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Deformacje schematów 0-wymiarowych i ich zastosowania

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Oświadczenie kierującego pracą

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Data

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Świadom odpowiedzialności prawnej oświadczam, że niniejsza praca dyplomowa została napisana przeze mnie samodzielnie i nie zawiera treści uzyskanych w sposób niezgodny z obowiązującymi przepisami.

Oświadczam również, że przedstawiona praca nie była wcześniej przedmiotem procedur związanych z uzyskaniem tytułu zawodowego w wyższej uczelni.

Oświadczam ponadto, że niniejsza wersja pracy jest identyczna z załączoną wersją elektroniczną.

Data

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Abstract

In this thesis we consider the geometry of the Hilbert scheme of points in $\mathbb{P}^n$, concentrating on the locus of points corresponding to the Gorenstein subschemes of $\mathbb{P}^n$. New results are given, most importantly we provide tools for constructing flat families and analysis of finite Gorenstein algebras and expose their efficiency by proving smoothability of certain families of algebras. Much of the existing theory and folklore is reviewed, providing a micro-encyclopaedic reference.

Słowa kluczowe

flatness, smoothability, finite Gorenstein scheme, Hilbert function decomposition, Hilbert scheme

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

Klasyfikacja tematyczna

13 Commutative Algebra
13D10 Homological methods: Deformations and infinitesimal methods
13H10 Local rings and semilocal rings: Special types (Cohen-Macaulay, Gorenstein, Buchsbaum, etc.)
14 Algebraic geometry
14C05 Cycles and subschemes: Parametrization (Chow and Hilbert schemes)
14D06 Families, fibrations: Fibrations, degenerations

Tytuł pracy w języku angielskim

Deformations of zero-dimensional schemes and applications

Contents

1. Introduction

The Hilbert scheme of r-points of $\mathbb{P}^n$ parametrises closed zero-dimensional subschemes of $\mathbb{P}^n$ of degree r, or more precisely flat families of closed subschemes of degree r. It is one of the most important moduli spaces. Constructed by Grothendieck [Gro95] in 1961 it still draws much of attention because many natural questions about its structure are still open. It is known to be projective, by construction, and connected by a result of Hartshorne [Har66]. It is unknown exactly in which cases it is irreducible, though if r and n large, it is not, see [CEVV09]. An intriguing phenomena is that all proofs of reducibility are somehow indirect, resting on a dimension count on a tangent space or more sophisticated invariants. Much work is done to develop direct criteria, see e.g. [EV10]. Similarly it is unknown whether the Hilbert scheme is reduced, though for large r, n it is believed that it has arbitrarily bad singularities such as non-reduced irreducible components.

The problems with describing the exact structure of the Hilbert scheme of points had drawn attention to its important open subschemes such as the open *Gorenstein locus* parametrising *Gorenstein* subschemes of $\mathbb{P}^n$, see [CN09]. Also this subset is known, for large r, n, to be reducible, see [Iar94]. In this paper we review much of the theory of zero-dimensional Gorenstein schemes and provide new tools for studying the Gorenstein locus in the aim to prove irreducibility of the locus for small r and arbitrary n.

This research is tightly connected with the study of secant and cactus varieties of a Veronese embedding $\nu : \mathbb{P}^n \to \mathbb{P}^N$: the zero-dimensional Gorenstein (resp. smoothable Gorenstein) subschemes of $\mathbb{P}^n$ of degree r are used to parametrise r-th cactus (resp. r-th secant) variety of $\nu(\mathbb{P}^n)$. See [BB10], especially Theorem 1.6 and Subsection 8.1, for details.

1.1. Notation and text structure

All considered rings and algebras are commutative and with unity which is preserved by homomorphisms. Throughout the paper k denotes a field and for a k-vector space V the symbol V^* denotes $\mathrm{Hom}_k(V, k)$; we will not use the star to refer to invertible elements of a ring. The expression $I \lhd A$ means "I is an ideal of a ring A". Where no confusion is likely to occur we use $(f_1, \ldots, f_n)$ to denote the ideal generated by rings elements $f_1, \ldots, f_n$. A *finite* module (or vector space) is by definition a finitely generated module. We will use the word *dimension* exclusively for Krull dimension. When speaking about linear dimension over k we will use the words rank and corank instead of dimension and codimension and the symbol rk_k instead of $\dim_k$. For example $\mathrm{Spec}\, k[\varepsilon]/\varepsilon^2$ has dimension zero and rank two over k.

The text is divided into three expository sections and a section presenting original research results. Sections 2 and 3 recall the notions of flatness and Hilbert scheme. The emphasis is on flatness of projective and affine morphisms, anticipating our interest in finite morphisms. Section 4 is almost entirely algebraic or rather linear-algebraic, being concerned with zero-dimensional Gorenstein rings and their exploration through Macaulay's inverse systems. The

paper culminates with the research Section 5 analysing the geometry of the Gorenstein locus of Hilbert points in $\mathbb{P}^n$.

Almost all results outside Section 5 are well-known or easy consequence of well-known results, however in many cases we did not find a suitable reference, thus being obliged to provide a proof. We have also provided proof of Theorem 2.3.9, whose original proof is complicated in the general setting of [Bjö79], and proofs of results from Subsection 4.4 which more or less follow [Iar94]. Moreover all the results, with the notable exception of Theorem 2.3.9 and its supporting theory, were previously used in the setting and are considered standard. On the other hand all results contained in Section 5 are, as far as we know, original. They are put merely as an illustration of the introduced theory. Some of the techniques used can be found in [Jel12], and a more thorough treatment will be given in a joint paper with Gf. Casnati and R. Notari.

2. Flatness

In this section we recall some standard material on flatness, mainly following [Har77, Section III.9].

Definition 2.0.1. *Let A be a ring and M be an A-module. We say that M is a* flat *A-module if the functor $(-) \otimes_A M$ is exact.*

This definition comes with a geometric counterpart, see Lemma 2.0.3 below for the connection:

Definition 2.0.2. *Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{F}$ be a quasi-coherent sheaf on X. We say that $\mathcal{F}$ is flat over Y at a point $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,y}$-module, where $y = f(x)$ and we consider $\mathcal{O}_{X,x}$ as an $\mathcal{O}_{Y,y}$-module via the natural map. We say that $\mathcal{F}$ is flat over Y if it is flat at every point of X. We say that the morphism $X \to Y$ is flat if and only if $\mathcal{O}_X$ is flat over Y.*

Below we recall some basic properties of flatness, for simplicity the first two are stated only for flat morphisms:

Lemma 2.0.3.   *1. Flatness is stable under base change: for flat $X \to Y$ and any $Y' \to Y$ the morphism $X \times_Y Y' \to Y'$ is flat;*

   2. composition of flat morphisms is flat;

   3. let A be a ring and M be an A-module. The sheaf $\mathcal{F} = \tilde{M}$ on the affine scheme $Y = \operatorname{Spec} A$ is flat over Y (with respect to $\operatorname{id} : Y \to Y$) if and only if M is a flat A-module.

Proof. See [Har77, Theorem III.9.2]. $\qquad\square$

2.1. Criteria for flatness

We will now investigate the flatness of modules. There are numerous criteria for flatness, we recall some of them below.

Lemma 2.1.1. *Let M be an A-module. If every finitely generated A-submodule of M is flat over A then M is flat over A.*

Proof. The module M is a direct limit of its finitely generated submodules M_n. Take any short exact sequence $\mathcal{S}$ of A-modules, then $M \otimes_A \mathcal{S} \simeq \left(\varinjlim M_n \right) \otimes_A \mathcal{S} \simeq \varinjlim (M_n \otimes_A \mathcal{S})$ and this is a short exact sequence of A-modules because it is a filtered direct limit of short exact sequences of A-modules. $\qquad\square$

Lemma 2.1.2. *A finitely generated module over a noetherian local ring is flat if and only if it is free.*

Proof. See [Wei94, Thm 3.2.7]. □

Lemma 2.1.3. *Flat modules are torsion free. A module over a principal ideal domain (PID) is flat if and only if it is torsion free. In particular a submodule of a flat module over a PID is a flat module.*

Proof. See [Eis95, Cor 6.3] and Lemma 2.1.1. Note that we do not assume that the module is finitely generated. □

We will be interested in connections of flatness and the behaviour of fibers of morphism. Here it is important to note, that we should make some additional assumptions on the morphism:

Example 2.1.4. *The hyperbola*

$$\operatorname{Spec} \frac{k[x,t]}{(tx-1)} \to \operatorname{Spec} k[t]$$

is flat over $\operatorname{Spec} k[t]$, *as* $k[x,t]/(tx-1) \simeq k[x,x^{-1}]$ *is torsion free over* $k[x^{-1}]$, *but on the other hand the fiber over* $(t-\alpha)$ *for an invertible* $\alpha \in k$ *is isomorphic to a point* $\operatorname{Spec} k$, *and over* (t) *the fiber is empty.*

On the other hand families with well-behaved fiber lengths are not necessary flat. For example we can add the origin to the aforementioned hyperbola, obtaining

$$\operatorname{Spec} \frac{k[x,t]}{(tx-1) \cap (t,x)} \to \operatorname{Spec} k[t],$$

which has fibers over all $(t-\alpha)$ *for* $\alpha \in k$ *isomorphic to* $\operatorname{Spec} k$. *But this morphism is not flat! In fact* $t \cdot (tx-1) \in (tx-1) \cap (t,x)$ *and* $tx-1 \notin (t,x)$, *thus* t *is a zero-divisor on* $k[x,t]/((tx-1) \cap (t,x))$, *contradicting Lemma 2.1.3. Geometrically this corresponds to the fact that the origin and the hyperbola are different connected components.*

2.2. Flatness for coherent sheaves on projective space

Example 2.1.4 shows that, as usually when thinking about fibers, one should assume that the morphism is proper. Since ultimately we are interested in finite morphisms, which are projective, we will now consider only the stronger notion of projective morphism.

Definition 2.2.1. *Let* T *be a noetherian scheme and* $\mathcal{F}$ *be a coherent sheaf on* $\mathbb{P}^n_T$. *Define, for any point* $t \in T$, *the* Hilbert polynomial P_t *of the sheaf* $\mathcal{F}$ *by the property*

$$P_t(m) = \operatorname{rk}_{k(t)} H^0\big((\mathbb{P}^n_T)_t, \mathcal{F}_t(m)\big) \text{ for } m \text{ large enough,}$$

where $\mathcal{F}_t$ *is the scheme-theoretic fiber of* $\mathcal{F}$ *over the point* $t \in T$. *If* $\mathcal{F} = \mathcal{O}_Z$ *is the structure sheaf of a projective scheme* Z, *then this agrees with the usual definition of the Hilbert polynomial of* Z.

By vanishing of higher cohomologies the definition is correct, see [Har77, III.5 Ex 5.2].

Theorem 2.2.2. *Let* T *be an integral noetherian scheme and* $\mathcal{F}$ *be a coherent sheaf on* $\mathbb{P}^n_T$. *Then the following are equivalent:*

    1. the sheaf $\mathcal{F}$ *is flat over* T;

2. *the Hilbert polynomial P_t is independent of the choice of $t \in T$.*

Proof. See [Har77, Thm III.9.9]. □

We are mostly interested in the case when the fibers of $\mathcal{F}$ are finite, so the Hilbert polynomial has degree zero.

Remark 2.2.3. *If $\mathcal{F}$ is flat and P_t is a polynomial of degree zero, then is it equal to $\mathrm{rk}_{k(t)} \Gamma(\mathcal{F}_t)$. This number is called the* rank of sheaf $\mathcal{F}$ at $t \in T$.

The typical case is $\mathcal{F}$ being the structure sheaf of a projective scheme $X \subseteq \mathbb{P}^n_T$ such that $X \to T$ is flat and with finite fibers. If T is of finite type over a field, then we have a useful corollary of Theorem 2.2.2:

Corollary 2.2.4. *Let T be an integral scheme of finite type over a field and $\mathcal{F} = \mathcal{O}_Z$ be the structure sheaf of a scheme Z projective over T. Suppose that for every closed $t \in T$ the rank of $\mathcal{F}$ is finite and does not depend on the choice of t. Then $\mathcal{F}$ is flat over T.*

Proof. By Theorem 2.2.2 and Remark 2.2.3 it is enough to show that $\mathrm{rk}_{k(t)} \Gamma(\mathcal{F}_t)$ is independent of the choice of point (not necessarily closed) $t \in T$. This is true for closed points, by assumption.

Note that any open subset of T contains a closed point. By semicontinuity of rank (see [Har77, II.5 Ex 5.8a]), the set of $t \in T$ such that the rank is minimal is open, thus it contains a closed point. But now it follows that it contains all closed points, so in fact it is equal to T. □

2.3. Flatness for filtered modules

Let V be a closed subscheme of $\mathbb{A}^n_k$. In this section we will investigate flatness of V under a projection $\mathbb{A}^n_k \to \mathbb{A}^1_k$, making use of the gradation on $\mathbb{A}^n_k$. Of course the gradation may not give a gradation on the algebra of global functions on V, but it always gives a filtration.

