Uniwersytet Warszawski
Wydział Matematyki, Informatyki i Mechaniki

Adam Michalik [1]
Nr albumu: 291698

Zastosowanie wiązki zerowej korelacji do znajdowania równań rozmaitości siecznych do zanurzeń Veronese przestrzeni rzutowej $\mathbb{P}^3$

Praca magisterska
na kierunku MATEMATYKA

Praca wykonana pod kierunkiem
dra Jarosława Buczyńskiego
Instytut Matematyczny Polskiej Akademii Nauk

Wrzesień 2014

[1] Praca wykonana przy wsparciu projektu „Secant varieties, computational complexity, and toric degenerations" realizowanego w ramach programu Homing Plus Fundacji na rzecz Nauki Polskiej, współfinansowanego z Funduszu Rozwoju Regionalnego Unii Europejskiej.

Oświadczenie kierującego pracą

Potwierdzam, że niniejsza praca została przygotowana pod moim kierunkiem i kwalifikuje się do przedstawienia jej w postępowaniu o nadanie tytułu zawodowego.

Data Podpis kierującego pracą

Oświadczenie autora (autorów) pracy

Świadom odpowiedzialności prawnej oświadczam, że niniejsza praca dyplomowa została napisana przeze mnie samodzielnie i nie zawiera treści uzyskanych w sposób niezgodny z obowiązującymi przepisami.

Oświadczam również, że przedstawiona praca nie była wcześniej przedmiotem procedur związanych z uzyskaniem tytułu zawodowego w wyższej uczelni.

Oświadczam ponadto, że niniejsza wersja pracy jest identyczna z załączoną wersją elektroniczną.

Data Podpis autora (autorów) pracy

Abstract

We find new classes of equations of secant varieties of Veronese embeddings of $\mathbb{P}^3$, by applying the method described by Joseph Landsberg and Giorgio Ottaviani in 2011 to the null-correlation bundle on $\mathbb{P}^3$. We also give a SAGE library which is useful in explicit calculations, and can be adapted to work with different bundles on projective spaces.

Słowa kluczowe

secant variety, vector bundle, Veronese variety

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

Klasyfikacja tematyczna

14 Algebraic geometry
14Nxx Projective and enumerative geometry
14N05 Projective techniques
14Mxx Special varieties
14M12 Determinantal varieties
14Jxx Surfaces and higher-dimensional varieties
14J60 Vector bundles on surfaces and higher-dimensional varieties, and their moduli

Tytuł pracy w języku angielskim

Finding equations of secant varieties to Veronese embeddings of $\mathbb{P}^3$ using null-correlation bundle

Contents

Introduction

Let $X \subset \mathbb{P}V$ be an irreducible projective variety. Any two different points $x_1, x_2 \in X$ span a secant line $l = \langle x_1, x_2 \rangle \subset \mathbb{P}V$. Consider the union of all such lines,

$$S_2(X) = \bigcup_{x_1, x_2 \in X} \langle x_1, x_2 \rangle \subset \mathbb{P}V.$$

It does not necessarily form a variety, but we can consider a smallest variety containing it, that is, its Zariski closure $\sigma_2(X) = \overline{S_2(X)}$. What are the equations describing variety $\sigma_2(X)$?

More generally, we can consider Zariski closure of the union of spans of k different points in X:

$$\sigma_k(X) = \overline{\bigcup_{x_1, \ldots, x_k \in X} \langle x_1, \ldots, x_k \rangle}.$$

Varieties defined in this way are called *secant varieties of* X. Studying them is a classical problem, and dates back to the Italian school in 19th century. They have been recently enjoying a renewed interest due to their use in solving problems in algebraic complexity theory, algebraic statistics, and other fields. For a survey of applications, see e.g. [Landsberg2012, Chapter 1].

One of the most important problems in the study of secant varieties is finding equations describing them. Not much is known about equations of varieties of secants to arbitrary varieties, and most results give equations for varieties of secants to some specific class of varieties – for instance, secants to Segre or Veronese embeddings, or some combination of the two.

In this paper we are working on the equations cutting out secant varieties to Veronese embeddings of $\mathbb{P}^3$. Our aim is to use techniques from [LandOtt11] to find equations cutting out secant varieties. We shall briefly describe the idea.

Let L be a very ample line bundle on X. Denote $V = H^0(L)$. We have a natural embedding:

$$X \hookrightarrow \mathbb{P}V^* = \mathbb{P}H^0(L)^*, \quad x \mapsto \{\phi \in H^0(L) : \phi(x) = 0\}.$$

To find equations describing $\sigma_k(X)$, authors [LandOtt11] give the following method. Let E be a vector bundle on X of rank e. For any $v \in H^0(L)^*$ we have a linear map

$$A_v^E : H^0(E) \to H^0(\mathcal{H}om(E, L))^*, \quad A_v^E(s)(\phi) = v(\phi(s)).$$

Define:

$$Rank_k(E) = \mathbb{P}\{v \in V : \operatorname{rank} A_v^E \leq ek\}$$

In [LandOtt11, Prop 5.1.1], authors prove the following

Proposition. *Let $X \subset \mathbb{P}V = \mathbb{P}H^0(L)^*$ be a variety, and E be a rank e vector bundle on it. Then:*

$$\sigma_r(X) \subset Rank_r(E).$$

In other words, minors of size $re + 1$ of A_v^E give equations for $\sigma_r(X)$.

As a bundle E we will use the null-correllation bundle on $\mathbb{P}^3$ and its twists. This choice allows us to characterize certain secant varieties, for example $\sigma_5(\nu_4(\mathbb{P}^3))$, as one of the irreducible components of the $Rank_r(E)$ variety defined above.

We will be working over an algebraically closed field of characteristic 0.

Chapter 1 recalls basic properties of secant varieties, gives some examples, and discusses the dimension of the secant varieties to Veronese embeddings.

Chapter 2 contains information about the null-correlation bundle, its construction and properties.

Chapter 3 gives more specific information about using vector bundles to find equations of secant varieties, and also applies these methods to show that some secant varieties to the Veronese embedding of $\mathbb{P}^3$ can given as an irreducible component of the rank variety of twisted null-correlation bundle. It also discusses cases when equations obtained from null-correlation bundle are not enough – in some of these cases we obtain appropriate secant variety as an irreducible component of the intersection of the rank variety of twisted null-correlation bundle and the catalecticant variety of catalecticant minors.

Finally, appendices contain Sage programs that are used in explicit calculations, and the results of the calculations.

Chapter 1

Basic properties of secant varieties

In this chapter we define secant varieties, recall their basic properties, and give some examples.

1.1. Definition

Recall the definition of the secant variety: let $X \subset \mathbb{P}^n$ be an irreducible projective variety that is not contained in a linear subspace. For $k \geq 2$, we define k-th secant variety to X by:

$$\sigma_k(X) = \overline{\bigcup_{x_1,\ldots,x_k \in X} \langle x_1, \ldots, x_k \rangle}.$$

Taking Zariski closure is necessary here: for example, consider the variety of secant lines to some curve C. A line L tangent at a point $x \in C$ clearly is a limit of secant lines, so it should belong to the secant variety. On the other hand, it may happen that x is the only point of the intersection $L \cap C$, so L is not a span of two points on C.

1.2. Example: Segre embedding

The classical example are the secant varieties to the two factor Segre embedding. It is important to us, because the approach we use in this paper to find the equations of secant varieties to other varieties is in a certain sense an extension of the method used for Segre embedding.

Let V, W be finite dimensional vector spaces. Consider Segre embedding:

$$\mathbb{P}V^* \times \mathbb{P}W \to \mathbb{P}(V^* \otimes W), \quad ([f], [w]) \mapsto [f \otimes w]$$

Its image will be denoted by $Seg(V^*, W)$.

There is a natural isomorphism $\Phi : V^* \otimes W \xrightarrow{\simeq} \mathrm{Hom}(V, W)$ defined as:

$$\Phi(f_1 \otimes w_1 + \ldots + f_n \otimes w_n)(v) = f_1(v) \cdot w_1 + \ldots + f_n(v) \cdot w_n \tag{1.1}$$

Elements of $V^* \otimes W$ can thus be viewed as linear maps $V \to W$, and if we choose coordinates on V and W, we can view elements of $V^* \otimes W$ as matrices of size $\dim W \times \dim V$. Also, the correspondence $V^* \otimes W \xrightarrow{\simeq} \mathrm{Hom}(V, W)$ makes it clear that the simple tensors $f \otimes w \in V^* \otimes W$ correspond precisely to the rank 1 maps, and so (after a choice of coordinates) to the rank 1 matrices. Since the image of the Segre embedding $\mathbb{P}V^* \times \mathbb{P}W \to \mathbb{P}(V^* \otimes W)$ are precisely (lines of) simple tensors, we see that the image of Segre embedding can be cut out (at least set-theoretically) by 2×2 minors of the generic $\dim W \times \dim V$ matrix.

Secant varieties $\sigma_k(Seg(V^*, W))$ are now easy to describe: the span of $f_1 \otimes w_1, f_2 \otimes w_2, \ldots, f_k \otimes w_k$ is precisely

$$\{a_1(f_1 \otimes w_1) + \ldots + a_k(f_k \otimes w_k) : a_1, \ldots, a_k \in k\} \tag{1.2}$$

Any element of (1.2) is sent by the correspondence (1.1) to the map $V \to W$, the rank of which is at most k – indeed, the image of that map will be contained in $\text{span}\langle w_1, \ldots, w_k\rangle$. On the other hand, any map $V \to W$ of rank at most k can be written as a sum of k maps of rank 1.

It follows that $\sigma_k(Seg(V^*, W))$ is equal to the closure of the set of linear maps of rank at most k. After a choice of coordinates in V and W, elements of $\text{Hom}(V, W)$ correspond to matrices $\dim W \times \dim V$, and the maps of rank at most k correspond to matrices of rank at most k. The set of such matrices is cut out (at least set-theoretically) by $(k+1) \times (k+1)$ size minors of a generic $\dim W \times \dim V$ matrix. It follows that it is Zariski closed, so $\sigma_k(Seg(V^*, W))$ is precisely the set of linear maps of rank at most k.

1.3. Dimension of secant varieties

Besides taking closure of all secant planes, there is another way of looking at $\sigma_k(X)$, which lets us prove some of its properties.

Let X be irreducible and nondegenerate (that is, not contained in any linear subspace of $\mathbb{P}^n$), and let $\dim X = s$. Consider k-fold product X^k, for $k \leq n+1$. There exists a rational map $s_k : X^k \dashrightarrow \mathbb{G}(k-1, n)$ to the Grassmanian of $k-1$-planes in $\mathbb{P}^n$, that takes k linearly independent points, and sends it to the unique $k-1$-plane that contains them. Let $\mathcal{S}_k(X)$ denote the closure of its image.

If $k - 1 > n - s$, the map s_k is dominant: indeed, any $k-1$-plane must then meet X at a variety of positive dimension. This means that the spans of k points of X fill out the whole $\mathbb{P}^n$, and so the secant varieties for $k > n - s + 1$ are not very interesting. We can therefore assume that $k \leq n - s + 1$. In this case, a general $k-1$-plane that is secant to X, intersects X only in finitely many points. It implies that a general secant $k-1$ plane can be obtained as a span of k different points of X only in finitely many ways, which means that that map s_k generically has finite fibers – there is an open nonempty subset $U \subset X^k$ such that restriction $U \to \mathbb{G}(k-1, n)$ has finite fibers.

Since X is irreducible, X^k is irreducible as well, so U is dense in X^k. This means that $\dim U = \dim X^k = k \dim X = ks$. Since the map $U \to \mathbb{G}(k-1, n)$ has finite fibers, the image has dimension ks as well. As U is dense in X^k, closure of the image of U is equal to $\mathcal{S}_k(X)$, and so $\dim \mathcal{S}_k(X) = sk$. Also, since U is irreducible, $\mathcal{S}_k(X)$ is irreducible as well.

