Ann}(f')^{\text{hom}} \subseteq \text{Ann}(G).$$

By Proposition 2.92, this contradicts the assumption that $[G]$ does not belong to $\sigma_{14}(\nu_d(\mathbb{P}^6))$. We showed that we have $[\text{Proj } T/\text{Ann}(f')^{\text{hom}}] \notin \mathcal{Hilb}_{14}^{\text{sm}}(\mathbb{P}^6)$ and therefore, condition (b) holds. Hence the algebra $\text{Apolar}(f')$ has Hilbert function $(1, 6, 6, 1)$ by [59, Thm. 1.1]. Thus, condition (a) also holds. \square

We present the aforementioned algorithm.

Theorem 5.27 ([40, Thm. 1.6]). *Let $T = \mathbb{C}[\alpha_0, \dots, \alpha_6]$ be a polynomial ring with graded dual ring $T^* = \mathbb{C}[x_0, \dots, x_6]$. Given an integer $d \geq 7$ and $[G] \in \kappa_{14}(\nu_d(\mathbb{P}T_1^*)) \subseteq \mathbb{P}T_d^*$ the following algorithm checks if $[G]$ belongs to $\sigma_{14}(\nu_d(\mathbb{P}T_1^*))$.*

Step 1 *Compute the ideal $\mathfrak{a} = \sqrt{((\text{Ann } G)_{\leq d-3})} \subseteq T$.*

Step 2 *If \mathfrak{a}_1 is not 6-dimensional, then $[G]$ is in $\sigma_{14}(\nu_d(\mathbb{P}T_1^*))$ and the algorithm terminates. Otherwise, compute $\{K \in T_1^* \mid \mathfrak{a}_1 \lrcorner K = 0\}$. Let y_0 be a generator of this one-dimensional \mathbb{C} -vector space.*

Step 3 *Let e be the maximal integer such that y_0^e divides G . If we have $e \neq d-3$, then $[G]$ is in $\sigma_{14}(\nu_d(\mathbb{P}T_1^*))$ and the algorithm terminates. Otherwise, let $G = y_0^{d-3}P$. Pick a basis y_0, y_1, \dots, y_6 of T_1^* and compute $f = P|_{y_0=1} \in R^* := \mathbb{C}[y_1, \dots, y_6]$.*

Step 4 *Let $I = \text{Ann}(f) \subseteq R$. If the (local) Hilbert function of R/I is not $(1, 6, 6, 1)$, then $[G]$ is in $\sigma_{14}(\nu_d(\mathbb{P}T_1^*))$, and the algorithm terminates.*

Step 5 *Compute $r = \dim_{\mathbb{C}} \text{Hom}_R(I, R/I)$. The point $[G]$ is in $\sigma_{14}(\nu_d(\mathbb{P}T_1^*))$ if and only if we have $r > 76$.*

Proof. Assume that $[G]$ does not belong to $\sigma_{14}(\nu_d(\mathbb{P}^6))$. There exist a basis (y_0, \dots, y_6) of T_1^* and $P \in T_3^*$ as in Lemma 5.26. Let $f = P|_{y_0=1}$ and define $f' = F'_3 + F'_2 + F'_1 + F'_0 \in \mathbb{C}[y_1, \dots, y_6]$ where F'_i equals $(d-i)!F_i$. We have $G = y_0^{[d-3]}F'_3 + y_0^{[d-2]}F'_2 + y_0^{[d-1]}F'_1 + y_0^{[d]}F'_0$. By Lemma 5.13(ii), there is an equality $\text{Ann}(G)_{\leq d-3} = (\text{Ann}(f')^{hom})_{\leq d-3}$. Moreover,

$$((\text{Ann}(f')^{hom})_{\leq d-3}) = \text{Ann}(f')^{hom}$$

holds since we have $d-3 \geq 4 > \deg(f')$. Therefore, we get

$$\mathfrak{a} = \sqrt{(\text{Ann}(G)_{\leq d-3})} = \sqrt{\text{Ann}(f')^{hom}} = (\beta_1, \dots, \beta_6),$$

where $\beta_1, \dots, \beta_6 \in T_1$ are dual to $y_1, \dots, y_6 \in T_1^*$. This shows that if the \mathbb{C} -linear space $(\sqrt{(\text{Ann}(G)_{\leq d-3})})_1$ is not 6-dimensional then $[G]$ is in $\sigma_{14}(\nu_d(\mathbb{P}^6))$. Therefore, in that case, algorithm stops correctly at Step 2.