Definition 2.3.1 (Filtration). *Let A be a ring. A* filtration *of A is an infinite sequence of abelian subgroups of A:*
$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \ldots$$
such that

1. $1 \in A_0$ *and* $A_n \cdot A_m \subseteq A_{n+m}$ *for all* $n, m \in \mathbb{Z}_{\geqslant 0}$;

2. $\bigcup A_n = A$.

Definition 2.3.2 (Rees algebra of a filtration). *If A is a ring filtered by A_n, then the abelian subgroup*
$$\mathcal{A} = \mathcal{A}(A_n) = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} A_n t^n \subseteq A[t]$$
of the polynomial ring $A[t]$ is actually a subring, called the Rees algebra of the filtration, *see [Eis95, Section 6.5]. We define the* associated graded ring of $\mathcal{A}$ *as*
$$\mathrm{gr}\, \mathcal{A} = \mathcal{A}/(t).$$

As an A_0-module $\mathrm{gr}\, \mathcal{A}$ is isomorphic to $\bigoplus_{n \geqslant 0} A_n/A_{n-1}$, where $A_{-1} = 0$. Note that $A = \mathcal{A}/(t-1)$.

Example 2.3.3. *1. Every ring A can be filtered trivially by letting $A_n = A$ for all $n \in \mathbb{Z}_{\geqslant 0}$, then the associated graded ring is just A.*

2. If $A = A^0 \oplus A^1 \oplus \ldots$ is a $\mathbb{Z}_{\geqslant 0}$-graded ring, then we obtain a filtration $A_n := \sum_{m \leqslant n} A^m$. In this situation we have a natural isomorphism of A_0-algebras $A \simeq \operatorname{gr} \mathcal{A}$ coming from $A_n / A_{n-1} \simeq (A^0 + \cdots + A^n) / (A^0 + \cdots + A^{n-1}) \simeq A^n$.

Definition 2.3.4 (Filtration on a module). *Let A be a filtered ring and M be an A-module. A filtration on M is an infinite sequence of abelian subgroups of M:*

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \ldots$$

such that $A_n \cdot M_m \subseteq M_{n+m}$ for all $n, m \in \mathbb{Z}_{\geqslant 0}$ and $\bigcup_n M_n = M$. To a filtered module M we can associate the Rees module

$$\mathcal{M} = \mathcal{M}(M_n) = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} M_n t^n \subseteq M[t]$$

which is naturally an $\mathcal{A}$-module, and the associated graded module $\operatorname{gr} \mathcal{M} = \mathcal{M}/(t) \cdot \mathcal{M}$ which is naturally a $\operatorname{gr} \mathcal{A}$-module. Note that for any $N \in \mathbb{Z}_{\geqslant 0}$ the subgroup $\mathcal{M}_{\geqslant N} := \bigoplus_{n \geqslant N} M_n t^n \subseteq \mathcal{M}$ is an $\mathcal{A}$-submodule of $\mathcal{M}$.

When speaking about the filtered rings and modules we will use the convention that $\mathcal{X}$ is the Rees algebra/module of an algebra/module X. One consequence of the above definitions is the following corollary.

Corollary 2.3.5. *Let A be a filtered ring and $B \subseteq A_0$ be a subring. Suppose that M is a filtered A-module. If M is finitely generated over A then $\operatorname{gr} \mathcal{M}$ and M have the same B-length (possibly infinite).*

Proof. Recall that B-length is additive on short exact sequences of B-modules by the Jordan-Hölder theorem, see [Eis95, Thm 2.13]. Fix a natural number n. The decomposition of M via the exact sequences

$$0 \to M_{n-1} \to M \to M/M_{n-1} \to 0,$$
$$0 \to M_{n-2} \to M_{n-1} \to M_{n-1}/M_{n-2} \to 0,$$
$$\ldots$$
$$0 \to M_0 \to M_1 \to M_1/M_0 \to 0$$

proves that if for infinitely many natural n the quotients M_n/M_{n-1} are non-zero then the B-lengths of $M, \operatorname{gr} \mathcal{M}$ are infinite. Consequently, we may assume that there exists n such that $M_n = M_{n+1} = \cdots = M$. Then the above decomposition proves that M and $\operatorname{gr} \mathcal{M}$ have equal B-lengths. $\square$

An important thing about the associated graded modules is that an exact sequence of filtered modules gives rise to an *exact* sequence of associated graded modules:

Proposition 2.3.6. *Let k be a field and A be a filtered k-algebra. Suppose that filtered A-modules M, N, P form an exact sequence*

$$0 \to N \to M \to P \to 0,$$

such that the filtration N_n is the preimage of M_n and P_n is the image of M_n for all $n \in \mathbb{Z}_{\geqslant 0}$. This sequence gives rise to exact sequences

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{P} \to 0 \quad and \quad 0 \to \operatorname{gr} \mathcal{N} \to \operatorname{gr} \mathcal{M} \to \operatorname{gr} \mathcal{P} \to 0.$$

Proof. The existence and exactness of the first sequence follows from assumptions. For the second one, note that $\mathcal{P} \subseteq P[t]$ is a torsion free $k[t]$-module, so it is flat. Now apply $(-) \otimes_{k[t]} k[t]/t$ obtaining an exact sequence

$$0 \to \mathcal{N}/t\mathcal{N} \to \mathcal{M}/t\mathcal{M} \to \mathcal{P}/t\mathcal{P} \to 0. \qquad \square$$

The following proposition computes the associated graded in the simplest case – for a cyclic submodule of a graded module.

Proposition 2.3.7. *Let* $A = \bigoplus A^n$ *be a* $\mathbb{Z}_{\geqslant 0}$*-graded ring and* $M = \bigoplus M^n$ *be a* $\mathbb{Z}_{\geqslant 0}$*-graded A-module. Suppose that* $(s) \lhd M$ *is generated by an element*

$$s = s_1 + \cdots + s_n,$$

where $s_i \in M^i$ *and* s_n *satisfies* $\mathrm{ann}_A(s_n) \subseteq \mathrm{ann}_A(s)$. *The ideal* (s) *has a filtration coming from the filtration on* M *given as in the Example 2.3.3, and, with respect to this filtration,*

$$\mathrm{gr}(s) \simeq (s_n).$$

Proof. The isomorphism $i : \mathrm{gr}\,\mathcal{M} \simeq M$ from Example 2.3.3 sends $\overline{m_0 + \cdots + m_n} \in M_n/M_{n-1}$, where $m_j \in M^j$, to m_n. Clearly $i(\mathrm{gr}(s))$ is a submodule of M containing (s_n). Take any $l \in \mathbb{Z}_{\geqslant 0}$ and a non-zero element $\overline{a \cdot s} \in M_l/M_{l-1}$. Let a' be the homogeneous component of a of maximal degree such that $a' \cdot s_n \neq 0$. From $\mathrm{ann}_A(s_n) \subseteq \mathrm{ann}_A(s)$ it follows that $a' \cdot s_n$ is the leading coefficient of $a \cdot s$. Under i the element $a \cdot s$ is mapped to $a' \cdot s_n \in (s_n)$, which proves $i(\mathrm{gr}(s)) \subseteq (s_n)$. $\qquad \square$

The presented methods, although very elementary, are quite handy when dealing with families of finite algebras, as the following example shows:

Example 2.3.8. *Let k be an algebraically closed field, then* $\mathrm{Spec}\,k[x,y,t]/(x^2 + t \cdot y \cdot x, y^2) \to \mathrm{Spec}\,k[t]$ *is flat.*

Indeed if we take the grading on $k[x,y,t]/y^2$ by powers of x, then the top degree form of $x^2 + t \cdot y \cdot x$ is equal to x^2 and thus independent of t. It follows that the fiber over each closed point $t \in T$ has k-rank equal to $\mathrm{rk}_k\,k[x,y]/(x^2,y^2)$ and by Corollary 2.2.4 we get that the morphism is flat.

A powerful generalization is given by the following theorem. Note that there are no assumptions neither on the ring nor on the module, in particular the module does not have to be finite.

Theorem 2.3.9. *Let A be a filtered ring and M be a filtered A-module. If $\mathrm{gr}\,M$ is flat over $\mathrm{gr}\,A$, then M is flat over A.*

Sketch of proof. This is a special case of [Bjö79, Prop 3.12].

We will prove the theorem with an additional assumption that A is an algebra over a field $k \subseteq A_0$. Choose a free resolution $\mathcal{P}_\bullet$ of the Rees module $\mathcal{M}$ in the category of $\mathcal{A}$-modules. Since $\mathcal{M} \subseteq M[t]$ is flat over $k[t]$ by Lemma 2.1.3 it follows that $\mathcal{P}_\bullet \otimes_{k[t]} k[t]/t$ is a free resolution of $\mathrm{gr}\,\mathcal{M}$. Choose any $I \lhd A$ and equip it with the induced filtration, obtaining the Rees module $\mathcal{I}$. Note that $\mathcal{I}$ is flat over $k[t]$, again by Lemma 2.1.3. Since $\mathrm{gr}\,\mathcal{M}$ is a flat $\mathrm{gr}\,\mathcal{A}$-module it follows that $\mathrm{gr}\,\mathcal{I} \otimes_{\mathrm{gr}\,\mathcal{A}} (-)$ is exact on $\mathcal{P}_\bullet \otimes_{k[t]} k[t]/t$, but as $\mathrm{gr}\,\mathcal{A} = \mathcal{A}/t$ this shows that

$$\mathcal{I} \otimes_{\mathcal{A}} \mathcal{P}_\bullet \otimes_{k[t]} k[t]/t \simeq \mathrm{gr}\,\mathcal{I} \otimes_{\mathrm{gr}\,\mathcal{A}} \left(\mathcal{P}_\bullet \otimes_{k[t]} k[t]/t\right) \tag{2.1}$$

is exact.

The module $\mathcal{M}$ is flat over $k[t]$, thus $\mathcal{M} \otimes \mathcal{I} \subseteq \mathcal{M}$ is also flat. Similarly for any i the module $\mathcal{I} \otimes_\mathcal{A} \mathcal{P}_i$ is flat over $k[t]$, in particular multiplication by t is injective on $\mathcal{I} \otimes_\mathcal{A} \mathcal{P}_i$. From (2.1) it follows that the homology modules $\mathcal{H}_i$ of $\mathcal{I} \otimes_\mathcal{A} \mathcal{P}_i$ satisfy $t \cdot \mathcal{H}_i = \mathcal{H}_i$. This means that $\operatorname{im} d_{i+1} + t \cdot \ker d_i = \ker d_i$, where $d_i : P_i \to P_{i-1}$. Since $\operatorname{im} d_{i+1}, \ker d_i$ are filtered A-modules one checks directly that $\mathcal{H}_i = 0$, thus $\mathcal{P}_i \otimes_\mathcal{A} \mathcal{I}$ is exact. Now the complex $\mathcal{P}_\bullet \otimes_\mathcal{A} \mathcal{I} \otimes_{k[t]} k[t]/(t-1) \simeq \mathcal{P}_\bullet/(t-1) \otimes_A I$ is exact and A-modules $\mathcal{P}_i/(t-1)$ form a free resolution of $\mathcal{M}/(t-1) = M$, hence we have $\operatorname{Tor}_1^A(I, M) = 0$ and this is a sufficient condition for A-flatness of M by [Wei94, Prop 3.2.4]. $\qquad\square$

Example 2.3.10. *1. The family*

$$\operatorname{Spec} k[x, y, t]/(x^2 + t \cdot y \cdot x) \to \operatorname{Spec} k[t] \text{ is flat.}$$

Indeed, we choose a trivial grading on $k[t]$, grading by powers of x on $k[x, y, t]$ and use Propositions 2.3.6, 2.3.7 to compute $\operatorname{gr} k[x, y, t]/(x^2 + t \cdot y \cdot x) \simeq k[x, y, t]/(x^2)$ which is clearly flat over $k[t]$. Note that the grading preserves $k[t]$ and so this is an isomorphism of $k[t]$-modules.

2. Suppose we would like to compute

$$\operatorname{gr}(x^2 - 1, xy - 1),$$

where $(x^2 - 1, xy - 1) \lhd k[x, y]$, with respect to the grading by total degree. Then

$$x - y = y \cdot (x^2 - 1) - x \cdot (xy - 1) \notin (x^2, xy)$$

is an element of $\operatorname{gr}(x^2 - 1, xy - 1)$, so the leading forms of $x^2 - 1, xy - 1$ do not generate $\operatorname{gr}(x^2 - 1, xy - 1)$. This suggests that $x^2 - 1, xy - 1$ is not a "good" generating set; a "better one" is $x - y, x^2 - 1$ because $\operatorname{gr}(x^2 - 1, xy - 1) = (x - y, x^2)$. One can see a connection with Gröbner bases.

3. The Hilbert scheme

In this section we follow a very accessible introduction by Strömme [Str96].

One advantage of flat projective families is that there exists a good scheme parametrising them, known as the Hilbert scheme. We need some definitions.