Now consider the incidence variety:

$$\mathcal{I}_k(X) = \{(p, V) \in \mathbb{P}^n \times \mathcal{S}_k(X) : p \in V\}$$

The image of the projection $\mathcal{I}_k(X) \to \mathbb{P}^n$ is precisely $\sigma_k(X)$. Consider the projection $\mathcal{I}_k(X) \to \mathcal{S}_k(X)$. Each fiber is isomorphic to $\mathbb{P}^{k-1}$, which is irreducible, and $\mathcal{S}_k(X)$ is also irreducible, so $\mathcal{I}_k(X)$ is irreducible of dimension $sk + k - 1$. From this we can conclude a few important properties of secant varieties:

Proposition 1.3.1. *Let $X \subset \mathbb{P}^n$ be an irreducible projective variety, and $\dim X = s$. Then $\dim \sigma_k(X) \leq sk + k - 1$, with equality holding if and only if a general point $p \in \sigma_k(X)$ lies only on finitely many secant planes. Also, X is irreducible.*

Proof. Indeed, $\sigma_k(X)$ is the image of $\mathcal{I}_k(X)$ under projection $\mathcal{I}_k(X) \to \mathbb{P}^n$. Since $\mathcal{I}_k(X)$ is irreducible of dimension $sk + k - 1$, the result follows. □

1.4. Defective secant variety to Veronese embedding

In Proposition 1.3.1 we obtained a bound on the dimension of secant variety. We will now see that this bound is not always attained.

Consider $\sigma_2(\nu_2(\mathbb{P}^2)) \subset \mathbb{P}^5$, the variety of lines secant to degree 2 Veronese embedding of projective plane. The bound in Proposition 1.3.1 says that $\dim \sigma_2(\nu_2(\mathbb{P}^2)) \leq 2 \cdot 2 + 2 - 1 = 5$, with equality if and only if a general point $x \in \sigma_2(\nu_2(\mathbb{P}^2))$ lies only on finitely many secant lines. We will show that a general point of $\sigma_2(\nu_2(\mathbb{P}^2))$ actually lies on a one dimensional family of lines secant to $\sigma_2(\nu_2(\mathbb{P}^2))$.

More precisely, note the image of any line $L \subset \mathbb{P}^2$ is a conic curve C contained in some plane $P \subset \mathbb{P}^5$ – indeed, by a change of coordinates, we can assume that $L = \{x_2 = 0\} = \{[x_0, x_1, 0] : x_0, x_1 \in k\}$. Then, since the Veronese map is defined by $\nu_2([x_0, x_1, x_2]) = [x_0^2, x_1^2, x_2^2, x_0x_1, x_1x_2, x_2x_0]$, the image of L under ν_2 is contained in a plane

$$P = \{y_2 = y_4 = y_5 = 0\} \subset \operatorname{Proj} k[y_0, y_1, y_2, y_3, y_4, y_5] = \mathbb{P}^5$$

Now take a general point $x \in \sigma_2(\nu_2(\mathbb{P}^2))$. Since it is general, it lies on some secant line $\langle \nu_2(p), \nu_2(q) \rangle$. Consider a curve $C \subset \mathbb{P}^5$ that is an image of $L = \langle p, q \rangle \subset \mathbb{P}^2$. By previous considerations, C lies in some plane $P \subset \mathbb{P}^5$. As $\nu_2(p)$ and $\nu_2(q)$ both belong to this plane, x also does. But since C is not a line, a general line through x in P intersects C in at least (in fact exactly) two points, so general line through x in P is secant to C, thus also to $\nu_2(\mathbb{P}^2)$.

This proves that the general point of $\sigma_2(\nu_2(\mathbb{P}^2))$ does not lie on only finitely many secant lines, and so the dimension of $\sigma_2(\nu_2(\mathbb{P}^2))$ is less than 5 that is suggested by Proposition 1.3.1. It is in fact not hard to show that the dimension is actually 4.

1.5. Expected dimension of secant varieties to Veronese variety

In Section 1.4 we saw that the dimension bound from Proposition 1.3.1 is not always attained. It was a long standing problem in algebraic geometry, to characterize the secant varieties to Veronese varieties, for which the dimension bound is attained. It was conjectured that it is attained always, except of some secant varieties to degree 2 Veronese, and some finite number of special cases, the full list of which was known at least since 1902 [Richmond].

Some progress on this conjecture has been made using classical methods, but full proof was obtained in the course of years 1985-1995 by Hirschowitz and Alexander using modern methods. Full historical account can be found in [BraOtt], where Brambilla and Ottaviani also give a shortened version of the proof. Specifically, the theorem says:

Theorem 1.5.1. *[BraOtt, Theorem 1.2] The secant varieties $\sigma_k(\nu_d(\mathbb{P}^n))$ attain dimension $\min(nk + k - 1, \binom{n+d}{n})$ except in following cases:*

- $d = 2, 2 \leq k \leq n$
- $n = 2, d = 4, k = 5$
- $n = 3, d = 4, k = 9$
- $n = 4, d = 3, k = 7$
- $n = 4, d = 4, k = 14$

This allows us from now on to call $nk + k - 1$ the *expected dimension* of $\sigma_k(\nu_d(\mathbb{P}^n))$.

Chapter 2

Null-correlation bundle

In this chapter we define the null-correlation bundle, and show its basic properties.

Unless stated otherwise, all bundles and sheaves are on $X = \mathbb{P}^3 = \operatorname{Proj} k[x_0, x_1, x_2, x_3]$, where k is algebraically closed field of characteristic 0.

2.1. Construction

Let $\omega : \mathcal{O}(1)^4 \to \mathcal{O}(2)$ be given as

$$\omega(s_0, s_1, s_2, s_3) = x_1 s_0 - x_0 s_1 + x_3 s_2 - x_2 s_3$$

Denote by F the coherent sheaf being the kernel of ω. Since ω is surjective, F is locally a free sheaf of rank 3, fitting into an exact sequence:

$$0 \to F \to \mathcal{O}(1)^4 \xrightarrow{\omega} \mathcal{O}(2) \to 0 \tag{2.1}$$

Define $t : \mathcal{O} \to \mathcal{O}(1)^4$ by $t(1) = (x_0, x_1, x_2, x_3)$. Notice that $\omega \circ t = 0$, thus t factors through F. Clearly t is injective. Let E fit in the exact sequence:

$$0 \to \mathcal{O} \xrightarrow{t} F \to E \to 0 \tag{2.2}$$

The sheaf E is also locally free: notice that cokernel of $t : \mathcal{O} \to \mathcal{O}(1)^4$ is $T\mathbb{P}^3$ – this is just the classical Euler sequence [Hart, Thm 8.13]:

$$0 \to \mathcal{O} \xrightarrow{t} \mathcal{O}(1)^4 \to T\mathbb{P}^3 \to 0.$$

Since $\omega \circ t = 0$, we have an induced surjection of locally free sheaves $\bar{\omega} : T\mathbb{P}^3 \to \mathcal{O}(2)$. Kernel of such surjection is also locally free – indeed, it is E. Since rank of $T\mathbb{P}^3$ is 3, and $\mathcal{O}(2)$ is line bundle, E is rank 2.

2.2. Trivialization of E

Let us write down a trivialization of E. Let $\phi_0' : \mathcal{O}(1)^4|_{D(x_0)} \to \mathcal{O}^2_{D(x_0)}$ be a map defined as:

$$\phi_0'(s_0, s_1, s_2, s_3) = \left(\frac{x_0 s_2 - x_2 s_0}{x_0^2}, \frac{x_0 s_3 - x_3 s_0}{x_0^2} \right)$$

Note that if $s = (s_0, s_1, s_2, s_3)$ is in the image of $t : \mathcal{O} \to \mathcal{O}(1)^4$, that is, $s = (x_0 f, x_1 f, x_2 f, x_3 f)$ for some $f \in \mathcal{O}(U), U \subset D(x_0)$, then $\phi_0'(s) = 0$. It follows that ϕ_0' induces a map on quotient

$(\mathcal{O}(1)^4/\operatorname{im} t)|_{D(x_0)} = T\mathbb{P}^3|_{D(x_0)} \to \mathcal{O}^2_{D(x_0)}$. Let us denote by ϕ_0 the restriction of this map to E, that is, the map $\phi_0 : E|_{D(x_0)} \to \mathcal{O}^2_{D(x_0)}$. We claim that this is isomorphism of sheaves. Indeed, its inverse is given by:

$$\psi_0 : \mathcal{O}^2_{D(x_0)} \to E|_{D(x_0)}, \quad , \psi_0(f,g) = (0, x_3 f - x_2 g, x_0 f, x_0 g) + \operatorname{im} t$$

We have:

$$\begin{aligned}
\psi_0\phi_0(s + \operatorname{im} t) &= \psi_0\phi_0((s_0, s_1, s_2, s_3) + \operatorname{im} t) \\
&= \psi_0\left(\frac{x_0 s_2 - x_2 s_0}{x_0^2}, \frac{x_0 s_3 - x_3 s_0}{x_0^2}\right) \\
&= (0, \frac{x_3 x_0 s_2 - x_2 x_0 s_3}{x_0^2}, \frac{x_0 s_2 - x_2 s_0}{x_0}, \frac{x_0 s_3 - x_3 s_0}{x_0}) \\
&= (0, \frac{x_3 s_2 - x_2 s_3}{x_0}, \frac{x_0 s_2 - x_2 s_0}{x_0}, \frac{x_0 s_3 - x_3 s_0}{x_0})
\end{aligned}$$

Now, since s is a section of F, we have:

$$x_3 s_2 - x_2 s_3 = x_0 s_1 - x_1 s_0,$$

so:

$$\begin{aligned}
\psi_0\phi_0(s + \operatorname{im} t) &= (0, \frac{x_3 s_2 - x_2 s_3}{x_0}, \frac{x_0 s_2 - x_2 s_0}{x_0}, \frac{x_0 s_3 - x_3 s_0}{x_0}) \\
&= (0, \frac{x_0 s_1 - x_1 s_0}{x_0}, \frac{x_0 s_2 - x_2 s_0}{x_0}, \frac{x_0 s_3 - x_3 s_0}{x_0}) \\
&= (0, s_1 - x_1\frac{s_0}{x_0}, s_2 - x_2\frac{s_0}{x_0}, s_3 - x_3\frac{s_0}{x_0})
\end{aligned}$$

But s_0 is a section of $\mathcal{O}(1)$ over some $U \subset D(x_0)$, so $\frac{s_0}{x_0} \in \mathcal{O}(U)$, and so $t(\frac{s_0}{x_0}) = (s_0, x_1\frac{s_0}{x_0}, x_2\frac{s_0}{x_0}, x_3\frac{s_0}{x_0})$. We see that:

$$(0, s_1 - x_1\frac{s_0}{x_0}, s_2 - x_2\frac{s_0}{x_0}, s_3 - x_3\frac{s_0}{x_0}) + \operatorname{im} t = (s_0, s_1, s_2, s_3) + \operatorname{im} t$$

which proves that $\psi_0\phi_0 = \mathrm{id}$.