Assume that the algorithm did not stop at Step 2. If G is of the form as in Lemma 5.26, then y_0 divides G exactly $(d-3)$ -times. Otherwise $[G]$ belongs to $\sigma_{14}(\nu_d(\mathbb{P}^6))$ and the algorithm stops correctly at Step 3.

Assume that the algorithm did not stop at Step 3. The algorithm does not stop at Step 4 if and only if condition (a) of Lemma 5.26 is fulfilled.

If the algorithm did not stop at Step 4, then P satisfies condition (a) from Lemma 5.26. Hence $[G]$ is in $\sigma_{14}(\nu_d(\mathbb{P}^6))$ if and only if P does not satisfy condition (b). The irreducible component $\mathcal{H}ilb_{1661}$ of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^6)$ is 76-dimensional and $\mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^6)$ is smooth at points in $\mathcal{H}ilb_{1661} \setminus \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^6)$ (see [59, Thm. 1.1 and Claim 3]). Thus, P does not satisfy condition (b) from Lemma 5.26 if and only if we have

$$\dim_{\mathbb{C}} \text{Hom}_R(I, R/I) > 76,$$

since the left term is the dimension of the tangent space $\mathbf{T}_{[\text{Spec } R/I]} \mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^6)$ by [49, Prop. 2.3]. \square

An implementation in Macaulay2 [43] of the algorithm from Theorem 5.27 is presented in [40, §A].

5.4 Distinguishing Grassmann secant from Grassmann cactus varieties

Let $d \geq 5$ be an integer. In this section, we state a theorem from [40] describing the irreducible component $\eta_{8,3}(\nu_d(\mathbb{P}^4))$ of the cactus variety $\kappa_{8,3}(\nu_d(\mathbb{P}^4))$ other than the secant variety $\sigma_{8,3}(\nu_d(\mathbb{P}^4))$. This is analogous to Theorem 5.25.

Theorem 5.28 ([40, Thm. 1.2]). *Let $d \geq 5$ be an integer and $T^* = \mathbb{C}[x_0, x_1, \dots, x_4]$. The Grassmann cactus variety $\kappa_{8,3}(\nu_d(\mathbb{P}T_1^*))$ has two irreducible components: the Grassmann secant variety $\sigma_{8,3}(\nu_d(\mathbb{P}T_1^*))$ and the other one, denoted by $\eta_{8,3}(\nu_d(\mathbb{P}T_1^*))$. Consider the map $\psi : \mathbb{P}T_1 \times \text{Gr}(3, T_2^*) \rightarrow \text{Gr}(3, T_d^*)$ given by $([y_0], [U]) \mapsto [y_0^{d-2}U]$. Its image is $\eta_{8,3}(\nu_d(\mathbb{P}T_1^*))$.*

Theorem 5.28 can be generalized for all $n \geq 4$ (see [40, Thm. 1.5]). However, we present the simpler version due to its similarity to Theorem 5.25.

By [20, Thm. 1.1], for all $r \leq 8$, the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^n)$ is reducible if and only if we have $n \geq 4$ and $r = 8$. Furthermore, for all $n \geq 4$ and $r = 8$, a general point of the non-smoothable irreducible component of $\mathcal{H}ilb_r(\mathbb{P}^n)$ corresponds to a subscheme whose coordinate ring has local Hilbert function $(1, 4, 3)$. Therefore, Theorem 5.28 describes the other irreducible component of the Grassmann cactus variety in a minimal case when such a component exists.

As in Section 5.3, we can characterize the points of $\eta_{8,3}(\nu_d(\mathbb{P}^n)) \setminus \sigma_{8,3}(\nu_d(\mathbb{P}^n))$ (see [40, Lem. 6.9]) and obtain an algorithm analogous to the one from Theorem 5.27 (see [40, Thm 6.8]).

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