Definition 3.0.11 (Hilbert functor)**.** *Let X be projective an S-scheme. Define a functor*

$$\underline{\text{Hilb}}_{X/S} : Sch_S^{op} \to Set$$

by putting, for any S-scheme T,

$$\underline{\text{Hilb}}_{X/S}(T) := \{ \text{ closed subschemes } Z \subseteq X \times_S T \text{ such that the projection } Z \to T \text{ is flat } \}$$

and for any morphism $T' \to T$ the map $\underline{\text{Hilb}}_{X/S}(T) \to \underline{\text{Hilb}}_{X/S}(T')$ sending $Z \subseteq X \times_S T$ to $Z' = Z \times_T T'$. In a similar way for any polynomial $P \in \mathbb{Q}[x]$ we can define a (sub)functor $\underline{\text{Hilb}}_{X/S}^P : Sch_S^{op} \to Set$ by letting

$$\underline{\text{Hilb}}_{X/S}^P(T) := \{ \text{ closed subschemes } Z \subseteq X \times_S T \text{ such that the projection } Z \to T \text{ is flat}$$
$$\text{and the Hilbert polynomial of } (\mathcal{O}_Z)_t \text{ is equal to } P \text{ for any } t \in T\},$$

see Theorem 2.2.2.

Theorem 3.0.12. *Suppose that X is projective over a noetherian scheme S. For any polynomial $P \in \mathbb{Q}[x]$ the functor $\underline{\text{Hilb}}_{X/S}^P$ is represented by a* projective *scheme $\text{Hilb}_{X/S}^P$ and the functor $\underline{\text{Hilb}}_{X/S}$ is represented by a scheme $\text{Hilb}_{X/S}$ equal to the countable disjoint union of all schemes $\text{Hilb}_{X/S}^P$ where $P \in \mathbb{Q}[x]$ (some of which are empty). We call this scheme the Hilbert scheme of X/S.*

Proof. See [Har10, Thm 1.1]. □

One important consequence of representability is the fact that S-points of $\text{Hilb}_{X/S}$ correspond to closed subschemes of X. For a closed subscheme $Z \subseteq X$ by $[Z] \in \text{Hilb}_{X/S}$ we denote the corresponding S-point.

Not much is known about the local geometry of the Hilbert scheme; for example in many cases we do not know if this scheme is reduced. However, we have the following result by Grothendieck, computing the tangent space at a point:

Theorem 3.0.13. *Let k be a field, X be projective over $\text{Spec}\, k$ and $Z \subseteq X$ be a closed subscheme given by the ideal sheaf $\mathcal{I}$. The tangent space to $\text{Hilb}_{X/k}$ at the point $[Z]$ is isomorphic to $H^0\left(Z, \mathcal{N}_{Z/X}\right)$, where $\mathcal{N}_{Z/X}$ is the normal sheaf of Z in X.*

Proof. By classifying flat families over $k[\varepsilon]/\varepsilon^2$; see [Har10, Thm I.2.4]. □

Being interested in finite schemes, we will use only a local consequence of this result:

Corollary 3.0.14. *Suppose that k is a field, X is projective over $\operatorname{Spec} k$ and let $Z \subseteq X$ be a closed scheme with support contained in an open affine subscheme $U = \operatorname{Spec} A$ of X, where the subscheme Z is cut out by an ideal $I \lhd A$. Then the tangent space to $\operatorname{Hilb}_{X/k}$ at $[Z]$ is isomorphic to $\operatorname{Hom}_{A/I}\left(I/I^2, A/I\right)$.*

Proof. Let $\mathcal{I}$ be the ideal sheaf of $Z \subseteq X$. Recall that $\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X/\mathcal{I}\right)$ may be regarded as a coherent sheaf on X, by [Har77, Remark II.8.9.1]. Clearly, its global sections are equal to its sections over U:

$$H^0\left(X, \mathcal{N}_{Z/X}\right) = H^0\left(U, \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X/\mathcal{I}\right)\right) \simeq \operatorname{Hom}_A\left(I/I^2, A/I\right) \simeq \operatorname{Hom}_{A/I}\left(I/I^2, A/I\right),$$

because on the affine scheme U we have the equivalence of categories of quasi-coherent sheaves and modules. $\qquad\square$

3.1. The smoothable component

Fix a natural number r and a projective variety X over an algebraically closed field k. In this subsection we investigate the scheme $\operatorname{Hilb}_{X/k}^r$, where we think of $r \in \mathbb{Z}$ as a constant polynomial. The scheme $\operatorname{Hilb}_{X/k}^r$ is commonly referred as to "the Hilbert scheme of r points of X". As the name suggests, the simplest example of a zero-dimensional subscheme of degree r is a disjoint union of r reduced points of X. The closure of points of $\operatorname{Hilb}_{X/k}^r$ corresponding to such subschemes will be called the smoothable component, see Definition 3.1.4. The main goal of this subsection is to show that it is indeed an irreducible component of $\operatorname{Hilb}_{X/k}^r$.

Lemma 3.1.1. *Let $\{p_1, \ldots, p_r\} = Z \subseteq X$ be a disjoint union of r closed points; then the tangent space to $\operatorname{Hilb}_{X/k}^r$ at the point $[Z]$ has k-rank equal to the sum of k-ranks of tangent spaces to X at $p_1, p_2, \ldots, p_r$.*

Proof. Since X is projective, we can find an affine open subset $\operatorname{Spec} A$ of X containing $p_1, \ldots, p_r$. By Corollary 3.0.14 it is enough to compute $\operatorname{rk}_k \operatorname{Hom}_{A/I}\left(I/I^2, A/I\right)$ where $I \lhd A$ is the (finitely generated) ideal of Z; in fact $I = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$ where $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are ideals corresponding to $p_1, \ldots, p_r$. The module I/I^2 is supported at $p_1, \ldots, p_r$, so

$$\operatorname{rk}_k \operatorname{Hom}_{A/I}\left(I/I^2, A/I\right) = \sum_{1 \leqslant i \leqslant r} \operatorname{rk}_k \operatorname{Hom}_{(A/I)_{\mathfrak{m}_i}}\left((I/I^2)_{\mathfrak{m}_i}, (A/I)_{\mathfrak{m}_i}\right)$$

$$= \sum_{1 \leqslant i \leqslant r} \operatorname{rk}_k \operatorname{Hom}_k\left(\mathfrak{m}_i/\mathfrak{m}_i^2, k\right),$$

since $(A/I)_{\mathfrak{m}_i} \simeq A/\mathfrak{m}_i \simeq k$ for every i. $\qquad\square$

Now we will describe an important open subset of $\operatorname{Hilb}_{X/k}^r$. Let $X_i = X$ for $i = 1, 2, \ldots, r$ and set

$$X^r := X_1 \times X_2 \times \cdots \times X_r$$

with projections $\pi_i : X^r \to X_i$ for $i = 1, \ldots, r$. Denote by $\Delta \subseteq X^r$ the union of all pullbacks of diagonals $\Delta_{ij} \subseteq X_i \times X_j$, where $1 \leqslant i < j \leqslant r$. Let X_{smooth} be the set of smooth points of X and set $U = X_{smooth}^r \setminus \Delta$; this is an open set in X^r. Let $Z_i \subseteq X^r \times X$ be the pullback of the diagonal $\Delta_i \subseteq X_i \times X$ and take

$$Z := Z_1 \cup \cdots \cup Z_r, \quad Z_U = Z \times_{X^r} U.$$

Proposition 3.1.2. *The family $Z_U \to U$ is flat over U and thus gives a morphism*

$$\varphi : U \to \mathrm{Hilb}^r_{X/k}.$$

Proof. Since $Z \to X$ is projective and $Z_U \to U$ is its pullback, so it is a projective morphism. The fiber over a closed point $(x_1, x_2, \ldots, x_r) \in U$ is just $\mathcal{O}_{\{x_1\} \cup \cdots \cup \{x_r\}}$, so the morphism is quasi-finite with fibers of rank r, thus finite by [GW10, Cor 12.89] and $\mathcal{O}_{Z_U}$ is a coherent sheaf over U. Since U is open in the variety X^r it is integral, so from Corollary 2.2.4 it follows that $Z_U \to U$ is flat. $\square$

Proposition 3.1.3. *The morphism $\varphi : U \to \mathrm{Hilb}^r_{X/k}$ defined in Proposition 3.1.2 is flat, in particular open. The image of φ has dimension $r \cdot \dim X$.*

Proof. Denote by $\overline{\mathrm{im}\,\varphi}$ the scheme-theoretic image of φ. The morphism φ has finite fibers over closed points, thus the dimension of $\overline{\mathrm{im}\,\varphi}$ is not less than $\dim U$, by [Eis95, Thm 10.10]. Since $\overline{\mathrm{im}\,\varphi}$ is projective and irreducible, its dimension is equal to the dimension of the local ring at any point of $\overline{\mathrm{im}\,\varphi}$. Take $u \in U$. By Lemma 3.1.1 we see that the tangent spaces at u and $\varphi(u)$ have the same k-rank. In particular

$$\dim U \leqslant \dim \overline{\mathrm{im}\,\varphi} = \dim \mathcal{O}_{\overline{\mathrm{im}\,\varphi}, \varphi(u)} \leqslant \dim \mathcal{O}_{\mathrm{Hilb}^r_{X/k}, \varphi(u)} \leqslant \mathrm{rk}_k\, T_{\mathrm{Hilb}^r_{X/k}, \varphi(u)} = \mathrm{rk}_k\, T_{U,u} = \dim U,$$

as U is smooth. This shows that $\mathrm{Hilb}^r_{X/k}$ is smooth at $\varphi(u)$. Now the result follows as a special case of [Eis95, Thm 18.16]. $\square$

Now we define, for a projective variety X, the smoothable component.

Definition 3.1.4. *Let $\mathrm{Hilb}^r_{X/k}{}^\circ$ be the irreducible component of the Hilbert scheme containing the image of the morphism $\varphi : U \to \mathrm{Hilb}^r_{X/k}$ defined in Proposition 3.1.2. We call $\mathrm{Hilb}^r_{X/k}{}^\circ$ the* smoothable component *of the Hilbert scheme $\mathrm{Hilb}^r_{X/k}$. It has dimension $r \cdot \dim X$.*

3.2. Smoothability

Let X be as in the previous subsection. One of the central questions of the theory of Hilbert schemes of points is "for which r it is true that $\mathrm{Hilb}^r_{X/k} = \mathrm{Hilb}^r_{X/k}{}^\circ$?" and its refinements. To give some down-to-earth conditions equivalent to this equality, we introduce the notion of smoothing.

Definition 3.2.1. *A (flat)* deformation *is a flat family $Y \to B$, where B is irreducible; we call B the base of deformation.*

Let R be a closed subscheme of X. A smoothing *of R in X is a closed subscheme $Y \subseteq B \times X$ such that the morphism $Y \to B$ is proper and a flat deformation with fiber $Y_b = \{b\} \times R$ at some closed point $b \in B$ and Y_η smooth at the generic point $\eta \in B$. We say that R is* smoothable *in X if there exists a smoothing of R in X.*

Intuitively the above definition says: to deduce that $[R] \in \mathrm{Hilb}^r_{X/k}{}^\circ$ we find a morphism $B \to \mathrm{Hilb}^r_{X/k}$ such the generic point of B maps into the interior of $\mathrm{Hilb}^r_{X/k}{}^\circ$ and a special point maps to $[R]$. The condition of being proper is technical, connected with the fact that the Hilbert scheme is defined only for projective X, but important — otherwise it would be easy to e.g. deform two points to only one point or one point to an empty scheme, see Example 2.1.4.

Remark 3.2.2. *The existence of smoothings $Y_1 \to B_1, \ldots, Y_n \to B_n$ of connected components of R implies the existence of a smoothing of R over $B_1 \times \cdots \times B_n$. This leads one to the conclusion that when considering smoothings of finite R it is enough to consider those R which are supported at a single point.*

The following proposition together with Proposition 3.2.5 shows a connection between smoothability and the smoothable component:

Proposition 3.2.3. *Let $X = \mathbb{P}_k^n$ and $R \subseteq X$ be a zero-dimensional closed subscheme of degree r. The following conditions are equivalent*

1. *$[R] \in \mathrm{Hilb}_{X/k}^{r}{}^{\circ}$;*

2. *there exists a smoothing of R in X;*

3. *there exists a smoothing of R in X over a base $D = \mathrm{Spec}\, k[\![x]\!]$.*

Proof. This is [CEVV09, Lem 4.1]. $\qquad\square$

It is natural to ask what is the class of schemes Z such that R is smoothable in Z. The following lemma shows that this class is in some sense directed; this answers [BGL10, Question 2.12].