It remains to prove that the other composition is also the identity:

$$\begin{aligned}
\phi_0\psi_0(f,g) &= \phi_0(0, x_3 f - x_2 g, x_0 f, x_0 g) \\
&= \left(\frac{x_0(x_0 f) - x_2 \cdot 0}{x_0^2}, \frac{x_0(x_0 g) - x_3 \cdot 0}{x_0^2}\right) \\
&= (f, g)
\end{aligned}$$

Similarly, we can find trivializations $\phi_i : E|_{D(x_i)} \to \mathcal{O}^2_{D(x_i)}$ for $i = 1, 2, 3$. Full list of trivializations and inverse maps is as follows:

$$\begin{aligned}
\phi_0((s_0,s_1,s_2,s_3)+\operatorname{im} t) &= \left(\frac{x_0s_2-x_2s_0}{x_0^2}, \frac{x_0s_3-x_3s_0}{x_0^2}\right)\\
\phi_0^{-1}(f,g) &= (0, x_3f-x_2g, x_0f, x_0g)+\operatorname{im} t\\
\phi_1((s_0,s_1,s_2,s_3)+\operatorname{im} t) &= \left(\frac{x_1s_2-x_2s_1}{x_1^2}, \frac{x_1s_3-x_3s_1}{x_1^2}\right)\\
\phi_1^{-1}(f,g) &= (x_2g-x_3f, 0, x_1f, x_1g)+\operatorname{im} t\\
\phi_2((s_0,s_1,s_2,s_3)+\operatorname{im} t) &= \left(\frac{x_2s_0-x_0s_2}{x_2^2}, \frac{x_2s_1-x_1s_2}{x_2^2}\right)\\
\phi_2^{-1}(f,g) &= (x_2f, x_2g, 0, x_1f-x_0g)+\operatorname{im} t\\
\phi_3((s_0,s_1,s_2,s_3)+\operatorname{im} t) &= \left(\frac{x_3s_0-x_0s_3}{x_3^2}, \frac{x_3s_1-x_1s_3}{x_3^2}\right)\\
\phi_3^{-1}(f,g) &= (x_3f, x_3g, x_0g-x_1f, 0)+\operatorname{im} t
\end{aligned}$$

2.3. Global sections of E

We will describe global sections of E. Since $H^1(\mathcal{O}) = 0$, from the long exact sequence corresponding to sequence (2.2) we deduce that the global sections of E are the global sections of F modulo sections comming from $\mathcal{O}$ by t. On the other hand, from sequence (2.1) we see that we can identify global sections of F with global sections $(s_0, s_1, s_2, s_3) \in H^0(\mathcal{O}(1)^4)$ satisfying

$$x_1s_0 - x_0s_1 + x_3s_2 - x_2s_3 = 0. \tag{2.3}$$

Since $H^0(\mathcal{O}(1)) = \operatorname{span}\langle x_0, x_1, x_2, x_3\rangle$, the global sections $H^0(\mathcal{O}(1)^4)$ can be identified with 4x4 matrices, where i-th column contains coefficients at x_0, x_1, x_2, x_3 in i-th direct summand of $H^0(\mathcal{O}(1)^4)$. Condition (2.3) says that $H^0(F)$ consists of the matrices of the form:

$$\begin{pmatrix} a & 0 & f & -e\\ 0 & a & -c & d\\ d & e & b & 0\\ c & f & 0 & b \end{pmatrix}. \tag{2.4}$$

The image of $t : H^0(\mathcal{O}) \to H^0(\mathcal{O}(1)^4)$ is the space of scalar matrices, so $H_0(E) = H^0(F)/\operatorname{im} t$ can be identified with matrices of the form (2.4) and trace 0, that is, the set of matrices of the form:

$$\begin{pmatrix} a & 0 & f & -e\\ 0 & a & -c & d\\ d & e & -a & 0\\ c & f & 0 & -a \end{pmatrix}. \tag{2.5}$$

Using similar method we can obtain a description of $H^0(E(k))$ for $k \in \mathbb{Z}$: twist (2.2) by k to obtain

$$0 \to \mathcal{O}(k) \xrightarrow{t} F(k) \to E(k) \to 0 \tag{2.6}$$

Again, $H^1(\mathcal{O}(k)) = 0$ (because we are working on $\mathbb{P}^3$), so it is enough to know the description of global sections of $F(k)$. From the exact sequence

$$0 \to F(k) \to \mathcal{O}(k+1)^4 \xrightarrow{\omega(k)} \mathcal{O}(k+2) \to 0 \tag{2.7}$$

we see that global sections of $F(k)$ can be identified with sections $(s_1, s_2, s_3, s_4) \in H^0(\mathcal{O}(k+1)^4)$ satisfying (2.3). We will only calculate the dimension of $H^0(E(k))$ for $k \geq 0$.

Consider a long exact sequence associated to (2.7):

$$0 \to H^0(F(k)) \to H^0(\mathcal{O}(k+1))^4 \xrightarrow{\omega(k)} H^0(\mathcal{O}(k+2)) \to H^1(F(k)) \to \dots$$

The map $H^0(\mathcal{O}(k+1))^4 \xrightarrow{\omega(k)} H^0(\mathcal{O}(k+2))$ is surjective: a global section $f \in H^0(\mathcal{O}(k+2))$ can be identified with a homogeneous polynomial of degree $k+2$ in variables x_0, x_1, x_2, x_3. It is clear that we can write any such polynomial f as

$$f = x_1 f_0 - x_0 f_1 + x_3 f_2 - x_2 f_3$$

for some homogenous polynomials f_0, f_1, f_2, f_3 of degree $k+1$.

The following sequence is thus exact:

$$0 \to H^0(F(k)) \to H^0(\mathcal{O}(k+1))^4 \xrightarrow{\omega(k)} H^0(\mathcal{O}(k+2)) \to 0$$

It follows that $h^0(F(k)) = 4h^0(\mathcal{O}(k+1)) - h^0(\mathcal{O}(k+2))$. On the other hand, since $H^1(\mathcal{O}(k)) = 0$, from (2.2) we see that $h^0(E(k)) = h^0(F(k)) - h^0(\mathcal{O}(k))$. Putting it together, we have the following

Proposition 2.3.1. *Dimension of the space of global sections of $E(k)$ for $k \in \mathbb{Z}$ is described by the following formula:*

$$\begin{aligned} h^0(E(k)) &= 4h^0(\mathcal{O}(k+1)) - h^0(\mathcal{O}(k+2)) - h^0(\mathcal{O}(k)) \\ &= 4\binom{k+4}{3} - \binom{k+5}{3} - \binom{k+3}{3} \\ &= \frac{1}{3}\left(k^3 + 9k^2 + 23k + 15\right) \end{aligned}$$

We can calculate the dimension easily for small k.

k	0	1	2	3	4	5	6	7	8	9
$h^0(E(k))$	5	16	35	64	105	160	231	320	429	560

Table 2.1: Dimensions of space of global sections of $E(k)$ for small k

We can prove an important and useful fact.

Proposition 2.3.2. *E is generated by global sections.*

Proof. We will work explicitly with trivialization. Let $f : H^0(E) \otimes \mathcal{O} \to E$ be the map induced by global sections. We need to show that f is surjective. It is enough to prove that it is surjective after restricting to $D(x_i)$ and composing with ϕ_i, that is, that map $\phi_i \circ f|_{D(x_i)} : H^0(E) \otimes \mathcal{O}_{D(x_i)} \to \mathcal{O}^2_{D(x_i)}$ is surjective.

We will deal now with the case of $i = 0$. As we calculated above, any global section $s \in H^0(E)$ can be written as:

$$s = (ax_0 + dx_2 + cx_3, ax_1 + ex_2 + fx_3, fx_0 - cx_1 - ax_2, -ex_0 + dx_1 - ax_3) + \operatorname{im} t$$

for some $a, c, d, e, f \in k$. The image of s under ϕ_0 is:

$$\begin{aligned} \phi_0(s) = \Bigg(&\frac{fx_0^2 - cx_0x_1 - ax_0x_2 - ax_0x_2 - dx_2^2 - cx_2x_3}{x_0^2}, \\ &\frac{-ex_0^2 + dx_0x_1 - ax_0x_3 - ax_0x_3 - dx_2x_3 - cx_3^2}{x_0^2} \Bigg) \end{aligned}$$

Let $s_a \in H^0(E)$ denote a section corresponding to $a = 1$ and $c = d = e = f = 0$, and $s_b, s_c, s_d, s_e \in H^0(E)$ analogously. We have:

$$\begin{aligned} \phi_0(s_f) &= (1,0) \\ \phi_0(s_e) &= (0,1) \end{aligned}$$

These sections clearly generate $\mathcal{O}^2_{D(x_0)}$.

Proof that E is globally generated at every point of $D(x_i)$ for $i = 1, 2, 3$ is similar. □

2.4. Codimension 2 locally complete intersection corresponding to E

There is another way of looking at E. Classical result (e.g. [Hart1978] or [OkSchnSp1980, §5]) says that there is a correspondence between vector bundles of rank 2 over $\mathbb{P}^n$, and codimension 2 locally complete intersections 2 in $\mathbb{P}^n$. The way it works is the following:

Let $\mathcal{E}$ be a rank 2 vector bundle over $\mathbb{P}^n$. The zero scheme $Y = V(s)$ of a general section $s \in H^0(E)$ is a locally complete intersection of codimension 2 (possibly singular and reducible). The bundle $\mathcal{E}$ can be recovered from Y – we have the following [OkSchnSp1980, Thm5.1.1]

Theorem 2.4.1. *Let Y be a locally complete intersection of codimension 2 in $\mathbb{P}^n$, $n \geq 3$, with sheaf of ideals $\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n}$. Assume that the determinant of a normal bundle to Y extends to a line bundle over the whole $\mathbb{P}^n$:*

$$\det \mathcal{N}_{Y/\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(k)|_Y.$$

Then there exists a vector bundle $\mathcal{E}$ of rank 2 over $\mathbb{P}^n$, and a section $s \in H^0(E)$, the scheme of zeros of which is Y. The section s induces an exact sequence:

$$0 \to \mathcal{O} \xrightarrow{s} \mathcal{E} \to \mathcal{I}_Y(k) \to 0,$$

and Chern classes of $\mathcal{E}$ are

$$c_1(E) = k, \quad c_2(E) = \deg Y$$

We shall now find an appropriate section $s \in H^0(E)$ such that the corresponding Y is easy to work with. Consider a section s corresponding to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.8}$$

Section s pulls back to the section $s' = (x_0, x_1, -x_2, -x_3) \in H^0(F) \subset H^0(\mathcal{O}(1)^4)$. Since E is obtained from F by taking quotient of the trace map, $s(x) = 0$ precisely when $s'(x) = \lambda x$ for some $\lambda \in k$. Over $D(x_0)$, we see that since $\operatorname{char} k \neq 2$, this can happen only if $x_2 = x_3 = 0$, and same for $D(x_1)$. Similarly, over $D(x_2)$ and $D(x_3)$, $s(x) = 0$ only if $x_0 = x_1 = 0$. We see thus that the zero set $Y = V(s)$ of s is the union of two lines $L_1 \cup L_2$, where $L_1 = \{x_0 = x_1 = 0\} = V(x_0, x_1)$, and $L_2 = \{x_2 = x_3 = 0\} = V(x_2, x_3)$.

Since L_1 is the intersection of two hyperplanes $H_1 = V(x_0)$ and $H_1' = V(x_1)$, its normal bundle is

$$\mathcal{N}_{L_1/\mathbb{P}^3} = \mathcal{N}_{H_1/\mathbb{P}^3}|_{L_1} \oplus \mathcal{N}_{H_1'/\mathbb{P}^3}|_{L_1} = \mathcal{O}_{\mathbb{P}^3}(1)|_{L_1} \oplus \mathcal{O}_{\mathbb{P}^3}(1)|_{L_1} = \mathcal{O}_{L_1}(1) \oplus \mathcal{O}_{L_1}(1)$$

Thus, the determinant of the normal bundle to L_1 is $\mathcal{O}_{L_1}(2)$. Same calculation for L_2 gives us that normal bundle to L_2 is also $\mathcal{O}_{L_2}(2)$, and so $\det \mathcal{N}_{Y/\mathbb{P}^3} = \mathcal{O}_Y(2)$.

Since $\mathcal{O}_Y(2) = \mathcal{O}_{\mathbb{P}^3}(2)|_Y$, we see that hypotheses of Theorem 2.4.1 are satisfied, so we could have constructed the bundle E by invoking the theorem.

As $\deg Y = \deg L_1 + \deg L_2 = 1 + 1 = 2$, it follows from Theorem 2.4.1 that Chern classes of E are:

$$c_1(E) = 2, \quad c_2(E) = \deg Y = 2$$

Knowledge of Chern classes lets us prove an important property of E:

Proposition 2.4.2. *The bundle E is indecomposable, that is, it is not a direct sum of line bundles.*

Proof. Indeed, if E was a direct sum of two line bundles, $E = \mathcal{O}(a) \oplus \mathcal{O}(b)$, its Chern classes would be $c_1(E) = a+b, c_2(E) = ab$. But there are no integers a, b satisfying $a+b = 2, ab = 2$. □

Also, as first Chern class of a vector bundle is a degree of its top exterior power, we see that

$$\det E = \bigwedge^2 E \simeq \mathcal{O}(c_1(E)) = \mathcal{O}(2) \tag{2.9}$$

For any rank 2 vector bundle, there is a natural isomorphism

$$E \to \mathcal{H}om(E, \det E) \quad s \mapsto (t \mapsto s \wedge t) \tag{2.10}$$

But

$$\mathcal{H}om(E, \det E) \simeq E^* \otimes \mathcal{O}(2) = E^*(2),$$

so we have isomorphism:

$$E \simeq E^*(2) \tag{2.11}$$

Twisting this isomorphism by -1 gives an isomorphism between $E(-1)$ and its dual $E^*(1)$. This gives another proof of the fact that E is indecomposable: if E splitted into direct sum of two line bundles, and $E(-1)$ was self-dual, E would have to be $\mathcal{O}(-a+1) \oplus \mathcal{O}(a+1)$. But for no $a \in \mathbb{Z}$ we have $h^0(\mathcal{O}(-a+1)) + h^0(\mathcal{O}(a+1)) = 5$.