Lemma 3.2.4. *Let $\varphi : Y_1 \to Y_2$ be an affine morphism of schemes and suppose that $R \subseteq Y_1$ is a zero-dimensional subscheme of Y_1 such that $\mathrm{im}\, \varphi(R) \simeq R$. If R is smoothable in Y_1, then it is smoothable in Y_2.*

Proof. Let $D = \mathrm{Spec}\, k[\![x]\!]$ and suppose that $Z \subseteq Y_1 \times D$ is a smoothing of R in Y_1. We claim that $Z' := \mathrm{im}(\varphi \times \mathrm{id})(Z) \subseteq Y_2 \times D$ is a smoothing of $\mathrm{im}\, \varphi(R)$ in Y_2. It would suffice to show that $\varphi' = \varphi \times \mathrm{id}_D$ gives an isomorphism of Z and Z' as schemes over D. Let $t \in D$ denote the closed point of D. Since $R = Z_t$, $\mathrm{im}\, \varphi(R) = Z'_t$ are finite and $Z \to D$ is proper, it follows that Z is finite over D, then Z' is also finite over D. In particular Z, Z' are affine. Now $Z_t \simeq Z'_t$ and Z is flat over D, so the result follows from Nakayama's lemma (see [Eis95, Cor 4.8]). $\quad\square$

At the end, we note the following important result from [BB10], which uses and generalizes [CN09, Lem 2.2]:

Proposition 3.2.5. *Suppose that X and Y are two projective varieties over an algebraically closed field of characteristic zero. Let R be a zero-dimensional scheme. Fix closed embeddings $R \subseteq X$ and $R \subseteq Y$. If R is smoothable in Y and $R \subseteq X$ is supported in the smooth locus of X, then R is smoothable in X.*

Proof. See [BB10, Prop 2.1]. $\qquad\square$

It is important to know that there are successful attempts of proving smoothability of zero-dimensional schemes by parametrising the Hilbert scheme and using computer algebra systems, see [BCR12].

4. Gorenstein schemes and algebras

This large section is devoted to proving various properties of Gorenstein schemes, needed for the later considerations on the points of the Hilbert scheme.

In this section we will consider only zero-dimensional noetherian schemes. They are affine, so we will talk of rings A rather than schemes $\operatorname{Spec} A$. Let us first state the preliminaries needed for the definition of a zero-dimensional Gorenstein module. We mainly follow [Eis95, Chapter 21].

If A is a ring, M is an A-module and $I \lhd A$ then we denote

$$\operatorname{ann}(M) := \{a \in A \mid aM = 0\}, \qquad \operatorname{ann}_M(I) := (m \in M \mid Im = 0).$$

Remark 4.0.6. *Finitely generated modules over a zero-dimensional noetherian ring are artinian.*

Definition 4.0.7. *Let $(A, \mathfrak{m}, k)$ be a zero-dimensional local ring and M be a finitely generated A-module. Define* the socle *of M to be the annihilator of $\mathfrak{m}$. We denote this submodule by $\operatorname{soc}(M) = \operatorname{ann}_M(\mathfrak{m})$.*

Recall that a submodule $N \subseteq M$ is an *essential submodule* of M if $N \cap M' \neq 0$ for any submodule $0 \neq M' \subseteq M$. The following lemma justifies the use of the word "socle"; it is a standard exercise in the theory of modules, see e.g. [Lam91, 2.4 Ex. 18].

Lemma 4.0.8. *Under the assumptions of Definition 4.0.7, the socle of M is the smallest, with respect to inclusion, essential submodule of M.*

Proof. First, let us show that $\operatorname{soc}(M)$ is essential. Take any nonzero $M' \subseteq M$. The sequence $M' \supseteq \mathfrak{m}M' \supseteq \mathfrak{m}^2 M' \supseteq \ldots$ is eventually constant: there exists n such that $\mathfrak{m}^n M' = \mathfrak{m}^{n+1} M'$ and by Nakayama's lemma (see [Eis95, Cor 4.8]) we get $\mathfrak{m}^n M' = 0$. Let $l \geqslant 0$ be the maximal natural number such that $\mathfrak{m}^l M' \neq 0$, then $\mathfrak{m}^l M' \subseteq M' \cap \operatorname{soc}(M)$.

Now, if $m \in \operatorname{soc}(M)$, then Am is a simple module, so any essential submodule of M contains it. $\qquad\square$

The proof of Lemma 4.0.8 motivates the following definition:

Definition 4.0.9. *Let $(A, \mathfrak{m}, k)$ be a zero-dimensional local ring and M be a finitely generated A-module. We say that M is* Gorenstein *if and only if $\operatorname{rk}_k \operatorname{soc}(M) = 1$. If M is Gorenstein then the* socle degree *of M is the maximal l such that $\mathfrak{m}^l M \neq 0$.*

Remark 4.0.10. *Since $\operatorname{soc}(M) \simeq \operatorname{Hom}_A(k, M)$ and any zero-dimensional local ring is (trivially) Cohen-Macaulay this definition agrees with the usual definition of Gorenstein rings. There is an important connection between Gorenstein rings and duality theory, see [Eis95, Section 21.3].*

Proposition 4.0.11. *Let* $(A, \mathfrak{m}, k)$ *be a zero-dimensional local ring. Then the following are equivalent*

1. *A is Gorenstein,*

2. *A is injective as an A-module.*

3. *the canonical module* ω_A *is isomorphic to A as an A-module.*

Proof. The canonical module is defined as a consequence of [Eis95, Prop 21.1] and the above statement is a part of [Eis95, Thm 21.5]. □

For arbitrary rings being Gorenstein is defined as a stalk-local property:

Definition 4.0.12. *Let A be a zero-dimensional ring and M be a finitely generated A-module. Then M is a* Gorenstein *A-module if and only if* $M_{\mathfrak{m}}$ *is a Gorenstein* $A_{\mathfrak{m}}$*-module for all maximal ideals* $\mathfrak{m}$ *of A.*

4.1. Gorenstein points of the Hilbert scheme

Let X be a projective scheme over a field k.

Lemma 4.1.1. *Let* $X \to Y$ *be a flat, quasi-finite morphism of projective schemes. Then the set of points* $y \in Y$ *such that the fiber* X_y *is Gorenstein is open in* Y.

Proof. Unfortunately, a detailed exposition would take us to far from our main subject, we only sketch the proof. The morphism $X \to Y$ is projective and quasi-finite, thus finite by [GW10, Cor 12.89]. The relative dualizing sheaf $\omega_{X/Y}$ of a morphism $X \to Y$ is introduced in [Kle80, Def 6] and it exists for $X \to Y$ by [Kle80, Cor 18]. Furthermore $\omega_{X/Y}$ is a quasi-coherent sheaf over X by definition and coherent by finiteness of the morphism and the properties from [Kle80, Def 1, Prop 9]. Now the locus where $\omega_{X/Y}$ is not invertible is closed in X, thus the locus of $y \in Y$ such that $\left(\omega_{X/Y} \right)_y$ is invertible is open in Y. The sheaf $\omega_{X/Y}$ is stable under base change by [Kle80, Prop 9], thus $\left(\omega_{X/Y} \right)_y \simeq \omega_{X_y/y}$ for every $y \in Y$.

Let k be a field and A be a finite k-algebra. Note that we have a canonical isomorphism $\mathrm{Hom}_A \left(M, \mathrm{Hom}_k \left(A, W \right) \right) \simeq \mathrm{Hom}_k \left(M, W \right)$ for every A-module M and k-module W. Let $X' = \mathrm{Spec}\, A$ and $Y' = \mathrm{Spec}\, k$. The relative dualizing sheaf $\omega_{X'/Y'}$ exists and it is isomorphic as a sheaf over X' to the sheafification of $\mathrm{Hom}_k \left(A, k \right)$ by [Kle80, Def 1, Prop 9] and the mentioned canonical isomorphism. The A-module $\mathrm{Hom}_k \left(A, k \right)$ is isomorphic to the canonical module ω_A, as defined in [Eis95, Prop 21.1], by the discussion after [Eis95, Cor 21.3]. Now the claim follows from Proposition 4.0.11. □

As an immediate consequence of Lemma 4.1.1 we see that there is a good locus of the Hilbert scheme parametrising Gorenstein subschemes of X.

Proposition 4.1.2. *Let* $\pi : \mathcal{U} \to \mathrm{Hilb}^r_{X/k}$ *be the projection from the universal family. The set of points* $h \in \mathrm{Hilb}^r_{X/k}$ *such that the fiber of* π *over* h *is Gorenstein is open in* $\mathrm{Hilb}^r_{X/k}$.

The tangent space to the Hilbert scheme at a k-point corresponding to a Gorenstein subscheme is also easier to calculate, thanks to the injectivity:

18

Proposition 4.1.3. *Let $Z \subseteq_{cl} X$ be a Gorenstein zero-dimensional closed scheme with support contained in an open affine subscheme $U = \operatorname{Spec} A$ of X, where the subscheme Z is cut out by an ideal $I \lhd A$. Then the tangent space to $\operatorname{Hilb}_{X/k}$ at $[Z]$ has rank $\operatorname{rk}_k I/I^2 = \operatorname{rk}_k A/I^2 - \operatorname{rk}_k A/I$.*

Proof. We can analyse the connected components of the support of Z separately, thus we assume that A/I is local. By Proposition 4.0.11 we see that the functor $\operatorname{Hom}_{A/I}(-, A/I)$ is exact. Now by induction on the length we prove that $\operatorname{rk}_k \operatorname{Hom}_{A/I}(M, A/I) = \operatorname{rk}_k M$ for any finitely generated A/I-module M. $\qquad\square$

4.2. Macaulay's inverse systems

From now on, we will assume that $(A, \mathfrak{m}, k)$ is a finite local k-algebra such that $k \to A/\mathfrak{m}$ is an isomorphism. We will say simply "let $(A, \mathfrak{m}, k)$ be a finite local k-algebra", implicitly assuming that $k \to A/\mathfrak{m}$ is an isomorphism.

So far we did not give any examples of Gorenstein rings. Macaulay's inverse systems give a rich source of such examples, to some extent classifying all local Gorenstein zero-dimensional k-algebras. First, we explain how to "intrinsically" put an algebra structure on the space of k-functionals on a linear space V, when V has an additional structure of a coalgebra. Since we will be mainly interested in the example $V = k[x_1, \ldots, x_n]$ and the result will be, at least in characteristic zero, $V^* \simeq k[\![\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]\!]$, the less curious reader should skip directly to Theorem 4.2.7.

We recall the definition of a coalgebra, for simplicity we deal only with cocommutative coalgebras:

Definition 4.2.1. *A k-module C with k-linear maps $\Delta : C \to C \otimes C$ (comultiplication) and $\varepsilon : C \to k$ (counity) is a (cocommutative) k-coalgebra if the following equations (dual to the commutative monoid equations) are satisfied:*

1. $(\varepsilon \otimes \operatorname{id}_C) \circ \Delta = (\operatorname{id}_C \otimes \varepsilon) \circ \Delta = \operatorname{id}_C$ (counitality),

2. $(\operatorname{id}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \operatorname{id}_C) \circ \Delta$ (coassociativity),

3. $\operatorname{swap} \circ \Delta = \Delta$ (cocommutativity), where $\operatorname{swap} : C \otimes C \ni \sum c_1 \otimes c_2 \mapsto \sum c_2 \otimes c_1 \in C \otimes C$.

Example 4.2.2. *In this paper we only encounter coalgebras of the form $\Gamma(S, \mathcal{O}_S)$ for an affine commutative monoid S, with Δ, ε induced by multiplication and identity of S.*

Corollary 4.2.3. *If C is cocommutative coalgebra then the k-linear space $C^* := \operatorname{Hom}(C, k)$ with the multiplication defined by*

$$(\varphi_1 \cdot \varphi_2)(c) = (\varphi_1 \otimes \varphi_2)(\Delta c),$$

is a commutative k-algebra and C with multiplication

$$\mu : C^* \otimes C \to C, \ \ defined \ by \ \mu(\varphi \otimes c) = (\varphi \otimes \operatorname{id}_C)(\Delta(c)),$$

is a C^*-module. We will write $\varphi \cdot c$ for $\mu(\varphi \otimes c)$.

Proof. The k-algebra structure follows directly from Definition 4.2.1. To check that C is a C^*-module, it is, by linearity, sufficient to prove $\varphi_1 \cdot (\varphi_2 \cdot c) = (\varphi_1 \cdot \varphi_2) \cdot c$ and this follows from definition of multiplication on C^* and coassociativity of C. $\qquad\square$

An enlightening exercise is to analyze dependencies between the subcoalgebras of C and C^*-submodules of C.

Remark 4.2.4. *We are primarily interested in the example $C = \Gamma\left(\mathbb{G}_a(k)^n\right) = k[x_1, \ldots, x_n]$, where $\mathbb{G}_a(k)$ is the affine line over k with addition. In this case the ring obtained in Corollary 4.2.3 is called the ring of divided powers, but we will not go into details here. Below we give an explicit description of C^* and its action on C when k is a field of characteristic zero.*

Proposition 4.2.5. *Let k be a field of characteristic zero. Fix the isomorphism*

$$\Gamma\left(\mathbb{G}_a(k)^n\right) \simeq k[x_1, \ldots, x_n] := C,$$

then C^* is isomorphic to $k[\![y_1, \ldots, y_n]\!]$, the ring of formal power series, by identifying a monomial $y_1^{a_1} \cdot \ldots \cdot y_n^{a_n}$ with the dual to the monomial $(a_1! \cdot a_2! \cdot \ldots \cdot a_n!)^{-1} \cdot x_1^{a_1} \ldots x_n^{a_n}$ in the monomial basis. The element y_i acts on $k[x_1, \ldots, x_n]$ by $f \mapsto y_i \lrcorner f := \frac{\partial f}{\partial x_i}$ and this extends to the action of C^* by multiplicativity and countable additivity.