Chapter 3

Finding equations of secant varieties

In this chapter we describe how one can use vector bundles to find equations of secant varieties. We apply the method to the null-correlation bundle, and give some examples where resulting rank variety has secant variety as an irreducible component. We also intersect the rank variety of null-correlation bundle with the catalecticant variety for symmetric catalecticant.

3.1. Obtaining equations from vector bundles

Let E be a vector bundle on X of rank e. For any $v \in H^0(L)^*$ we have a linear map

$$A_v^E : H^0(E) \to H^0(\mathcal{H}om(E, L))^*$$

defined by

$$A_v^E(s)(\phi) = v(\phi(s))$$

where $s \in H^0(E), \phi \in H^0(\mathcal{H}om(E, L))$.

Define:

$$Rank_k(E) = \mathbb{P}\{v \in V : \operatorname{rank} A_v^E \leq ek\}$$

The following theorem [LandOtt11, Prop 5.1.1] is the essence:

Theorem 3.1.1. *Let $X \subset \mathbb{P}V = \mathbb{P}H^0(L)^*$ be a variety, and E be a rank e vector bundle on it. Then:*

$$\sigma_r(X) \subset Rank_r(E).$$

In other words, minors of size $re + 1$ of A_v^E give equations for $\sigma_r(X)$.

Proof. By definition of A_v^E we have that for $x = [v] \in X$, any $s \in H^0(E)$ such that $s(x) = 0$ satisfies $A_v^E(s) = 0$. Indeed, in the embedding

$$X \hookrightarrow \mathbb{P}H^0(L)^*, \quad x \mapsto \{\psi \in H^0(L) : \psi(x) = 0\},$$

the image of x in $\mathbb{P}H^0(L)^*$ is precisely the line of functionals that vanish on sections that vanish on x. But if $s \in H^0(E)$ vanishes on x, then since $\phi \in H^0(\mathcal{H}om(E, L))$ is a morphism of vector bundles, for any such ϕ, the section $\phi(s) \in H^0(L)$ also vanishes on x. Thus v vanishes on $\phi(s)$. This means that $A_v^E(s) = v(\phi(s)) = 0$ for any $\phi \in H^0(\mathcal{H}om(E, L))$, so $A_v^E(s) = 0 \in H^0(\mathcal{H}om(E, L))^*$.

We see thus that:

$$H^0(\mathcal{I}_x \otimes E) \subset \ker A_v^E$$

where $\mathcal{I}_x$ is an ideal sheaf of functions vanishing at x. The subspace $H^0(\mathcal{I}_x \otimes E) \subset H^0(E)$ of sections vanishing at x has a codimension at most $\operatorname{rank} E = e$ – indeed, for any subspace $V \subset H^0(E)$, when $\dim V > e$, as the fiber of E over x, E_x, has dimension e, the linear map $V \to E_x, s \mapsto s(x)$ must have a nontrivial kernel. It follows that V must intersect nontrivially $H^0(\mathcal{I}_x \otimes E)$, and so $\operatorname{codim} H^0(\mathcal{I}_x \otimes E) \leq e$.

Since $H^0(\mathcal{I}_x \otimes E) \subset \ker A_v^E$ and $\operatorname{codim} H^0(\mathcal{I}_x \otimes E) \leq e$, we must have that $\operatorname{rank} A_v^E \leq e$. Now, general $v \in \tilde{\sigma}_r(X)$ in the affine cone over secant variety can be expressed as $v = x_1 + \ldots + x_r$, for $x_1, \ldots, x_r \in \tilde{X}$, the affine cone over $X \subset \mathbb{P}H^0(L)^*$. Since we have:

$$A_v^E = A_{x_1 + \ldots + x_r}^E = A_{x_1}^E + \ldots + A_{x_r}^E,$$

it follows that for $v \in \tilde{\sigma}_r(X)$, we have:

$$\operatorname{rank} A_v^E = \operatorname{rank} A_{x_1 + \ldots + x_r}^E = \operatorname{rank}\left(\sum_{i=1}^r A_{x_i}^E\right) \leq \sum_{i=1}^r \operatorname{rank} A_{x_i}^E \leq \sum_{i=1}^r e = re$$

The point $v \in \tilde{\sigma}_r(X)$ was general, so we have proved that for a dense set of points $x = [v] \in \sigma_r(X)$, we have $\operatorname{rank} A_v^E \leq re$. But $\operatorname{rank} A_v^E \leq re$ is a closed condition on v, so it also holds on a closure of this set of points, which is exactly $\sigma_r(X)$. □

In practice, the description of $H^0(E)$ and $H^0(\mathcal{H}om(E, L))$ may not be easy to find and work with. Because of that, frequently it is easier to work with presentations.

Definition 3.1.2. *Presentation of a vector bundle $\mathcal{E}$ over a projective space $\mathbb{P}^n$ is a morphism:*

$$p_{\mathcal{E}} : L_1 \to L_0$$

where L_1, L_0 are direct sums of line bundles, and

$$\operatorname{im} p_E = \mathcal{E}.$$

The Theorem 3.1.1 has its analogue in the presentation setting. Take any $v \in H^0(L)^*$. We have a natural map

$$P_v^E : H^0(L_1) \to H^0(\mathcal{H}om(L_0, L))^*, \quad P_v^E(s)(\phi) = v(\phi(p_E(s))).$$

Theroem 3.1.1 now takes form [LandOtt11, Prop. 8.4.1]

Theorem 3.1.3. *Assume that maps $H^0(L_1) \xrightarrow{i} H^0(E)$ and $H^0(\mathcal{H}om(L_0, L)) \xrightarrow{j} H^0(\mathcal{H}om(E, L))$ are surjective. Then the rank of A_v^E equals the rank of P_v^E, so that the size $ke + 1$ minors give equations for $\sigma_k(X)$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} H^0(L_1) & \xrightarrow{P_v^E} & H^0(\mathcal{H}om(L_0, L))^* \\ \downarrow i & & \uparrow j^* \\ H^0(E) & \xrightarrow{P_v^E} & H^0(\mathcal{H}om(E, L))^* \end{array}$$

Since by assumption i and j are surjective, j^* is injective, so it follows that $\operatorname{rank} A_v^E = \operatorname{rank} P_v^E$. □

It is also useful to know when the rank conditions give enough equations to cut out the secant variety. For this, there is a following theorem:

Theorem 3.1.4. *[LandOtt11, 8.4.2] Notations as above. Let $v \in \sigma_k(X)$. Assume that maps $H^0(L_1) \xrightarrow{i} H^0(E)$ and $H^0(\mathcal{H}om(L_0, L)) \xrightarrow{j} H^0(\mathcal{H}om(E, L))$ are surjective. Define pairing:*

$$g_v : \ker P_v \otimes (\operatorname{im} P_v)^{\perp} \to H^0(L)^{**}$$

by

$$g_v(f, h)(\phi) = P_\phi(f)(h)$$

If the rank of g_v equals the codimension of $\sigma_k(X)$, then $\sigma_k(X)$ is an irreducible component of $Rank_k(E)$ passing through $[v] \in \mathbb{P}H^0(L)^*$.

The reason why such pairing g_v appears here is the following. We can naturally identify $H^0(L)^{**}$ with $H^0(L)$. Then for v such that $[v]$ is a smooth point of $Rank_k(E)$, the image of g_v in $H^0(L)$ is precisely the affine conormal space of $Rank_k(E)$ at $[v]$.

3.2. Presentation of null-correlation bundle

We shall now find a nice presentation of null-correlation bundle E that allows us to easily check the assumptions of Theorem 3.1.3.

Since E is generated by global sections, we have a surjection

$$H^0(E) \otimes \mathcal{O} \simeq \mathcal{O}^5 \to E \to 0. \tag{3.1}$$

On the other hand, by our construction, E is subbundle of $T\mathbb{P}^3$. Now, on $\mathbb{P}^3$ there is a Koszul complex:

$$0 \to \bigoplus_{0 \le i<j<k<l \le 3} \mathcal{O}(-4) \to \bigoplus_{0 \le i<j<k \le 3} \mathcal{O}(-3) \to \bigoplus_{0 \le i<j \le 3} \mathcal{O}(-2) \to \bigoplus_{0 \le i \le 3} \mathcal{O}(-1) \to \mathcal{O} \to 0 \tag{3.2}$$

Let U be the kernel of $\bigoplus_{0 \le i \le 3} \mathcal{O}(-1) \to \mathcal{O}$, so that we have an exact sequence:

$$0 \to U \to \bigoplus_{0 \le i \le 3} \mathcal{O}(-1) \to \mathcal{O} \to 0$$

Since it is exact, U is locally free as a kernel of surjection of vector bundles. Consider the dual sequence:

$$0 \to \mathcal{O} \to \bigoplus_{0 \le i \le 3} \mathcal{O}(1) \to U^* \to 0$$

The map $\mathcal{O} \to \bigoplus_{0 \le i \le 3} \mathcal{O}(1)$ is just the trace map $t : \mathcal{O} \to \mathcal{O}(1)^4$, so $U^* \simeq T\mathbb{P}^3$. Thus $U = \Omega^1$, the sheaf of 1-forms on $\mathbb{P}^3$.

Now, by exactness of Koszul complex (3.2), we have a surjection:

$$\bigoplus_{0 \le i<j \le 3} \mathcal{O}(-2) \to U = \Omega^1.$$

Dually, we have an injection:

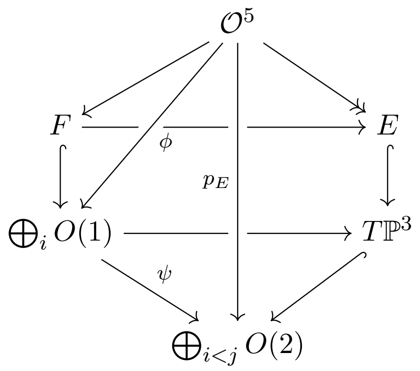
$$0 \to T\mathbb{P}^3 \to \bigoplus_{0 \le i<j \le 3} \mathcal{O}(2). \tag{3.3}$$

Putting together (3.1) and (3.3), we obtain a presentation of E:

$$p_E : \mathcal{O}^5 \twoheadrightarrow E \hookrightarrow T\mathbb{P}^3 \hookrightarrow \bigoplus_{0 \leq i < j \leq 3} \mathcal{O}(2).$$

Let us write p_E explicitly.

We have a following commutative diagram:



(3.4)

where the horizontal maps are the cokernel maps of the trace maps $\mathcal{O} \hookrightarrow F$ and $\mathcal{O} \hookrightarrow \bigoplus_i \mathcal{O}(1)$, and the map $\mathcal{O}^5 \to F$ is induced by the choice of liftings of the 5 global sections of E that generate $H^0(E)$. It follows that we can calculate p_E as a composition:

$$p_E : \mathcal{O}^5 \xrightarrow{\phi} \bigoplus_i \mathcal{O}(1) \xrightarrow{\psi} \bigoplus_{i<j} \mathcal{O}(2)$$

The map $\phi : \mathcal{O}^5 \to \bigoplus_i \mathcal{O}(1)$ is given by $1_i \mapsto u_i$, where 1_i is a unit in i-th summand of $H^0(\mathcal{O}^5)$, and u_i, $i = 1, \dots, 5$ are sections which map to a basis of $H^0(E)$. In our case, we can assume that the sections u_i, $i = 1, \dots, 5$ map to s_a, s_c, s_d, s_e, s_f, where $s_a \in H^0(E)$ denote a section corresponding to $a = 1$ and $c = d = e = f = 0$, and $s_b, s_c, s_d, s_e \in H^0(E)$ analogously, as in Proposition 2.3.2. Thus, $\phi : \mathcal{O}^5 \to \bigoplus_i \mathcal{O}(1)$ is given by a matrix:

$$\begin{pmatrix} x_0 & x_3 & x_2 & 0 & 0 \\ x_1 & 0 & 0 & x_2 & x_3 \\ -x_2 & -x_1 & 0 & 0 & x_0 \\ -x_3 & 0 & x_1 & -x_0 & 0 \end{pmatrix} \tag{3.5}$$

As discussed earlier, the map $\psi : \bigoplus_i \mathcal{O}(1) \to \bigoplus_{i<j} \mathcal{O}(2)$ is dual to a Koszul complex map $\bigoplus_{i<j} \mathcal{O}(-2) \twoheadrightarrow \Omega^1 \hookrightarrow \bigoplus_i \mathcal{O}(-1)$, which corresponds to the following map of graded $S = k[x_0, x_1, x_2, x_3]$-modules:

$$\bigoplus_{i<j} S(-2) \to \bigoplus_i S(-1), \quad e_{ij} \mapsto x_j e_i - x_i e_j \tag{3.6}$$

which, as a matrix, is given by:

$$\begin{pmatrix} x_1 & x_2 & x_3 & 0 & 0 & 0 \\ -x_0 & 0 & 0 & x_2 & x_3 & 0 \\ 0 & -x_0 & 0 & -x_1 & 0 & x_3 \\ 0 & 0 & -x_0 & 0 & -x_1 & -x_2 \end{pmatrix}. \tag{3.7}$$

Dualizing, the map $\psi : \bigoplus_i \mathcal{O}(1) \hookrightarrow \bigoplus_{i<j} \mathcal{O}(2)$ is given by:

$$\begin{pmatrix} x_1 & -x_0 & 0 & 0 \\ x_2 & 0 & -x_0 & 0 \\ x_3 & 0 & 0 & -x_0 \\ 0 & x_2 & -x_1 & 0 \\ 0 & x_3 & 0 & -x_1 \\ 0 & 0 & x_3 & -x_2 \end{pmatrix}. \tag{3.8}$$

We can now calculate the matrix of $p_E : \mathcal{O}^5 \to \bigoplus_{i<j} \mathcal{O}(2)$ – it is the product of (3.8) and (3.5):

$$\begin{pmatrix} 0 & x_1x_3 & x_1x_2 & x_0x_2 & -x_0x_3 \\ 2x_0x_2 & x_2x_3 + x_0x_1 & x_2^2 & 0 & -x_0^2 \\ 2x_0x_3 & x_3^2 & x_2x_3 - x_0x_1 & x_0^2 & 0 \\ 2x_1x_2 & x_1^2 & 0 & x_2^2 & x_2x_3 - x_0x_1 \\ 2x_1x_3 & 0 & -x_1^2 & x_2x_3 + x_0x_1 & x_3^2 \\ 0 & -x_1x_3 & -x_1x_2 & x_0x_2 & x_0x_3 \end{pmatrix} \tag{3.9}$$

We shall now obtain some more information about the presentation p_E, and use it to obtain a better presentation.

Consider an inclusion $T\mathbb{P}^3 \hookrightarrow \bigoplus_{i<j} \mathcal{O}(2)$. To find the cokernel, we dualize and take a look at the kernel: the dual map $\bigoplus_{i<j} \mathcal{O}(-2) \to \Omega^1$ comes from the Koszul complex map $\bigoplus_{i<j} S(-2) \to \bigoplus_i S(-1)$, and its kernel is equal to the image of $\bigoplus_{i<j<k} S(-3) \to \bigoplus_{i<j} S(-2)$, which is the same as cokernel of $\bigoplus_{i<j<k<l} S(-4) \to \bigoplus_{i<j<k} S(-3)$. But this map is the dual of the map $\bigoplus_i S(-1) \to S$ shifted by 4. Since we know that its kernel is Ω^1, the cokernel of $\bigoplus_{i<j<k<l} S(-4) \to \bigoplus_{i<j<k} S(-3)$ is $T\mathbb{P}^3(-4)$. It can be illustrated by the following diagram, in which the horizontal sequence is the Koszul complex:

$$\begin{array}{ccccccccc} & 0 & & & 0 & & & & \\ & & \searrow & \nearrow & & & & & \\ & & & T\mathbb{P}^3(-4) & & & & & \\ & & \nearrow & & \searrow & & & & \\ 0 \to \mathcal{O}(-4) \to & \bigoplus_{i<j<k} \mathcal{O}(-3) & & \longrightarrow & \bigoplus_{i<j} \mathcal{O}(-2) & \longrightarrow & \bigoplus_{i} \mathcal{O}(-1) & \to \mathcal{O} \to 0 & \\ & & & & & \searrow & \nearrow & & \\ & & & & & \Omega^1 & & & \\ & & & & \nearrow & & \searrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{3.10}$$

It follows that we have a following exact sequence of locally free sheaves:

$$0 \to T\mathbb{P}^3(-4) \to \bigoplus_{i<j} \mathcal{O}(-2) \to \Omega^1 \to 0$$

Dualizing, we obtain:

$$0 \to T\mathbb{P}^3 \to \bigoplus_{i<j} \mathcal{O}(2) \to \Omega^1(4) \to 0$$

Let C be the cokernel of inclusion $E \to \bigoplus_{i<j} \mathcal{O}(2)$. By previous considerations, we have the following commutative diagram, in which all rows and columns are exact:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & T\mathbb{P}^3 & \xrightarrow{\omega} & \mathcal{O}(2) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & \bigoplus_{i<j} \mathcal{O}(2) & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \Omega^1(4) & = & \Omega^1(4) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{3.11}$$

We see that C is an extension of $\Omega^1(4)$ by $\mathcal{O}(2)$. To understand C, we will calculate $\mathrm{Ext}^1(\Omega^1(4), \mathcal{O}(2)) = \mathrm{Ext}^1(\Omega^1, \mathcal{O}(-2))$. From Euler sequence:

$$0 \to \Omega^1 \to \mathcal{O}(-1)^4 \to \mathcal{O} \to 0$$

we have a long exact Ext sequence:

$$\ldots \to \mathrm{Ext}^1(\mathcal{O}(-1)^4, \mathcal{O}(-2)) \to \mathrm{Ext}^1(\Omega^1, \mathcal{O}(-2)) \to \mathrm{Ext}^2(\mathcal{O}, \mathcal{O}(-2)) \to \ldots$$

We have:

$$\mathrm{Ext}^1(\mathcal{O}(-1)^4, \mathcal{O}(-2)) = \mathrm{Ext}^1(\mathcal{O}(-1), \mathcal{O}(-2))^4 = \mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(-1))^4 = H^1(\mathcal{O}(-1))^4 = 0$$

$$\mathrm{Ext}^2(\mathcal{O}, \mathcal{O}(-2)) = H^2(\mathcal{O}(-2)) = 0$$

Thus $\mathrm{Ext}^1(\Omega^1(4), \mathcal{O}(2)) = \mathrm{Ext}^1(\Omega^1, \mathcal{O}(-2)) = 0$, and so $C \simeq \Omega^1(4) \oplus \mathcal{O}(2)$.

This additional $\mathcal{O}(2)$ summand suggests that there is some redundancy in p_E. Indeed, if we look back at the matrix (3.9), we see that row 1 (that is, the first summand in $\bigoplus_{i<j} \mathcal{O}(2)$) is redundant. More precisely, if we change the order of the summands in $\mathcal{O}^5$, so that p_E is represented by the matrix:

$$\begin{pmatrix} -x_0x_2 & -x_0x_3 & x_1x_2 & x_1x_3 & 0 \\ 0 & -x_0^2 & x_2^2 & x_0x_1 + x_2x_3 & 2x_0x_2 \\ x_0^2 & 0 & x_2x_3 - x_0x_1 & x_3^2 & 2x_0x_3 \\ x_2^2 & x_2x_3 - x_0x_1 & 0 & x_1^2 & 2x_1x_2 \\ x_0x_1 + x_2x_3 & x_3^2 & -x_1^2 & 0 & 2x_1x_3 \\ x_0x_2 & x_0x_3 & -x_1x_2 & -x_1x_3 & 0 \end{pmatrix} \tag{3.12}$$

and if we compose it with an automorphism of $\mathcal{O}^5$ that multiplies 5th summand by $-\frac{1}{2}$, and also with an automorphism of $\mathcal{O}(2)^6$ that multiplies 4th and 5th summand by -1, we obtain a map $q : \mathcal{O}^5 \to \bigoplus_{i<j} \mathcal{O}(2)$ represented by a matrix:

$$\begin{pmatrix} -x_0x_2 & -x_0x_3 & x_1x_2 & x_1x_3 & 0 \\ 0 & -x_0^2 & x_2^2 & x_0x_1 + x_2x_3 & -x_0x_2 \\ x_0^2 & 0 & x_2x_3 - x_0x_1 & x_3^2 & -x_0x_3 \\ -x_2^2 & x_0x_1 - x_2x_3 & 0 & -x_1^2 & x_1x_2 \\ -x_0x_1 - x_2x_3 & -x_3^2 & x_1^2 & 0 & x_1x_3 \\ x_0x_2 & x_0x_3 & -x_1x_2 & -x_1x_3 & 0 \end{pmatrix} \tag{3.13}$$

which still has the property that $\operatorname{im} q \simeq E$. Its first row, however, is the transposition of the last column (up to scalar). Thus, the projection map $\bigoplus_{i<j} \mathcal{O}(2) \to \mathcal{O}(2)^5$ that drops the first coordinate (corresponding to $i = 0, j = 1$) is injective on E, and so composition of this projection with q has its image still isomorphic to E. **From now on, we shall call this map** $p_E : \mathcal{O}^5 \to \mathcal{O}(2)^5$. Its matrix is:

$$\begin{pmatrix} 0 & -x_0^2 & x_2^2 & x_0x_1 + x_2x_3 & -x_0x_2 \\ x_0^2 & 0 & x_2x_3 - x_0x_1 & x_3^2 & -x_0x_3 \\ -x_2^2 & x_0x_1 - x_2x_3 & 0 & -x_1^2 & x_1x_2 \\ -x_0x_1 - x_2x_3 & -x_3^2 & x_1^2 & 0 & x_1x_3 \\ x_0x_2 & x_0x_3 & -x_1x_2 & -x_1x_3 & 0 \end{pmatrix} \tag{3.14}$$

Note that the matrix is antisymmetric. Twisting it by -1 we can obtain a self-dual presentation of $E(-1) \simeq E^*(1)$:

$$p_E(-1) : \mathcal{O}(-1)^5 \to \mathcal{O}(1)^5, \quad (p_E(-1))^* = -p_E(-1).$$

By previous calculations, p_E fits in an exact sequence:

$$\mathcal{O}^5 \xrightarrow{p_E} \mathcal{O}(2)^5 \to \Omega^1(4) \to 0 \tag{3.15}$$

To find the kernel of p_E, notice that the dual map p_E^* is just $-p_E(-2)$, so if we dualize (3.15), we obtain:

$$0 \to T\mathbb{P}^3(-4) \to \mathcal{O}(-2)^5 \xrightarrow{-p_E(-2)} \mathcal{O}^5 \tag{3.16}$$

Now twist it by 2 to obtain:

$$0 \to T\mathbb{P}^3(-2) \to \mathcal{O}^5 \xrightarrow{-p_E} \mathcal{O}(2)^5 \tag{3.17}$$

Since the kernel of $-p_E$ is the same as the kernel of p_E, we have an exact sequence:

$$0 \to T\mathbb{P}^3(-2) \to \mathcal{O}^5 \xrightarrow{p_E} \mathcal{O}(2)^5 \to \Omega^1(4) \to 0 \tag{3.18}$$

3.3. Using presentation of E to find equations

Recall Theorem 3.1.3:

Theorem 3.3.1. *Assume that maps $H^0(L_1) \xrightarrow{i} H^0(E)$ and $H^0(\mathcal{H}om(L_0, L)) \xrightarrow{j} H^0(\mathcal{H}om(E, L))$ are surjective. Then the rank of A_v^E equals the rank of P_v^E, so that the size $ke+1$ minors give equations for $\sigma_k(X)$.*