Proof. We need the multi-index notation: let $\mathbf{x} := (x_1, \ldots, x_n)$, then by $\mathbf{x^a}$ we mean the monomial $\prod x_i^{a_i}$, by $\mathbf{a}!$ the product of $a_i!$ and by $\binom{\mathbf{a}}{\mathbf{b}}$ the expression $\mathbf{a}!/(\mathbf{b}!\,(\mathbf{a} - \mathbf{b})!)$, which is defined when $a_i \geqslant b_i$ for all i.

Let us fix the isomorphism of k-vector spaces $C^* \simeq k[\![z_1, \ldots, z_n]\!]$ by identifying $\mathbf{z^a}$ with the dual to $\mathbf{x^a}$ in the monomial basis. We observe that $\Delta : C \to C \otimes C$ is defined by setting $\Delta(x_i) := 1 \otimes x_i + x_i \otimes 1$ and extending to a k-algebra homomorphism, which gives

$$\Delta\left(\mathbf{x^a}\right) = \sum_{\mathbf{b,c}:\ \mathbf{b+c=a}} \binom{\mathbf{a}}{\mathbf{b}} \mathbf{x^b} \otimes \mathbf{x^c}, \text{ so that}$$

$$\mathbf{z^b} \cdot \mathbf{z^c} = \binom{\mathbf{b+c}}{\mathbf{b}} \mathbf{z^{b+c}}.$$

Now define a k-linear map $f : k[\![y_1, \ldots, y_n]\!] \to C^*$ by $f\left(\mathbf{y^a}\right) = \mathbf{a}! \cdot \mathbf{z^a}$. Since k is of characteristic 0 it is an isomorphism. From the above multiplication law it follows that f is a k-algebra homomorphism. To finish the proof it is sufficient to note that

$$y_i \lrcorner \mathbf{x^a} = \binom{\mathbf{a}}{1_i} \mathbf{x^{a-1_i}} = a_i \cdot \mathbf{x^{a-1_i}} = \frac{\partial \mathbf{x^a}}{\partial x_i},$$

where $1_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on i-th position. $\qquad\square$

Now we will prove the main theorem of Macaulay's inverse systems. The duality coming from this theorem is, as far as we know, used only in the case of $C = \Gamma\left(\mathbb{G}_a(k)^n\right)$, but we will prove it in the language of coalgebras. The reasons are twofold. Firstly, the proof is essentially the same and clearer to prove in the abstract setting. Secondly, even in the case $C = \Gamma\left(\mathbb{G}_a(k)^n\right)$, where k has positive characteristic, the algebra C^* is quite complicated to deal by hand.

Recall that for a ring A and A-modules $N \subseteq M$ we say that N is a *small* A-submodule of M if for every A-submodule $M' \subseteq M$ the equality $M' + N = M$ implies $M' = M$. As we will see, this is "dual" to being essential.

For a coalgebra C and $D \subseteq C$ by $D^\perp$ we denote $\{\varphi \in C^* \mid \varphi(d) = 0 \text{ for all } d \in D\}$, similarly for $I \subseteq C^*$ by $I^\perp$ we denote $\{c \in C \mid i(c) = 0 \text{ for all } i \in I\}$, where $\varphi(d)$ is the value of a functional $\varphi \in C^*$ on an element $d \in C$.

Theorem 4.2.6 (Macaulay's inverse systems). *Let C be a cocommutative k-coalgebra. Suppose that*

1. *there is a rank one subcoalgebra $C_0 \subseteq C$, contained in each non-zero C^*-submodule of C;*

2. *for every ideal I of C^* such that C^*/I is finite (over k) there is a finite (over k) C^*-submodule $D \subseteq C$ such that $D^\perp \subseteq I$.*

Then

1. *there is a bijection*

$$\{I \triangleleft C^* \mid \operatorname{rk}_k C^*/I < \infty\} \longleftrightarrow \{C^*\text{-submodules of } C \text{ finite over } k\}$$
$$f_1 : I \triangleleft C^* \longmapsto I^\perp$$
$$f_2 : D^\perp \triangleleft C^* \longleftarrow D \subseteq C;$$

2. *the above bijection preserves k-rank:*

$$\operatorname{rk}_k C^*/I = \operatorname{rk}_k I^\perp \text{ and } \operatorname{rk}_k D = \operatorname{rk}_k C^*/D^\perp;$$

3. *the algebra C^* is local with residue field k;*

4. *If $D, D' \subseteq C$ are finitely generated C^*-modules, then $D' \subseteq D$ is small if and only if the image of $D'^\perp$ in $C^*/D^\perp$ is essential;*

5. *the above bijection restricts to a bijection between Gorenstein quotients of C^* and cyclic C^*-submodules of C.*

Proof. We view C as contained in C^{**} via the canonical map.

1. Clearly $D \subseteq f_1 \circ f_2(D) = \left(D^{\perp\perp} \cap C\right)$ and if $c \in C \setminus D$ then we can find $\varphi \in D^\perp$ such that $\varphi(c) \neq 0$, this proves $f_1 \circ f_2 = \operatorname{id}$. Similarly, $I \subseteq f_2 \circ f_1(I) = \left(I^\perp \cap C\right)^\perp$. Take $D \subseteq C$ from Condition 2, then we have a perfect pairing between finite vector spaces $C^*/D^\perp$ and D, which restricts to a perfect pairing of C^*/I with $f_1(I)$; this proves $f_2 \circ f_1(I) = I$ by counting k-ranks.

2. It is sufficient to check this for f_2 and then it is trivial as D is finite over k and $C^*/D^\perp \simeq D^*$.

3. As $C_0 \subseteq D \subseteq C$ for any D, we see that $C_0^\perp$ is the largest ideal of C^* such that the quotient is finite over k, thus the largest ideal of C^*.

4. Let $I := D^\perp$. Suppose that $J/I \subseteq C^*/I$ is essential and set $E := f_1(J) \subseteq D$. Take any submodule $E' \subsetneq D$ with corresponding ideal $J'/I \subseteq C^*/I$. Since $J'/I \neq 0$ we have $J'/I \cap J/I \neq 0$ and so $(J' \cap J)^\perp = E' + E$ is not the whole of D. This argument can be reversed.

5. Gorenstein quotients are precisely those containing a rank one essential submodule, so by Point 4 it is enough to prove that cyclic modules are precisely those having a corank one small submodules. Indeed by Nakayama's lemma (see [Eis95, Cor 4.8]) a cyclic module C^*c has corank one small submodule $\mathfrak{m}_{C^*}c$. Suppose now that $D \subseteq C$ contains a corank one small submodule $D' \subseteq D$ and choose $c \in D \setminus D'$; then $C^*c + D' = D$, so $C^*c = D$. $\qquad\square$

21

The corollary of Theorem 4.2.6 are the classical Macaulay's inverse systems:

Theorem 4.2.7 (Macaulay's inverse systems for $\mathbb{G}_a^n$)**.** *Let k be a field of characteristic zero and V be a k-vector space with basis $x_1, \ldots, x_n$. Let $S = \mathrm{Sym}(V) = k[x_1, \cdots, x_n]$ be a polynomial k-algebra and $S^* := k[\![y_1, \cdots, y_n]\!]$ be a ring of power series. We view S^* as acting on S by identifying y_i with $\frac{\partial}{\partial x_i}$; denote this action by $\lrcorner : S^* \otimes S \to S$. There is a bijection*

$$\{I \lhd S^* \mid \mathrm{rk}_k\, S^*/I < \infty\} \longleftrightarrow \{\text{ finitely generated } S^*\text{-submodules of } S\}$$

$$f_1 : I \lhd S^* \longmapsto \mathrm{ann}_S(I) =: I^\perp$$

$$f_2 : M^\perp := \mathrm{ann}_{S^*}(M) \longleftarrow\!\shortmid M \subseteq S;$$

preserving k-rank and restricting to a bijection between Gorenstein quotients of S^* and cyclic S^*-submodules of S. Moreover I is homogeneous if and only if $\mathrm{ann}_S(I)$ is generated by homogeneous polynomials.

Proof. We will use Theorem 4.2.6 together with Proposition 4.2.5. First let us check the assumptions of Theorem 4.2.6. First, $k \subseteq S$ is contained in any non-zero S^*-submodule of S. Next, take $I \lhd S^*$ such that S^*/I is finite over k. The algebra S^*/I is local and artinian, thus $\mathfrak{m}_{S^*}^{m+1} \subseteq I$ for some m. But $\mathfrak{m}_{S^*}^{m+1}$ is the ideal orthogonal to the S^*-submodule of S consisting of polynomials of degree at most m.

Note that each cyclic S^*-submodule of S is finite over k, so that a S^*-submodule $M \subseteq S$ is finite over k if and only if it is a finitely generated S^*-module. Now the main claims follow from Theorem 4.2.6 together with Proposition 4.2.5, we left the homogeneity fact for the reader. $\qquad\square$

Definition 4.2.8. *For an ideal $I \lhd S^*$, any $f \in S$ such that $f^\perp = I$ is called the* dual socle generator *of S^*/I. Conversely for $f \in S$ the quotient $S^*/f^\perp$ is called the* apolar algebra of *$f \in S$.*

Remark 4.2.9. *Every local finite Gorenstein algebra $(A, \mathfrak{m}, k)$ may be viewed as a quotient of S^*, and the most important part of Theorem 4.2.7 is that any such algebra is isomorphic to an algebra of the form $S^*/f^\perp$ for some polynomial $f \in S$. We will explore this ideas in Subsection 4.5.*

Example 4.2.10. *If $f = x_1^2 + x_1 \cdot x_2$, then $I = \left(\left(\frac{\partial}{\partial x_2}\right)^2, \left(\frac{\partial}{\partial x_1}\right)^2 - 2 \cdot \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} \right)$. The socle of the dual algebra is generated by the image of e.g. $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}$.*

4.3. Perfect pairings

The following proposition states, in the linear-algebra language, that a finite local Gorenstein algebra A is isomorphic to its canonical module, see Proposition 4.0.11). It will be used in the following subsection.

Proposition 4.3.1. *Let $(A, \mathfrak{m}, k)$ be a finite local Gorenstein k-algebra. Choose any splitting into k-modules $A = V \oplus \mathrm{soc}(A)$ and a projection $\pi : A \to \mathrm{soc}(A) \simeq k$. Then the pairing*

$$m : A \times A \ni (a, b) \longmapsto \pi(ab) \in k$$

is perfect.

Proof. It is enough to prove that for any $a \in A$ we have $m(\{ka\} \times A) \neq 0$. But $m(\{ka\} \times A) = \pi(Aa) \neq 0$ as the ideal Aa contains an element from the socle by Lemma 4.0.8. $\qquad\square$

Remark 4.3.2. *Even though the choice of splitting and π clearly affects the pairing, we see that for any ideal I the orthogonal complement of I is the annihilator $\mathrm{ann}\,(I)$, in particular it does not depend on these choices.*

4.4. Local Hilbert function of a Gorenstein algebra

In this and next subsections we follow a foundational paper of the theory of zero-dimensional Gorenstein k-algebras by Iarrobino [Iar94]. The idea is to exploit the perfect pairing from Proposition 4.3.1 to gain information about the powers of the maximal ideal and thus about the Hilbert function. The most important result is the decomposition of the Hilbert function, Theorem 4.4.3.

Definition 4.4.1. *Let $(A, \mathfrak{m}, k)$ be a local ring. The* local Hilbert function *of A is defined by*

$$h_A : \mathbb{Z}_{\geqslant 0} \ni n \longmapsto \mathrm{rk}_k\, \mathfrak{m}^n/\mathfrak{m}^{n+1} \in \mathbb{Z}_{\geqslant 0}.$$

If A is a finite k-algebra, then $h(n) = 0$ for $n \gg 0$ and we will identify the function with the vector of its nonzero values.

Note that the local ring $(A, \mathfrak{m}, k)$ has a natural filtration by powers of maximal ideal $\mathfrak{m}$ and that the local Hilbert function is computed from $\mathrm{gr}\,\mathcal{A}$ with respect to this filtration.

Let us denote $(\mathfrak{m}^m)^\perp := \mathrm{ann}\,(\mathfrak{m}^m)$ for any $m \geqslant 0$ and adopt the convention that non-positive powers of ideals are equal to the whole ring. A direct consequence of Proposition 4.3.1 with Remark 4.3.2 is:

Corollary 4.4.2. *Let $(A, \mathfrak{m}, k)$ be a finite local Gorenstein k-algebra. Then for every $m \geqslant 0$ we have*

$$h_A(m) = \mathrm{rk}_k\, \frac{(\mathfrak{m}^{m+1})^\perp}{(\mathfrak{m}^m)^\perp}.$$

Proof. Indeed, fixing any pairing as in Proposition 4.3.1 we see that

$$\left(\frac{\mathfrak{m}^m}{\mathfrak{m}^{m+1}}\right)^* \simeq \frac{(\mathfrak{m}^{m+1})^\perp}{(\mathfrak{m}^m)^\perp}. \qquad\square$$

Iarrobino noticed that a Gorenstein algebra satisfies strong symmetry conditions on the local Hilbert function:

Theorem 4.4.3 (Hilbert function decomposition)**.** *Suppose that $(A, \mathfrak{m}, k)$ is a finite local Gorenstein k-algebra with socle degree (see 4.0.9) equal to j. Define*

$$A_{m,n} := \mathfrak{m}^m \cap (\mathfrak{m}^n)^\perp \ \text{ and } \ Q_{m,n} := \frac{A_{m,n}}{A_{m+1,n} + A_{m,n-1}} \ \text{ for all } 0 \leqslant m, n \leqslant j+1.$$

Denote

$$\Delta_{A,s}(t) = \Delta_s(t) := \begin{cases} \mathrm{rk}_k\, Q_{t, j+1-(s+t)} & \text{for } \ 0 \leqslant s \leqslant j, \ 0 \leqslant t \leqslant j - s, \\ 0 & \text{otherwise} \end{cases}$$

and call this the Hilbert function decomposition *with rows Δ_s. It is convenient to note that $\mathrm{rk}_k\, Q_{m,n} = \Delta_{j+1-(m+n)}(m)$.*

1. $Q_{m,0} = Q_{j+1,m} = 0$ for all m. If $m < j+1$, then $Q_{0,m} = 0$.

2. If $m + n > j + 1$, then $A_{m,n} = A_{m,n-1}$, so $Q_{m,n} = 0$.

3. Fix any pairing $A \times A \to k$ defined as in Proposition 4.3.1. It induces a perfect pairing
$$Q_{m,n} \times Q_{n-1,m+1} \to k,$$
in particular $\Delta_s(t) = \Delta_s((j-s)-t))$, we say that Δ_s is symmetric with respect to $\frac{j-s}{2}$.

4. For $j \geqslant a \geqslant 0$ denote $h_a(t) := \sum_{i=0}^{a} \Delta_i(t)$. The quotient
$$\frac{\operatorname{gr} \mathcal{A}}{C_a = \bigoplus_{0 \leqslant m \leqslant j} C_a^m}$$
has Hilbert function h_a, where $C_a^m \subseteq \mathfrak{m}^m/\mathfrak{m}^{m+1}$ is the image of $A_{m,j-(m+a)} = \mathfrak{m}^m \cap \left(\mathfrak{m}^{j-(m+a)}\right)^{\perp}$.

5. The local Hilbert function h_A satisfies $h_A(t) = 0$ for $t > j$ and
$$h_A(t) = \sum_{i=0}^{j-t} \Delta_i(t) \quad \text{for } 0 \leqslant t \leqslant j.$$

Remark 4.4.4. *Some of the claims of the theorem may be summarized by saying that the following tables are respectively "symmetric up to taking duals" (see Point 3 of the theorem) and symmetric with respect to their anti-diagonals:*

Proof of Theorem 4.4.3. 1. Obviously $A_{m,0} = A_{m,-1} = \mathfrak{m}^m$ and $\mathfrak{m}^{j+1} = 0$, this proves the first equality. For the second equality, note that if $m < j+1$, then $\mathfrak{m}^m \neq 0$ and so $(\mathfrak{m}^m)^{\perp} \subseteq \mathfrak{m}$.

2. By assumption $\mathfrak{m}^{m+n-1} = 0$, thus $\mathfrak{m}^m \cap (\mathfrak{m}^n)^\perp = \mathfrak{m}^m \cap (\mathfrak{m}^{n-1})^\perp = \mathfrak{m}^m$.

3. Recall that k-modules are *modular* i.e. if M, N, P are k-modules and $M \subseteq P$ then $(M + N) \cap P = M + N \cap P$. In particular, as for any n, m we have $\mathfrak{m}^n \subseteq \mathfrak{m}^{n-1}$ and $(\mathfrak{m}^m)^\perp \subseteq (\mathfrak{m}^{m+1})^\perp$ it follows that

$$\left(\mathfrak{m}^{n+1} + \left(\mathfrak{m}^{m-1} \right)^\perp \right) \cap \mathfrak{m}^n = \mathfrak{m}^{n+1} + \mathfrak{m}^n \cap \left(\mathfrak{m}^{m-1} \right)^\perp,$$

$$\left(\mathfrak{m}^{n+1} + \mathfrak{m}^n \cap \left(\mathfrak{m}^{m-1} \right)^\perp \right) \cap (\mathfrak{m}^m)^\perp = \mathfrak{m}^{n+1} \cap (\mathfrak{m}^m)^\perp + \mathfrak{m}^n \cap \left(\mathfrak{m}^{m-1} \right)^\perp. \tag{4.1}$$

Below, for an A-module M by M^* we denote $\mathrm{Hom}_k(M, k)$ and for $N \subseteq M$ by $N^\perp \subseteq M^*$ we denote functionals which are zero on N. The fixed perfect pairing $A \times A \to k$ gives an isomorphism of k-modules $A \simeq A^*$. From Remark 4.3.2 it follows that

$$\left(\mathfrak{m}^{n-1} \cap \left(\mathfrak{m}^{m+1} \right)^\perp \right)^\perp \simeq \mathfrak{m}^{m+1} + \left(\mathfrak{m}^{n-1} \right)^\perp, \quad \text{and} \quad \left(\mathfrak{m}^n + (\mathfrak{m}^m)^\perp \right)^\perp \simeq \mathfrak{m}^m \cap (\mathfrak{m}^n)^\perp, \quad \text{thus}$$

$$\left(\frac{\mathfrak{m}^{n-1} \cap (\mathfrak{m}^{m+1})^\perp}{\left(\mathfrak{m}^{n-1} \cap (\mathfrak{m}^{m+1})^\perp \right) \cap \left(\mathfrak{m}^n + (\mathfrak{m}^m)^\perp \right)} \right)^* \simeq \frac{\left(\mathfrak{m}^{n-1} \cap (\mathfrak{m}^{m+1})^\perp \right)^\perp + \left(\mathfrak{m}^n + (\mathfrak{m}^m)^\perp \right)^\perp}{\left(\mathfrak{m}^{n-1} \cap (\mathfrak{m}^{m+1})^\perp \right)^\perp} \simeq$$

$$\simeq \frac{\left(\mathfrak{m}^{m+1} + (\mathfrak{m}^{n-1})^\perp \right) + \left(\mathfrak{m}^m \cap (\mathfrak{m}^n)^\perp \right)}{\mathfrak{m}^{m+1} + (\mathfrak{m}^{n-1})^\perp} \simeq \frac{\mathfrak{m}^m \cap (\mathfrak{m}^n)^\perp}{\mathfrak{m}^m \cap (\mathfrak{m}^n)^\perp \cap \left(\mathfrak{m}^{m+1} + (\mathfrak{m}^{n-1})^\perp \right)},$$

which is, by Equation (4.1), equivalent to $Q_{n-1,m+1} \simeq Q_{m,n}^*$. Now $\Delta_s(t) = \mathrm{rk}_k Q_{t,j+1-(s+t)} = Q_{j-(s+t),t+1} = \Delta_s(j-(s+t))$.

4. For any $m \in \mathbb{Z}_{\geqslant 0}$ we have $\mathfrak{m} \cdot A_{m,j-(m+a)} \subseteq A_{m+1,j-(m+1+a)}$ which proves that C_a is an ideal in $\mathrm{gr}\,\mathcal{A}$. Let us fix $e := j - (m + a)$ and look at the filtration

$$A_{m,e} \subseteq A_{m,e+1} \subseteq A_{m,e+2} \subseteq \ldots A_{m,j} \subseteq A_{m,j+1} = \mathfrak{m}^m.$$

Since $\mathfrak{m}^{m+1} \cap A_{m,n} = A_{m+1,n}$, this gives a filtration

$$0 = \frac{A_{m,e}}{A_{m+1,e} + A_{m,e}} \subseteq \frac{A_{m,e+1}}{A_{m+1,e+1} + A_{m,e}} \subseteq \cdots \subseteq \frac{A_{m,j+1}}{A_{m+1,j+1} + A_{m,e}} = \frac{\mathfrak{m}^m}{\mathfrak{m}^{m+1} + \mathfrak{m}^m \cap (\mathfrak{m}^e)^\perp},$$

with associated graded $Q_{m,e+1} \oplus Q_{m,e+2} \oplus \ldots Q_{m,j+1}$. Since the modules are finitely generated over k from Corollary 2.3.5 it follows that

$$h_{\mathrm{gr}\,\mathcal{A}/C_a}(m) = \sum_{e+1 \leqslant n} Q_{m,n} = \sum_{n=e+1}^{j+1-m} Q_{m,n} = \sum_{i=0}^{j+1-m-(e+1)} \Delta_i(m) = \sum_{i=0}^{a} \Delta_i(m).$$

5. The ideal $A_{m,j-(m+j)} = A_{m,-m}$ is zero; thus $I_j = 0$ and $A = A/I_j$. Now take $a = j$ in Point 4. $\qquad \square$

Remark 4.4.5. *It follows from the proof that $\Delta_j = (0)$ and $\Delta_{j-1} = (0,0)$ for any algebra, so we will ignore these vectors. The maximal a such that Δ_a may have a non-zero entry is $a = j - 2$ and indeed $\Delta_{j-2} = (0, *, 0)$ contains useful information about the algebra, as we will see in Proposition 5.1.1.*

Example 4.4.6. *Below we write the $Q_{m,n}$ table for $j = 3$, which is the smallest nontrivial case.*

$\mathrm{rk}_k \frac{A}{\mathfrak{m}}$	0	0	0
0	$\mathrm{rk}_k \frac{\mathfrak{m}\cap(\mathfrak{m}^3)^{\perp}}{\mathfrak{m}^2\cap(\mathfrak{m}^3)^{\perp}+\mathfrak{m}\cap(\mathfrak{m}^2)^{\perp}}$	0	0
0	$\mathrm{rk}_k \frac{\mathfrak{m}\cap(\mathfrak{m}^2)^{\perp}}{\mathfrak{m}^2\cap(\mathfrak{m}^2)^{\perp}+\mathfrak{m}\cap(\mathfrak{m})^{\perp}}$	$\mathrm{rk}_k \frac{\mathfrak{m}^2\cap(\mathfrak{m}^2)^{\perp}}{\mathfrak{m}^3\cap(\mathfrak{m}^2)^{\perp}+\mathfrak{m}^2\cap(\mathfrak{m})^{\perp}}$	0
0	0	0	$\mathrm{rk}_k\, \mathfrak{m}^3 \cap (\mathfrak{m})^{\perp}$

Hence the Hilbert function has decomposition $\Delta_0 = (1, h(2), h(2), 1)$, $\Delta_1 = (0, h(1) - h(2), 0)$ thus satisfying $h(1) \geqslant h(2)$.

Example 4.4.7. *From Point 4 of Theorem 4.4.3 it follows that there does not exist a local Gorenstein algebra A with Hilbert function decomposition*

$$
\begin{aligned}
\Delta_0 &= & (1, & 1, & 1, & 1, & 1, & 1) \\
\Delta_1 &= & (0, & 0, & 1, & 0, & 0) \\
\Delta_2 &= & (0, & 0, & 0, & 0) \\
\Delta_3 &= & (0, & 1, & 0).
\end{aligned}
$$

Indeed $v = \Delta_0 + \Delta_1 = (1, 1, 2, 1, 1, 1)$ should be a local Hilbert function of some finite k-algebra B and this is impossible because $v(1) = 1$ and $v(2) > 1$. Note that $h_A = (1, 2, 2, 1, 1, 1)$ seems possible to obtain and indeed there exist Gorenstein algebras with such function and decomposition

$$
\begin{aligned}
\Delta_0 &= & (1, & 1, & 1, & 1, & 1, & 1) \\
\Delta_1 &= & (0, & 0, & 0, & 0, & 0) \\
\Delta_2 &= & (0, & 1, & 1, & 0) \\
\Delta_3 &= & (0, & 0, & 0).
\end{aligned}
$$

4.5. Relations between the Hilbert function and the dual generator

Macaulay's inverse systems and the local Hilbert function decomposition provide us with multitude of information about the local finite Gorenstein algebra. In this section we will draw various conclusions about the structure of the algebra from the knowledge of its dual socle generator. Understanding a Gorenstein algebra A by analysis of the dual socle generator $f \in S$ of a chosen presentation $A \simeq S^*/f^{\perp}$ is elementary. The main problem of this method is embarras de richesses coming from the fact that we have many choices of f. To avoid it we classify polynomials, provide normal forms etc. The most important results in this direction are Lemma 4.5.5 and Theorem 4.5.9.