If we want to use our presentation $p_E : L_1 \to L_0$, where $L_1 = \mathcal{O}^5$ and $L_0 = \mathcal{O}(2)^5$, we need to check that $H^0(L_1) \xrightarrow{i} H^0(E)$ and $H^0(\mathcal{H}om(L_0, L)) \xrightarrow{j} H^0(\mathcal{H}om(E, L))$ are surjective. But from (3.18) we have exact sequences:

$$0 \to T\mathbb{P}^3(-2) \to \mathcal{O}^5 \to E \to 0 \tag{3.19}$$

$$0 \to E \to \mathcal{O}(2)^5 \to \Omega^1(4) \to 0 \tag{3.20}$$

Applying $\mathcal{H}om(-, \mathcal{O}(d))$ to (3.20), which is the same as dualizing and twisting by d, we get:

$$0 \to T\mathbb{P}^3(-4+d) \to \mathcal{H}om(\mathcal{O}(2)^5, \mathcal{O}(d)) \to \mathcal{H}om(E, \mathcal{O}(d)) \to 0 \tag{3.21}$$

Since $H^1(T\mathbb{P}^3(k)) = 0$ for any $k \in \mathbb{Z}$ (as one can easily deduce from Euler sequence), considering long exact cohomology sequences of (3.19) and (3.21), we see that assumptions of the Theorem 3.1.3 are satisfied for p_E and all of its twists. Thus, we have a following theorem:

Theorem 3.3.2. *Let $p_E : \mathcal{O}^5 \to \mathcal{O}(2)^5$ be the presentation of null-correlation bundle, as above. Let $k \in \mathbb{Z}$, $d, r \in \mathbb{N}$ be arbitrary. Then the size $2r+1$ minors of the evaluation map:*

$$P_v^{E(k)} : H^0(\mathcal{O}(k)^5) \to H^0(\mathcal{H}om(\mathcal{O}(k+2)^5, \mathcal{O}(d)))$$

give equations for $\sigma_r(\nu_d(\mathbb{P}^3))$.

3.4. Explicit calculations for $\sigma_5(\nu_4(\mathbb{P}^3))$

We shall now use Theorem 3.3.2 to find explicit equations for $\sigma_5(\nu_4(\mathbb{P}^3))$. Theorem says that we need to use minors of size $2 \cdot 5 + 1 = 11$. From table (2.1) we see that since $h^0(E) = 5$, the map P_v^E has rank ≤ 5 for any $v \in H^0(\mathcal{O}(4))^*$, so we cannot possibly obtain any new information. Because of that, instead of E, we shall use $E(1)$ with presentation $p_{E(1)} : \mathcal{O}(1)^5 \to \mathcal{O}(3)^5$ – the matrix representing the presentation $p_{E(1)}$ of $E(1)$ is the same as matrix of p_E, but we now interpret it as a map $\mathcal{O}(1)^5 \to \mathcal{O}(3)^5$.

The corresponding $P_v^{E(1)}$ maps now is:

$$P_v^{E(1)} : H^0(\mathcal{O}(1)^5) \to H^0(\mathcal{H}om(\mathcal{O}(3)^5, \mathcal{O}(4)))^*, \quad P_v^{E(1)}(s)(\phi) = v(\phi(p_{E(1)}(s)))$$

and $h^0(\mathcal{O}(1)^5) = h^0(\mathcal{H}om(\mathcal{O}(3)^5, \mathcal{O}(4))) = 20$.

As a basis of $H^0(\mathcal{O}(k))$ we choose homogenous degree k polynomials in variables x_0, x_1, x_2, x_3. The basis of the dual $H^0(\mathcal{O}(k))^*$ will be the dual basis – the dual to $x_0^{a_0} x_1^{a_1} x_2^{a_2} x_3^{a_3}$ will be denoted by ϕ_{a_0,a_1,a_2,a_3}.

In these bases, for a given presentation $p_F : L_1 \to L_0$, we can obtain explicit matrix of P_ϕ^F as follows [LandOtt11, 8.3]: let $L_1 = \bigoplus_{j=1}^{m_1} \mathcal{O}(b_j)$, $L_0 = \bigoplus_{i=1}^{m_0} \mathcal{O}(b_i)$, and let $p_{ij} \in H^0(\mathcal{O}(a_i - b_j))$ denote the map $\mathcal{O}(b_j) \to \mathcal{O}(a_i)$ given by composition:

$$\mathcal{O}(b_j) \to L_1 \to L_0 \to \mathcal{O}(a_i)$$

Then for any $\phi \in H^0(L)^*$, the matrix of P_ϕ^F is obtained by taking a matrix of $m_1 \times m_0$ blocks, the (i,j)-th block representing the catalecticant contracted by p_{ij}.

In Appendix B we describe, how we used the program described in Appendix A to calculate the matrix of $P_v^{E(1)}$ the minors of which give equations for $\sigma_5(\nu_4(\mathbb{P}^3))$, as in Theorem 3.1.3. We also calculated the rank of the conormal pairing, as in Theorem 3.1.4.

For randomly chosen general point $[v] \in \sigma_5(\nu_4(\mathbb{P}^3))$, the rank of the conormal pairing is 15. Since $\sigma_5(\nu_4(\mathbb{P}^3))$ is not on the exceptions list from Theorem 1.5.1, it has the expected dimension, which is $5 \cdot 3 + 5 - 1 = 19$. On the other hand, the projective space that $\sigma_5(\nu_4(\mathbb{P}^3))$ is embedded in has dimension $\binom{4+3}{4} - 1 = 34$. The codimension of $\sigma_5(\nu_4(\mathbb{P}^3))$ is thus 15, and since it is equal to the rank of conormal pairing, by Theorem 3.1.4 we obtain the following:

Theorem 3.4.1. *Let $p_{E(1)} : \mathcal{O}(1)^5 \to \mathcal{O}(3)^5$ be the presentation of the null-correlation bundle twisted by 1, $E(1)$, given by the matrix (3.14). Then $\sigma_5(\nu_4(\mathbb{P}^3))$ is an irreducible component of the variety cut out by 11×11 size minors of $P_v^{E(1)} : H^0(\mathcal{O}(1)^5) \to H^0(\mathcal{O}(1)^5)$*

It is also easy to see that the matrix of $P_v^{E(1)}$ occuring in the Theorem 3.4.1 is actually skew symmetric, so instead of 11×11 size minors, we can take 12×12 Pfaffians.

3.5. Calculations for other secant varieties

Similarly to $\sigma_5(\nu_4(\mathbb{P}^3))$, we also calculated, when $\sigma_k(\nu_d(\mathbb{P}^3))$ is an irreducible component of $Rank_k(E(l))$, where $E(l)$ is the null-correlation bundle twisted by l. The calculations where performed for $3 \leq d \leq 8$ and $1 \leq l \leq 4$. The results can be found in Appendix C.

3.6. Intersection with rank variety of catalecticant maps

Classical way of finding equations of secant varieties to Veronese embedding was by studying the catalecticant maps. These were obtained via natural embedding $S^dV \hookrightarrow S^aV \otimes S^{d-a}V$ – for $\phi \in S^dV$, one considers the corresponding map $\phi_{a,d-a} : S^aV^* \to S^{d-a}V$. If $\phi \in \sigma_r(\nu_d(\mathbb{P}V))$, then rank $\phi_{a,d-a} \leq r$. Thus, if one chooses basis of V and writes down the corresponding matrix of $\phi_{a,d-a}$, its $(r+1) \times (r+1)$ size minors give equations of $\sigma_r(\nu_d(\mathbb{P}V))$. For explicit examples, and history, see [Geramita].

This approach can be viewed as using the vector bundle method above – indeed, if one chooses $X = \mathbb{P}V$, $L = \mathcal{O}(d)$, and $E = \mathcal{O}(a)$, the corresponding map $A_v^E : H^0(E) \to H^0(\mathcal{H}om(E,L)^*$ is just a map $H^0(\mathcal{O}(a)) \to H^0(\mathcal{O}(a)^* \otimes \mathcal{O}(d))^*$, and $H^0(\mathcal{O}(a)) = S^aV^*$, while $H^0(\mathcal{O}(a)^* \otimes \mathcal{O}(d))^* = H^0(\mathcal{O}(d-a))^* = S^{d-a}V^{**} = S^{d-a}V$, and careful consideration of the maps involved makes it clear that these are just the catalecticants described above.

In some cases it is known that catalecticant minors cut out secant variety as a scheme, or at least give equations for variety such that secant variety is its irreducible component. For instance, size 5×5 minors of $\phi_{a,d-a}$ for $a = \lfloor \frac{d}{2} \rfloor$ cut out $\sigma_4(\nu_d(\mathbb{P}^2))$ as a scheme. For more examples, see table in [LandOtt11, p. 2]

However, just like the equations stemming from the null-correlation bundle above, the catalecticant equations are not always enough – calculating the conormal space shows that the corresponding rank variety is sometimes too large. However, we can intersect the rank variety of null-correlation bundle with the rank variety of catalecticant. The conormal space to the intersection is a sum of conormal spaces to the factors. We can therefore calculate the conormal space to the intersection at some element of the secant variety, and check whether its dimension is equal to the codimension of the secant variety. When this is the case, the secant variety is an irreducible component of the intersection.

We performed the calculation in some cases when neither null-correlation bundle, nor symmetric catalecticants (that is, for $a = \lfloor \frac{d}{2} \rfloor$) are by themselves enough to cut out the secant variety. In many cases, the sum of conormal spaces had dimension equal to the codimension of the secant variety. When this is the case, the secant variety is an irreducible component of the intersection.

For example:

- $\sigma_6(\nu_4(\mathbb{P}^3))$: its dimension is 23, codimension 11. We randomly chose some generic element x of the secant variety. The conormal space of rank variety of null-correlation bundle twisted by 1 at x has dimension 6. The corresponding conormal space for catalecticant minors has dimension 10, and their sum has dimension 11, which is equal to the codimension.

- $\sigma_{14}(\nu_6(\mathbb{P}^3))$: dimension is 55, codimension is 28. The rank variety of $E(2)$, the null-correlation bundle twisted by 2, has 21-dimensional conormal space at a randomly chosen element of the secant variety. The conormal space of catalecticant variety at the same point is also 21, and their sum has dimension 28, which is equal to the codimension.

- $\sigma_{15}(\nu_6(\mathbb{P}^3))$: dimension is 59, codimension is 24. The conormal of null-correlation has dimension 10, the conormal of catalecticant has dimension 15, their sum has dimension 24, which is equal to the codimension.

On the other hand, there are also cases when even the two classes of equations aren't enough:

- $\sigma_{16}(\nu_6(\mathbb{P}^3))$: dimension is 63, codimension is 20. At a randomly chosen element, the conormal space of the rank variety of null-correlation bundle twisted by 2 has dimension 3, while the conormal space to the catalecticant variety has dimension 10. Their sum has dimension 13, which is smaller than the codimension.