It is important to understand that most complications arise from the fact that the dual socle generator is in general not homogeneous. For the homogeneous case see Lemma 4.5.3.

Fix $S = k[x_1, \cdots, x_n]$ and $S^* = k[\![y_1, \cdots, y_n]\!]$ as defined in the Theorem 4.2.7. By S^m we denote the homogeneous polynomials of total degree m is S. Fix an element $f \in S$ and a local finite Gorenstein k-algebra $(A, \mathfrak{m}, k)$ isomorphic to $S^*/f^{\perp}$. Let j be the socle degree of A, it is equal to the degree of the polynomial f. It is important to know that, although using

Theorem 4.2.7 we are bound to consider only characteristic zero, the facts from this section remain true if we replace $k[\![y_1, \cdots, y_n]\!]$ by the divided powers ring and use Theorem 4.2.6.

Define $(S^*f)_n$ to be the submodule of S^*f spanned by elements of degree less than n and $(S^*f)^m := \mathfrak{m}_{S^*}^m f$, finally put $(S^*f)_n^m := (S^*f)_n \cap (S^*f)^m$.

Proposition 4.5.1. *We have an isomorphism of S^*-modules*

$$i : A \to S^*f \subseteq S, \;\; defined \; by \; \partial \mapsto \partial \lrcorner f.$$

The submodule $A_{m,n}$, defined in Theorem 4.4.3, maps to $(S^*f)_n^m$ under this isomorphism and thus the k-rank of $Q_{m,n}$, also defined in Theorem 4.4.3, is equal to

$$\mathrm{rk}_k \frac{(S^*f)_n^m}{(S^*f)_{n-1}^m + (S^*f)_n^{m+1}} = \mathrm{rk}_k(S^*f)_n^m - \mathrm{rk}_k\left((S^*f)_{n-1}^m + (S^*f)_n^{m+1}\right).$$

Furthermore, the value of the Hilbert function $h_A(t)$ is equal to both $\mathrm{rk}_k(S^*f)^t - \mathrm{rk}_k(S^*f)^{t+1}$ and $\mathrm{rk}_k(S^*f)_{t+1} - \mathrm{rk}_k(S^*f)_t$.

Proof. The claim on $A_{m,n}$ follows directly from definitions. For the part concerned with the Hilbert function check Corollary 4.4.2. $\qquad\square$

Example 4.5.2. *Using Proposition 4.5.1 one can compute the Hilbert function of the apolar algebra A of $x_1^5 + x_2^4 + x_3^4$. We have $(S^*f)_1 = k, (S^*f)_2 = k \oplus kx_1 \oplus kx_2 \oplus kx_3$ and so on, hence we compute $h_A = (1, 3, 3, 3, 1, 1)$.*

It is important to see that computing $\mathrm{rk}_k A_{m,n}$ or h_A via Proposition 4.5.1 we are mainly interested in the top degree forms of the derivatives.

The situation is particularly easy when the dual socle generator is homogeneous.

Lemma 4.5.3. *If f is homogeneous then $\Delta_m(n) = 0$ for all $m > 0$ and n. In particular the local Hilbert function of A is equal to Δ_0.*

Proof. Since for m, n such that $m + n < j + 1$ we have $(S^*f)_n^m = 0$ the claim follows from Proposition 4.5.1. $\qquad\square$

Remark 4.5.4. *One might hope that, for every $a \geqslant 0$, if f is a sum of polynomials of degrees at least $j - a$, then Δ_b has zero entries for all $b > a$. But this is not true. Take $f = x_1^4 - 12x_1^2x_2$, then $(y_1^2 + y_2)\lrcorner f = -24x_2$ and one may check that this is a non-zero element of $(S^*f)_2^1/\left((S^*f)_1^1 + (S^*f)_2^2\right)$. In particular $\Delta_2 \neq (0, 0, 0)$ and in fact the decomposition is*

$$\begin{aligned}
\Delta_0 &= & (1, & 1, & 1, & 1, & 1) \\
\Delta_1 &= & (0, & 0, & 0, & 0) \\
\Delta_2 &= & (0, & 1, & 0).
\end{aligned}$$

Intuitively, low degree homogeneous terms of f contribute only to $(S^*f)_n^m$ for which $m+n$ is small. The following lemma captures this intuition and says "if you add low degree term, you change only Δ_s for $s \gg 0$".

Lemma 4.5.5. *Suppose that polynomials $f_1, f_2 \in S$ of degree j are such that $\deg(f_1 - f_2) \leqslant j - a$. Denote $\Delta = \Delta_{S^*/f_1^\perp}$ and $\Delta' = \Delta_{S^*/f_2^\perp}$, then*

$$\Delta_m(n) = \Delta'_m(n) \;\; for \; all \; m \leqslant a - 1 \; and \; all \; n.$$

Proof. In the computation of $\Delta_m(n)$, where $m \leqslant a-1$, only $Q_{m,n}$ such that $(j+1)-(m+n) \leqslant a-1$ are used. By Proposition 4.5.1 k-ranks of such $Q_{m,n}$ depend only on k-ranks of $(S^*f_1)_n^m$ such that $(j+1)-(m+n) \leqslant a$. Let us fix m and take two canonical epimorphisms

$$\pi_i : \mathfrak{m}_{S^*}^m \to S^*f_i, \quad \pi_i(\partial) = \partial \lrcorner f_i \quad \text{for} \quad i = 1, 2.$$

We claim that if $(j+1)-(m+n) \leqslant a$ then the preimages $\pi_1^{-1}(S^*f_1)_n^m, \pi_2^{-1}(S^*f_2)_n^m$ are equal. First, if $\partial \in \mathfrak{m}_{S^*}^m$, then $\deg \partial \lrcorner (f_1 - f_2) \leqslant j-a-m$. An element $\partial \in \mathfrak{m}_{S^*}^m$ satisfies $\pi_1(\partial) \in (S^*f_1)_n^m$ if and only if $\deg \partial \lrcorner f_1 < n$. Similarly $\pi_2(\partial) \in (S^*f_2)_n^m$ if and only if $\deg \partial \lrcorner f_2 < n$. But $n \geqslant j+1-a-m > \deg \partial \lrcorner (f_1 - f_2)$ so both conditions are equivalent.

Now

$$\frac{(S^*f_1)_n^m}{(S^*f_1)_{n-1}^m} \simeq \frac{\pi_1^{-1}\left((S^*f_1)_n^m\right)}{\pi_1^{-1}\left((S^*f_1)_{n-1}^m\right)} = \frac{\pi_2^{-1}\left((S^*f_2)_n^m\right)}{\pi_2^{-1}\left((S^*f_2)_{n-1}^m\right)} \simeq \frac{(S^*f_2)_n^m}{(S^*f_2)_{n-1}^m}. \tag{4.2}$$

Since $(S^*f)_{n-1}^m \cap (S^*f)_n^{m+1} = (S^*f)_{n-1}^{m+1}$ we have inclusions

$$\frac{(S^*f_1)_n^{m+1}}{(S^*f_1)_{n-1}^{m+1}} \hookrightarrow \frac{(S^*f_1)_n^m}{(S^*f_1)_{n-1}^m} \quad \text{and} \quad \frac{(S^*f_2)_n^{m+1}}{(S^*f_2)_{n-1}^{m+1}} \hookrightarrow \frac{(S^*f_2)_n^m}{(S^*f_2)_{n-1}^m},$$

thus the claim follows from isomorphisms (4.2). $\qquad\square$

Example 4.5.7 shows that the inequality $m \leqslant a-1$ is strict.

Corollary 4.5.6. *1. The sequence $\Delta_{S^*/f^\perp, 0}$ is the Hilbert function of the top degree form of f.*

2. If the Hilbert function of $S^*/f^\perp$ is symmetric i.e. $h(t) = h(j-t)$ for all $0 \leqslant t \leqslant j$, then it is equal to Δ_0.

Proof. 1. Let f_{top} be the top degree form of f, then $\deg f - f_{top} \leqslant j-1$ and from Lemma 4.5.5 it follows that Δ_0 are equal for f and f_{top}. But f_{top} is homogeneous so from Lemma 4.5.3 it follows that Δ_0 is the Hilbert function of f_{top}.

2. The proof is purely combinatorial — the only important fact is that Δ_i are symmetric with respect to $\frac{j-i}{2}$ for $i \geqslant 0$, this follows from Theorem 4.4.3. $\qquad\square$

Example 4.5.7. *Let us take $n = 2$, $S^* = k[\![y_1, y_2]\!]$ as defined in Theorem 4.2.7 and an apolar algebra $A = S^*/(x_1^3 - x_2^2)^\perp$. Since Δ_0 is the Hilbert function of $S^*/(x_1^3)^\perp$ is it equal to $(1, 1, 1, 1)$. Moreover as $h_A(1) = 2$ and $\Delta_1 = (0, a, 0)$ we see that $1 + a = 2$, so that $a = 1$ and the full decomposition is*

$$\begin{aligned} \Delta_0 = & \quad (1, \quad 1, \quad 1, \quad 1) \\ \Delta_1 = & \quad (0, \quad 1, \quad 0) \end{aligned}$$

yielding $\mathrm{rk}_k A = 5$ and $h_A = (1, 2, 1, 1)$.

On the other hand let us take $B = S^*/(x_1^3 - x_1 \cdot x_2)^\perp$, $f := x_1^3 - x_1 x_2$. Then one calculates that $(S^*f)^1 = k \oplus kx_1$, $x_1 x_2$ is "shadowed by x_1^3" and so the Hilbert function decomposition is

$$\begin{aligned} \Delta_0 = & \quad (1, \quad 1, \quad 1, \quad 1) \\ \Delta_1 = & \quad (0, \quad 0, \quad 0). \end{aligned}$$

We will now see that this difficulty can be overrun by standardising f. Unfortunately this method works only for problems occurring in low degrees, in general the problem of removing "exotic summands" is difficult, see [BMR12].

Theorem 4.2.7 presents any finite local Gorenstein k-algebra as $S^*/f^\perp$, but the choice of f is not unique. However we have particularly good choices of f, each of which is named the standard form of f.

Definition 4.5.8. *For a finite Gorenstein k-algebra $A \simeq S^*/I$ of socle degree j let $\Delta_\bullet$ be the decomposition of the Hilbert function of A and $e(i) := \sum_{t=0}^{i} \Delta_t(1)$. If $f \in S$ is a dual socle generator of S^*/I then f is in the standard form iff*

$$f = f_0 + f_1 + f_2 + f_3 + \cdots + f_j \text{ for } f_i \in S^i \cap k[x_1, \ldots, x_{e(j-i)}]$$

and either the socle degree of A is at most one or $f_0 = f_1 = 0$.

The following theorem proves that a standard form exists for any $f \in S$.

Theorem 4.5.9 (Standard form of a dual socle generator)**.** *Let $(A, \mathfrak{m}, k)$ be a local finite Gorenstein k-algebra of socle degree j with $h_A(1) = n$. Set $S = k[x_1, \cdots, x_n]$ and $S^* = k[\![y_1, \cdots, y_n]\!]$. Then there exists $f = f_0 + \cdots + f_j \in S$, such that*

1. $S^*/f^\perp \simeq A$,

2. $f_i \in S^i \cap k[x_1, \ldots, x_{e(j-i)}]$, where $e(i) = \sum_{t=0}^{i} \Delta_t(1)$.

If $j \geqslant 2$ then one can assume $f_0 = f_1 = 0$.

Proof. Choose first any $f' \in S$ such that $S^*/(f')^\perp \simeq A$; this is possible by Theorem 4.2.7. Write $f' = f'_0 + \cdots + f'_j$, where $f'_i \in S^i$ for $i = 0, \ldots j$. Consider the sequence of ideals

$$\mathfrak{m}_{S^*} \supseteq \mathrm{ann}_{S^*}\left(S^* f'_j\right) \supseteq \mathrm{ann}_{S^*}\left(S^* f'_j \oplus S^* f'_{j-1}\right) \supseteq \ldots \mathrm{ann}_{S^*}\left(S^* f'_j \oplus \cdots \oplus S^* f'_0\right) \supseteq 0,$$

then

$$\mathrm{rk}_k \frac{\mathrm{ann}_{S^*}\left(S^* f'_j \oplus \cdots \oplus S^* f'_{j-a}\right) + \mathfrak{m}^2_{S^*}}{\mathfrak{m}^2_{S^*}} = \mathrm{rk}_k \frac{(S^* f)^1_{j-a-1} + \mathfrak{m}^2_{S^*} f}{\mathfrak{m}^2_{S^*} f} =$$

$$\mathrm{rk}_k \frac{\mathfrak{m} \cap (\mathfrak{m}^{j-a-1})^\perp + \mathfrak{m}^2}{\mathfrak{m}^2} = \mathrm{rk}_k \frac{\mathfrak{m}}{\mathfrak{m}^2} - e(a),$$

where the last equality follows from Theorem 4.4.3 Point 4. Now we choose $z_1, \ldots, z_n \in \mathfrak{m}_{S^*}$ such that

$$\mathrm{ann}_{S^*}\left(S^* f'_j \oplus \cdots \oplus S^* f'_{j-a}\right) + \mathfrak{m}^2_{S^*} = k z_n + \cdots + k z_{e(a)+1} + \mathfrak{m}^2_{S^*}$$

for all $0 \leqslant a \leqslant j$ and fix an automorphism of S^* sending y_i to z_i (choose additional z_i if necessary). This automorphism sends $f'^\perp$ to some ideal $I \lhd S^*$ such that the quotient by I is Gorenstein of socle degree j. This ideal is, by Theorem 4.2.7, the annihilator of some $f = f_0 + \cdots + f_j$ where $f_i \in S^i$ and it is easy to check that $f_i \in k[x_1, \ldots, x_{e(j-i)}]$, where $x_1, \ldots, x_n$ is the basis dual to $z_1, \ldots, z_n$.