Appendix A

Sage library for explicit calculations

This chapters contains a Sage library that can be used to calculate explicit equations of a secant variety to Veronese embedding, given a presentation of a vector bundle.

```
def contract(m, d):
    '''
    contract(polynomial, diff) -- contract polynomial by polynomial

    Let S = k[x_0, ..., x_n] be a polynomial ring. S is naturally
    a graded S-module, but we can also introduce a nonstandard
    module structure like this -- let S' = k[y_0, ..., y_n] be a
    polynomial ring with renamed variables. Let S act on S' as if x_i
    was d/dx_i, but without differentiation constant. In other words,
    x_i acts on monomial f(y_0, ..., y_n) by reducing the degree in
    which the variable y_i occurs in f, or giving 0 if y_i doesn't
    occur in f. For example:
    x_0 * y_0^n = y_0^{n-1}
    or
    x_0 x_1^2 * y_0^2 y_1^4 = y_0 y_1^2
    or
    x_0 * y^1 = 0

    This function implements this operation. For example:
    sage: R.<x0, x1, x2> = QQ[]
    sage: contract(5*x0^2*x1^3, 3*x0*x1^2)
    15*x0*x1
    '''

    R = m.parent()
    d = R(d)
    mdict = m.dict()
    ddict = d.dict()
    outd = {}
    for dmon in ddict:
        for mmon in mdict:
            contracted = tuple((x-y for (x, y) in zip(mmon, dmon)))
            if all((z >= 0 for z in contracted)):
```

```
                coeff = ddict[dmon]*mdict[mmon]
                outd[contracted] = outd.get(contracted, 0) + coeff
    return R(outd)

def all_monomials(vars, deg):
    '''
    all_monomials(vars, deg) -- return all monomials in given vars of
    a given degree

    For example:
    sage: R.<x0, x1, x2> = QQ[]
    sage: all_monomials([x0,x1,x2], 3)
    [x0^3, x0^2*x1, x0*x1^2, x1^3, x0^2*x2, x0*x1*x2, x1^2*x2,
     x0*x2^2, x1*x2^2, x2^3]
    '''

    n = len(vars)
    vs = WeightedIntegerVectors(deg, n*[1])
    mons = []
    for vec in vs:
        m = 1
        for var, exp in zip(vars, vec):
            m *= var^exp
        mons.append(m)
    mons.reverse()
    return mons

def exponents_to_str(exps):
    v = ""
    for i, e in enumerate(exps):
        v += str(i)*e
    return v

def mon_to_str(mon):
    exps = mon.exponents()[0]
    return exponents_to_str(exps)

def exponents_to_dual_str(exps):
    return "phi_%s" % exponents_to_str(exps)

def mon_to_dual_str(mon):
    '''
    exponents_to_dual_str(exps) -- converts a tuple of exponents of a
    monomial at given variable to the name of an element in a dual
    basis

    Let V be a vector space with basis x0, ..., xn. Having chosen a
    basis, we obtain an isomorphism of symmetric power S^k V with a
    vector space of degree k homogenous polynomials in variables x0,
```

```
    ..., xn. Naturally, degree k monomials in variables x0, ..., xn
    form a basis of S^k V.

    For us, it will be useful to work also with a dual space to S^k V
    -- indeed, the degree k Veronese reembedding embeds PV* as a
    subvariety of P (S^k V)*. This function lets us obtain a name of a
    dual basis element dual to a given monomial in S^k V. For example,

    sage: mon_to_dual_str(x0^2 * x1 * x2)
    'phi_0012'
    '''

    exps = mon.exponents()[0]
    return exponents_to_dual_str(exps)

def dual_vars_and_mons(vars, deg):
    mons = all_monomials(vars, deg)
    return [(mon_to_dual_str(mon), mon) for mon in mons]

def dual_var_names(vars, deg):
    '''
    dual_var_names(vars, deg) -- returns a list of names of vectors in
    dual basis to a standard basis in S^deg span<vars>

    For example:
    sage: dual_var_names([x0,x1,x2], 3)
    ['phi_000', 'phi_001', 'phi_011', 'phi_111', 'phi_002',
     'phi_012', 'phi_112', 'phi_022', 'phi_122', 'phi_222']
    '''

    return [vm[0] for vm in dual_vars_and_mons(vars, deg)]

def polynomial_to_dual(poly, vars):
    '''
    polynomial_to_dual(poly, vars) -- converts a homogenous polynomial
    to a dual representation

    For example, generic degree 3 form in variables x0, x1, x2 has
    form
    phi_000 x0^3 + phi_111 x1^3 + phi_222 x2^3 +
    + 3 phi_001 x0^2 x1 + 3 phi_112 x1^2 x2 + 3 phi_022 x2^2 x0 +
    + 6 phi_012 x0 x1 x2

    This functions converts a polynomial in variables vars to the
    dictionary of its dual coefficients. Remember, though, that these
    are not exactly the coefficients in the dual basis, as we also
    divide by multinomial coefficients.

    For example:
```

```
    sage: polynomial_to_dual((x0+x1+x2)^3, [x0,x1,x2])
    {'phi_000': 1, 'phi_001': 1, 'phi_002': 1, 'phi_011': 1,
    'phi_012': 1, 'phi_022': 1, 'phi_111': 1, 'phi_112': 1,
    'phi_122': 1, 'phi_222': 1}

    sage: polynomial_to_dual(x0^2 + x0 * x1 + 3 * x0 * x2, [x0,x1,x2])
    {'phi_00': 1, 'phi_01': 1/2, 'phi_02': 3/2,
     'phi_11': 0, 'phi_12': 0, 'phi_22': 0}
    '''

    assert poly.is_homogeneous()
    deg = poly.degree()
    pd = poly.dict()
    var_names = dual_var_names(vars, deg)
    v = dict(zip(var_names, [0]*len(var_names)))
    for mon_exps in pd:
        # we divide by multinomial, because our basis in dual is
        # x_{a_1}^{b_1} * ... * x_{a_k}^{b^k} |-->
        # |-- > (deg \choose b_1, b_2, ..., b_k) phi_{a_1(b_1 times) ... a_k(b_k times)}
        # e.g. x_0^2 x_2 |--> (3 \choose 2, 1) phi_002 = 3 phi_002
        multi = multinomial(*mon_exps)
        cf = pd[mon_exps]/multi

        dual_var_name = exponents_to_dual_str(mon_exps)
        v[dual_var_name] += cf
    return v

def random_linear_polynomial(vars, hi=10):
    '''
    random_linear_polynomial(vars, hi=10) -- returns a random linear
    combination of variables in vars, with integral coefficients from
    1 to hi (default from 1 to 10)

    For example:
    sage: random_linear_polynomial([x0,x1,x2])
    x0 + 2*x1 + 7*x2
    sage: random_linear_polynomial([x0,x1,x2])
    6*x0 + 4*x1 + 5*x2
    '''

    r = 0
    for v in vars:
        # we don't want to get 0 here as it would be nongeneric
        r += ZZ.random_element(1, hi)*v
    return r


def generic_element_of_veronese_secant(vars, d, k):
    '''
```

```
    generic_element_of_veronese_secant(vars, d, k) -- returns a
    generic element of k-th secant variety to degree d Veronese
    reembedding of P(span<vars>*)

    Degree d Veronese reembedding of PV* embeds its as a subvariety of
    P (S^d V)*. Form f \in (S^d V)* is in the image of degree d
    Veronese precisely when it is a d-th power of a linear form. Thus,
    a generic element of k-th secant variety to degree k Veronese
    embedding is precisely a form which is a sum of k d-th powers of
    linear forms.

    Since the projective space of lines in V is PV*, we need to
    convert a sum of k d-th powers to its dual representation, and
    since our dual basis in V* is not exactly the basis dual to the
    monomials, we also divide by appropriate multinomial coefficient.
    '''

    f = 0
    for i in range(k):
        f += random_linear_polynomial(vars)^d
    return polynomial_to_dual(f, vars)

def general_homogeneous_form(vars, deg):
    '''
    general_homogeneous_form(vars, deg) -- returns a general
    homogenous form in variables vars of degree deg.

    For example:
    sage: general_homogeneous_form([x0,x1,x2], 3)
    phi_000*x0^3 + phi_001*x0^2*x1 + phi_011*x0*x1^2 + phi_111*x1^3 +
    + phi_002*x0^2*x2 + phi_012*x0*x1*x2 + phi_112*x1^2*x2 +
    + phi_022*x0*x2^2 + phi_122*x1*x2^2 + phi_222*x2^3
    '''

    mons = all_monomials(vars, deg)
    lm = len(mons)

    old_vars = map(str,vars)
    lv = len(old_vars)

    gen_vars = ["phi_%s" % mon_to_str(mon) for mon in mons]
    Sbase = PolynomialRing(vars[0].base_ring(), names = gen_vars)
    Sgen = PolynomialRing(Sbase, names = old_vars)
    return sum((Sgen(Sbase.gen(i))*Sgen(mons[i]) for i in range(lm)))

def catalecticant(f, vars, a, b):
    '''
    catalecticant(f, vars, a, b) -- given f \in S^d span<x0, ..., xk>,
    calculates a matrix of a corresponding catalecticant
```

```
    f_a,b: S^a span<x0, ..., xk> -> S^b span<x0, ..., xk>

    Obviously, for it to make sense we must have a+b = deg f.

    The result is (a+k \choose a) x (b+k \choose b) matrix
    representing the catalecticant in the natural bases of S^a
    span<x0, ..., xk> and S^b<x0, ..., xk>, that is, the bases of
    degree a and degree b monomials respectively.

    For example:
    sage: f = general_homogeneous_form([x0,x1,x2], 3)
    sage: f
    phi_000*x0^3 + phi_001*x0^2*x1 + phi_011*x0*x1^2 + phi_111*x1^3
    + phi_002*x0^2*x2 + phi_012*x0*x1*x2 + phi_112*x1^2*x2 + phi_022*x0*x2^2
    + phi_122*x1*x2^2 + phi_222*x2^3
    sage: cf = contract(f, x0)
    sage: cf
    phi_000*x0^2 + phi_001*x0*x1 + phi_011*x1^2 + phi_002*x0*x2
    + phi_012*x1*x2 + phi_022*x2^2
    sage: catalecticant(cf, [x0,x1,x2], 1, 1)
    [phi_000 phi_001 phi_002]
    [phi_001 phi_011 phi_012]
    [phi_002 phi_012 phi_022]
    '''

    ma = all_monomials(vars, a)
    la = len(ma)
    mb = all_monomials(vars, b)
    lb = len(mb)
    assert f.is_homogeneous()
    assert f == 0 or f.degree() == a+b
    S = f.parent()
    rows = []
    for i in range(la):
        row = []
        for j in range(lb):
            monij = ma[i]*mb[j]
            row.append(f.monomial_coefficient(S(monij)))
        rows.append(row)
    return matrix(rows)

def calculate_pev(pe, vars, bjs, ais, d):
    '''
    calculate_pev(pe, vars, bjs, ais, d) -- calculate the matrix of a
    natural morphism P^E_v: H^0(L_1) -> H^0(Hom(L_0, L))* coming from
    the presentation p_E: L_1 -> L_0 of a vector bundle E.

    Let p_E: L_1 -> L_0 be a presentation of a vector bundle E -- that
    is, let L_1 = (+)_j O(b_j), L_0 = (+)_i O(a_i) be direct sums of
```

```
line bundles on P^n, and let p_E: L_1 -> L_0 be a map between them
such that im p_E is a vector bundle E. Let also L = O(d) be some
very ample line bundle on P^n, and let v be any element of
H^0(L)*. Then we have a natural map

  P^E_v: H^0(L_1) -> H^0(Hom(L_0, L))*

defined as:

  P^E_v(s)(phi) = v(phi(p_E(s)))

This function calculates the matrix of P^E_v in natural bases of
H^0(L_1) and H^0(Hom(L_0, L))*.
'pe' is a matrix that represents a presentation map p_E
'vars' are projective coordinates, i.e. basis of H^0(O(1))
'bjs' and ais are lists of degrees of line bundles being summands of
'L_1' and 'L_0', respectively
'd' is a degree of a bundle L that is used here.

For example:
sage: R.<x0, x1, x2> = QQ[]
sage: pe = matrix([[0 , x2, -x1], [-x2, 0, x0], [x1, -x0, 0]])
sage: pe
[  0  x2 -x1]
[-x2   0  x0]
[ x1 -x0   0]
sage: pev = calculate_pev(pe, [x0,x1,x2], [1,1,1], [2,2,2], 3)
sage: pev
[       0        0        0  phi_002  phi_012  phi_022 -phi_001 -phi_011 -phi_012]
[       0        0        0  phi_012  phi_112  phi_122 -phi_011 -phi_111 -phi_112]
[       0        0        0  phi_022  phi_122  phi_222 -phi_012 -phi_112 -phi_122]
[-phi_002 -phi_012 -phi_022        0        0        0  phi_000  phi_001  phi_002]
[-phi_012 -phi_112 -phi_122        0        0        0  phi_001  phi_011  phi_012]
[-phi_022 -phi_122 -phi_222        0        0        0  phi_002  phi_012  phi_022]
[ phi_001  phi_011  phi_012 -phi_000 -phi_001 -phi_002        0        0        0]
[ phi_011  phi_111  phi_112 -phi_001 -phi_011 -phi_012        0        0        0]
[ phi_012  phi_112  phi_122 -phi_002 -phi_012 -phi_022        0        0        0]

'''

gf = general_homogeneous_form(vars, d)
rows = []
for j, bj in enumerate(bjs):
    row = []
    for i, ai in enumerate(ais):
        pij = pe[i][j]
        c = catalecticant(contract(gf, pij), vars, bj, d-ai)
        row.append(c)
    rows.append(row)
```

```
    return block_matrix(rows, subdivide=False).transpose()


def find_rank_conormal_pairing(pev, v, vars, deg):
    '''
    find_rank_conormal_pairing(pev, v, vars, deg) -- finds the rank of a
    conormal space to rank variety at v.

    The function calculate_pev defined above calculates a matrix of a
    map
      P^E_v: H^0(L_1) -> H^0(Hom(L_0, L))*
    originating form a presentation p_E: L_1 -> L_0 of a vector bundle
    E of rank e.

    Consider a rank k variety corresponding to P^E_v:

      Rank_k(E) = { [v] \in P H^0(L)*: rank P^E_v <= e*k }

    The theorem says (under some assumptions), that if we embed the
    variety X using line bundle L, its r-th secant variety
    sigma_r(X) \subset P H^0(L)* is contained in Rank_r(E) \subset P H^0(L)*.

    Another theorem gives (under some assumptions) a condition for
    sigma_r(X) to be an irreducible component of Rank_r(E). Indeed, we
    have a pairing

      g_v: ker P^E_v (x) (im P^E_v)^T -> H^0(L)**

    defined by:
      g_v(f, h)(phi) = P^E_phi(f)(h)

    For a smooth point [v] \in Rank_r(E), the image of g_v, mapped by
    natural isomorphism H^0(L)** -> H^0(L), is precisely the affine
    conormal space of Rank_r(E). The theorem says that if v \in
    sigma_k(X) is a smooth point of Rank_k(E), and the rank of g_v is
    equal to the codimension of sigma_k(X), then sigma_k(X) is an
    irreducible component of Rank_r(E) passing through [v].

    This function, given matrix 'pev' repesenting a map P^E_v, a vector
    'v' \in H^0(L), projective coordinates 'vars' of a projective space
    P^n, and the degree 'deg' of Veronese reembedding of P^n,
    calculates the rank of a conormal map. One can then compare it
    with the codimension of sigma_k(v_deg(P^n)).
    '''

    gvars = dual_var_names(vars, deg) # names of vectors in basis of H^0(L)*
    gvn = len(gvars)
    H0LDD = QQ^gvn
    zerop = dict(zip(gvars, [0]*gvn)) # zero functional in H^0(L)*
```

```
hld_basis = [] # basis of H^0(L)*
for gv in gvars:
    hld_v = zerop.copy()
    hld_v[gv] = 1
    hld_basis.append(hld_v)

pv = matrix(QQ, pev(**v)) # P_v

# in Sage, .kernel() is "left kernel" e.g. { w: wA = 0 }
# so m.kernel() is actually kernel of the transpose of m
ker_pv_basis = pv.transpose().kernel().basis()
im_pv_orth_basis = pv.kernel().basis()
conormal_gens = []
# now iterate over all pairs
A = pev.base_ring()
for f in ker_pv_basis:
    for h in im_pv_orth_basis:
        l = [0]*gvn
        g_fh = (matrix(A, h) * pev * matrix(A, f).transpose())[0][0]
        for i, hlv in enumerate(hld_basis):
            l[i] = g_fh(**hlv)
        conormal_gens.append(H0LDD(l))
return H0LDD.span(conormal_gens)
```

Appendix B

Explicit calculations for $\sigma_5(\nu_4(\mathbb{P}^3))$

Here we give a program that uses the presentation of the null-correlation bundle obtained in this paper to find equations of $\sigma_5(\nu_4(\mathbb{P}^3))$, and show that it is an irreducible component of a rank variety.

```
load("secant_equations.sage")

R.<x0, x1, x2, x3> = QQ[]

proj_coords = [x0, x1, x2, x3]
proj_dim = len(proj_coords) - 1
print "Working on P^%d " % proj_dim

pe = matrix([[0 , -x0^2 , x2^2 , x0*x1 + x2*x3 , -x0*x2],
    [x0^2 , 0 , x2*x3 - x0*x1 , x3^2 , -x0*x3],
    [-x2^2 , x0*x1 - x2*x3, 0 , -x1^2 , x1*x2],
    [-x0*x1 -x2*x3 , -x3^2 , x1^2 , 0 , x1*x3],
    [x0*x2 , x0*x3 , -x1*x2 , -x1*x3 , 0]])
l1 = [1,1,1,1,1]
l0 = [3,3,3,3,3]
print "Presentation is %s -> %s" % (l1, l0)
print "Presentation matrix: "
print pe

veronese_deg = 4
print "Degree of Veronese embedding: %d" % veronese_deg
pev = calculate_pev(pe, proj_coords, l1, l0, veronese_deg)
print "P^E_v:"
print pev

kth_secant = 5
expected_dim = kth_secant*(proj_dim+1) - 1
print "Working with s_%d(v_%d(P^%d)), its expected dimension is %d" \
  % (kth_secant, veronese_deg, proj_dim, expected_dim)

v = generic_element_of_veronese_secant(proj_coords,
        d=veronese_deg, k=kth_secant)
```

```
print "Calculating rank of conormal space at: "
print v

print "Conormal space has rank %d" % find_rank_conormal_pairing(
    pev, v, [x0,x1,x2,x3], veronese_deg).rank()
veronese_dim = binomial(proj_dim+veronese_deg, veronese_deg) - 1
print "Codimension of secant variety is %d - %d = %d" \
  % (veronese_dim, expected_dim, veronese_dim - expected_dim)
```

The output of this program is following:

```
Working on P^3

Presentation is [1, 1, 1, 1, 1] -> [3, 3, 3, 3, 3]

Presentation matrix:
[            0         -x0^2          x2^2  x0*x1 + x2*x3        -x0*x2]
[         x0^2             0 -x0*x1 + x2*x3          x3^2        -x0*x3]
[        -x2^2  x0*x1 - x2*x3             0         -x1^2         x1*x2]
[-x0*x1 - x2*x3         -x3^2          x1^2             0         x1*x3]
[        x0*x2         x0*x3        -x1*x2        -x1*x3             0]
Degree of Veronese embedding: 4
P^E_v:

20 x 20 dense matrix over Multivariate Polynomial Ring in phi_0000, (...), phi_3333
over Rational Field (type 'print pev.str()' to see all of the entries)

Working with s_5(v_4(P^3)), its expected dimension is 19

Calculating rank of conormal space at:
{'phi_0233': 2037, 'phi_1122': 10836, 'phi_1123': 3276, 'phi_0112': 3024,
'phi_0113': 1092, 'phi_0111': 1705, 'phi_0033': 1421, 'phi_0013': 1338,
'phi_1222': 16230, 'phi_0011': 1781, 'phi_0133': 1174, 'phi_3333': 4465,
'phi_2233': 3697, 'phi_2333': 3011, 'phi_1223': 5082, 'phi_1111': 5985,
'phi_1113': 2272, 'phi_1112': 7812, 'phi_1133': 1476, 'phi_0222': 11856,
'phi_0223': 4029, 'phi_0022': 8406, 'phi_0023': 2781, 'phi_1333': 2050,
'phi_0123': 2034, 'phi_0122': 5814, 'phi_0002': 6678, 'phi_0003': 2109,
'phi_2222': 26500, 'phi_2223': 8531, 'phi_0333': 2019, 'phi_0000': 5475,
'phi_0012': 3834, 'phi_1233': 2262, 'phi_0001': 3007}

Conormal space has rank 15
Codimension of secant variety is 34 - 19 = 15
```

Appendix C

Results of calculations

We calculated the dimension of a conormal space of the rank variety of various twists of the null-correlation bundle at the element of secant variety to Veronese embedding. If the dimension equals the codimension of the secant variety, it means that the secant variety is an irreducible component of the secant variety. We present the results as the following tables. Entry Y means that the corresponding secant variety is an irreducible component of the corresponding rank variety.

	v_3	v_4	v_5	v_6	v_7	v_8
σ_2	N	Y	Y	Y	Y	Y
σ_3	N	Y	Y	Y	Y	Y
σ_4	N	Y	Y	Y	Y	Y
σ_5		Y	Y	Y	Y	Y
σ_6		N	Y	Y	Y	Y
σ_7		N	Y	Y	Y	Y
σ_8		N	N	N	N	N

Table C.1: Results for E(1)

	v_3	v_4	v_5	v_6	v_7	v_8
σ_2	N	N	Y	Y	Y	Y
σ_3	N	N	Y	Y	Y	Y
σ_4	N	N	Y	Y	Y	Y
σ_5		N	Y	Y	Y	Y
σ_6		N	Y	Y	Y	Y
σ_7		N	Y	Y	Y	Y
σ_8		N	N	Y	Y	Y
σ_9			N	Y	Y	Y
σ_{10}			N	Y	Y	Y
σ_{11}			N	Y	Y	Y
σ_{12}			N	Y	Y	Y
σ_{13}			N	Y	Y	Y
σ_{14}				N	Y	Y
σ_{15}				N	Y	Y
σ_{16}				N	Y	Y
σ_{17}				N	N	N

Table C.2: Results for E(2)

	v_3	v_4	v_5	v_6	v_7	v_8
σ_2			N	Y	Y	Y
σ_3			N	Y	Y	Y
σ_4			N	Y	Y	Y
σ_5			N	Y	Y	Y
σ_6			N	Y	Y	Y
σ_7			N	Y	Y	Y
σ_8			N	N	Y	Y
σ_9			N	N	Y	Y
σ_{10}			N	N	Y	Y
σ_{11}			N	N	Y	Y
σ_{12}			N	N	Y	Y
σ_{13}			N	N	Y	Y
σ_{14}				N	Y	Y
σ_{15}				N	Y	Y
σ_{16}				N	Y	Y
σ_{17}				N	N	Y
σ_{18}					N	Y
σ_{19}					N	Y
σ_{20}					N	Y
σ_{21}					N	Y
σ_{22}					N	Y
σ_{23}					N	Y
σ_{24}					N	Y
σ_{25}					N	Y
σ_{26}					N	Y
σ_{27}					N	N
σ_{28}					N	N
σ_{29}					N	N

Table C.3: Results for E(3)

	v_3	v_4	v_5	v_6	v_7	v_8
σ_2		N	N	N	Y	Y
σ_3		N	N	N	Y	Y
σ_4		N	N	N	Y	Y
σ_5		N	N	N	Y	Y
σ_6		N	N	N	Y	Y
σ_7		N	N	N	Y	Y
σ_8		N	N	N	N	Y
σ_9			N	N	N	Y
σ_{10}			N	N	N	Y
σ_{11}			N	N	N	Y
σ_{12}			N	N	N	Y
σ_{13}			N	N	N	Y
σ_{14}				N	N	Y
σ_{15}				N	N	Y
σ_{16}				N	N	Y
σ_{17}				N	N	N
σ_{18}				N	N	N
σ_{19}				N	N	N
σ_{20}					N	N
σ_{21}					N	N
σ_{22}					N	N
σ_{23}					N	N
σ_{24}					N	N

Table C.4: Results for E(4)

Bibliography

[LandOtt11] J.M. Landsberg, Giorgio Ottaviani, *Equations for secant varieties of Veronese and other varieties*, arXiv:1111.4567 [math.AG]

[Hart1978] Robin Hartshorne *Stable vector bundles of rank 2 on* $\mathbb{P}^3$, Mathematische Annalen, Volume 238, Issue 3, pp 229-280, 1978

[Hart] Robin Hartshorne *Algebraic Geometry*, Berlin, New York: Springer-Verlag, 1977

[OkSchnSp1980] Christian Okonek, Michael Schneider, Heinz Spindler, *Vector Bundles on Complex Projective Spaces*, Birkhäuser, 2011

[Landsberg2012] J.M. Landsberg *Tensors: Geometry and Applications*, American Mathematical Society, 2012

[Richmond] H.W. Richmond, *On canonical forms*, Quart. J. Math. 33 (1902), 331-340

[BraOtt] Maria Chiara Brambilla, Giorgio Ottaviani, *On the Alexander-Hirschowitz Theorem* arXiv:math/0701409 [math.AG]

[Geramita] Anthony V. Geramita, *Catalecticant varieties*, Commutative algebra and algebraic geometry (Ferrara), volume dedicated to M. Fiorentini, Lecture Notes in Pure and Appl. Math., vol. 206, Dekker, New York, 1999, pp. 143 – 156.