Now suppose $j \geqslant 2$. Since $h_A(1) = n$, by Proposition 4.5.1 we see that any polynomial of degree at most one is a derivative of f. In particular $f_0 + f_1 = \partial \lrcorner f$ for some $\partial \in S^*$. Since $j \geqslant 2$ we have $\partial \in \mathfrak{m}_{S^*}$, then $f - (f_0 + f_1) = (1 - \partial) \lrcorner f$ generates the same S^* submodule of S as f and $f^\perp = (f - f_0 - f_1)^\perp$ follows. $\qquad \square$

4.6. Families of dual generators and morphisms to Hilbert scheme

In this subsection we obtain tools for the reasoning "since the set of dual generators is irreducible, the corresponding apolar algebras form an irreducible subset of the Hilbert scheme". But the idea is deeper — we obtain morphisms from affine varieties to the Hilbert scheme, which give information about its geometry.

Proposition 4.6.1. *Fix positive integers j, r and a point $p \in \mathbb{P}^n$ with an affine neighbourhood U. View $S^{\leqslant j}$ as an affine space and let $V \subseteq S^{\leqslant j}$ be a Zariski-constructible subset such that for every closed point $f \in V$ the apolar algebra of f has rank r. Then V induces a morphism*

$$\varphi : V \to \mathrm{HilbGor}^r_{\mathbb{P}^n}$$

such that for every closed point $f \in V$ the image $\varphi(f) \in \mathrm{HilbGor}^r_{\mathbb{P}^n}$ corresponds to a closed subscheme $W_f \subseteq \mathbb{P}^n$ supported at p and with $\Gamma(U, W_f)$ isomorphic to the apolar algebra of f.

Proof. We will construct a closed subscheme $Z \subseteq \mathbb{P}^n \times V$ flat over V and with suitable fibers, then the required morphism will follow from the universal property of the Hilbert scheme. We can view $R := \mathrm{Spec}\left(S^*/\mathfrak{m}_{S^*}^{j+1}\right)$ as a closed subscheme of $\mathbb{P}^n$ supported at p and we will in fact construct $Z \subseteq R \times V$.

We first treat $S^*/\mathfrak{m}_{S^*}^{j+1}$ formally. Let $W \to S^*/\mathfrak{m}_{S^*}^{j+1}$ be an isomorphism of k-vector spaces. Consider the vector bundle $\mathcal{W} = \mathrm{Spec}\left(\mathrm{Sym}\, W^*\right) \times V$ over V. Define a subvariety $\mathcal{A}nn \subseteq \mathcal{W}$ by

$$\mathcal{A}nn = \{(\partial, f) \in \mathcal{W} \mid \partial \lrcorner f = 0\},$$

then the fibers of $\mathcal{A}nn$ over closed points of V are affine spaces of constant dimension, so $\mathcal{A}nn$ is a subbundle of $\mathcal{W}$. Let $\mathcal{Q}$ be the quotient bundle, then dualizing we have an inclusion of vector bundles $\mathcal{Q}^* \subseteq \mathcal{W}^*$.

The bundle $\mathcal{W}^*$ is canonically isomorphic to $\mathrm{Spec}\left(\mathrm{Sym}\, W\right) \times V$. The k-linear isomorphism $W \to S^*/\mathfrak{m}_{S^*}^{j+1}$ induces morphisms $R \to \mathrm{Spec}\left(\mathrm{Sym}\, W\right)$ and $R \times V \to \mathcal{W}^*$. We claim that the pullback

$$Z := (R \times V) \times_{\mathcal{W}^*} \mathcal{Q}^* \subseteq R \times V$$

has the properties required. Choose a closed point $f \in V$. The fiber of $\mathcal{Q}$ over f is the affine space $\mathcal{W}_f/\mathcal{A}nn_f$ and the fiber of $\mathcal{Q}^*$ over f is the affine space $\mathrm{Hom}_k\left(\mathcal{W}_f/\mathcal{A}nn_f, k\right)$, which is defined by the ideal of $\mathrm{Sym}\, W$ generated by the linear subspace $\mathrm{ann}_R(f) \subseteq W$. It follows that the fiber of Z over f is isomorphic to $\mathrm{Spec}\left(R/\mathrm{ann}_R(f)\right)$, the affine scheme of the apolar algebra of f. The scheme Z is finite over V and its fibers over closed points of V have rank r. From [Har77, II.5 Ex 5.8bc] it follows that $\mathcal{O}_Z$ is a locally free V module, thus in particular Z is flat over V. $\square$

Proposition 4.6.2. *Let j be a positive integer and $V \subseteq S^{\leqslant j}$ be a Zariski-constructible subset. Then the set of $f \in V$ such that the apolar algebra of f has maximal rank over k is open in V.*

Proof. This is a direct consequence of semi-continuity of dimension of fibers and the proof of Proposition 4.6.1. $\square$

5. Applications

In this section for simplicity we assume that $k = \mathbb{C}$, though many of the results may be proved with much weaker assumptions. Our objective is to present applications of the above theory, in particular we will be interested in finding deformations and smoothings of zero-dimensional Gorenstein subschemes of $\mathbb{P}^n = \mathbb{P}^n_k$. Supported by Proposition 3.2.5 we will investigate $\mathrm{HilbGor}^r_{\mathbb{P}^n}$. The picture here was mainly drawn by Iarrobino [Iar94], who proved that "the majority" of algebras of large enough rank is not smoothable. In particular he showed an example of non-smoothable Gorenstein algebra of rank 14 by computing that the tangent space rank of the corresponding point of the Hilbert scheme of $\mathbb{P}^6$ is less than $6 \cdot 14$, which is the dimension of the smoothable component. On the other hand Casnati and Notari (see [CN11]) proved that algebras of rank at most 10 are smoothable. It would be interesting to know what is the minimal rank of a non-smoothable Gorenstein k-algebra, we hope to answer this question in a joint paper with Gf. Casnati and R. Notari. In this section we content ourselves with analyzing three examples.

To prove that a subset $V \subseteq \mathrm{HilbGor}^r_{\mathbb{P}^n}$ lies in $\mathrm{HilbGor}^r_{\mathbb{P}^n}{}^\circ := \mathrm{HilbGor}^r_{\mathbb{P}^n} \cap \mathrm{Hilb}^r_{\mathbb{P}^n}{}^\circ$ we will apply ternary approaches:

1. Prove directly that for any $[R] \in V$ the scheme $R \subseteq \mathbb{P}^n$ is smoothable;

2. Prove that V is irreducible and find a smooth point of the Hilbert scheme contained in $V \cap \mathrm{HilbGor}^r_{\mathbb{P}^n}{}^\circ$;

3. Prove that V is irreducible and that a general point of V lies in $\mathrm{HilbGor}^r_{\mathbb{P}^n}{}^\circ$.

We will use the first approach to prove smoothability of Gorenstein algebras with Hilbert function $(1, 5, 4, 1)$, the second approach to prove smoothability of Gorenstein algebras having Hilbert function $(1, 4, 4, 3, 1)$ and the third approach to prove smoothability of *graded* Gorenstein algebras with Hilbert function $(1, 3, 3, 3, 3, 1)$.

The most important facts that we will use without further reference are Proposition 3.2.5, which allows us not to care too much about the ambient space and the Definition 3.1.4 of the smoothable component, which implies that to prove smoothability of X it is sufficient to show a deformation with a fiber X over a closed point and a general fiber *smoothable*, not necessarily smooth.

Before proving our main results we provide the technical background by further analyzing the dual socle generators and apolar ideals in the general situation covering the above cases.

5.1. Enhancements of Iarrobino symmetric decompositions

The following proposition further standardises the dual generator. It is used in proof of Theorem 5.2.2.

Proposition 5.1.1. *Let $f \in S$ be a polynomial of degree j such that the Hilbert function decomposition from Theorem 4.4.3 has $\Delta_{j-2} = (0, q, 0)$. Then the apolar algebra of f is isomorphic to the apolar algebra of $g \in S$, where $g = g_j + g_{j-1} + \cdots + g_2$ is the standard form from Theorem 4.5.9 and furthermore g_2 is a sum of q squares of variables not appearing in $g_{\geqslant 3}$ and a quadric in variables appearing in $g_{\geqslant 3}$.*

Proof. Applying Theorem 4.5.9 we obtain g in the required form except from, perhaps, the assumptions on g_2. We will prove the theorem when $j \geqslant 3$, the case $j = 2$ is easy and we leave it to the reader. Let $f := e(j - 3) = \sum_{t=0}^{j-3} \Delta_t (1)$, $e := f + q$, then $g_{\geqslant 3} \in k[x_1, \ldots, x_f]$ and $g_2 \in k[x_1, \ldots, x_e]$. We can diagonalize g_2 with respect to $x_{f+1}, \ldots, x_e$ obtaining

$$g_2 = \sum_{i=f+1}^{e} \lambda_i \cdot x_i^2 + Q$$

where $Q \in (x_1, \ldots, x_f)$ and $\lambda_i \in \{0, 1\}$. Note that after this operation g is still in the standard form. If all $\lambda_{\bullet}$ are equal to 1, then by another linear change of coordinates $x_{f+1}, \ldots, x_e$ we obtain $g_2 = \sum_{i=f+1}^{e} x_i'^2 + Q'$ where $Q' \in k[x_1, \ldots, x_f]$, thus the claim follows. If for some i we have $\lambda_i = 0$ then $y_i^* \lrcorner g \in k[x_1, \ldots, x_f]$, thus from a k-rank count it follows that $\mathrm{rk}_k(S^* f)_2 g < n + 1$ but this contradicts the fact that $\mathrm{rk}_k(S^* f)_2 g = h(0) + h(1) = n + 1$, see Proposition 4.5.1. $\square$

5.2. Application of flatness criterion from Theorem 2.3.9

There exist polynomials $f \in S$ such that there is an easily described flat family with general member reducible and a special member isomorphic to the apolar algebra of f. We present them below.

Proposition 5.2.1. *Let $S = k[x_1, \cdots, x_n]$ and $T = S[X]$. Let $f \in S$ and $\partial \in S^*$ be such that $\partial^2 \lrcorner f = 0$. Take a natural number $m \geqslant 2$ and set $g := f + X^m \cdot \partial \lrcorner f$, then*

$$\mathrm{ann}_{T^*}(g) = T^* \cdot \mathrm{ann}_{S^*}(f) + T^* \cdot \alpha \cdot \mathrm{ann}_{S^*}(\partial \lrcorner f) + (\alpha^m - m! \cdot \partial),$$

where $\alpha \in T^*$ is the dual to X.

Proof. Clearly the element $\alpha^m - m! \cdot \partial$ annihilates g, and reducing modulo it, we may investigate only elements of the form $\sigma_0 + \sigma_1 \cdot \alpha + \cdots + \sigma_{m-1}\alpha^{m-1}$, where $\sigma_i \in S^*$. The action of such element on g is given by

$$\left(\sigma_0 + \sigma_1 \cdot \alpha + \cdots + \sigma_{m-1}\alpha^{m-1}\right) \lrcorner g = $$
$$\sigma_0 \lrcorner f + X^m(\sigma_0 \partial \lrcorner f) + m \cdot X^{m-1}(\sigma_1 \partial \lrcorner f) + \cdots + m! \cdot X(\sigma_{m-1}\partial \lrcorner f),$$

thus the element belongs to $\mathrm{ann}_{T^*}(g)$ if and only if $\sigma_0 \lrcorner f = 0, \sigma_1 \partial \lrcorner f = 0, \ldots, \sigma_{m-1}\partial \lrcorner f = 0$, and the claim follows. $\square$

In the following theorem we use Theorem 2.3.9 in an essential way – without any knowledge of the fibers, we prove that a certain family is flat. This theorem will act as the main ingredient of almost all proofs of smoothability, allowing one to reduce the question to the smoothability of algebras of lower k-rank.

Theorem 5.2.2. *Let* $T^* = k[\![y_1, \ldots, y_n, \alpha]\!]$ *and* $A = T^*/I$ *be a finite* k-*algebra defined by the ideal*

$$I = (\alpha^o - q) + J,$$

where $o \geqslant 2, q \in k[y_1, \ldots, y_n]$ *and* $J \lhd T^*$ *is generated by elements of the ideal* $(y_1, \ldots, y_n) \lhd k[y_1, \ldots, y_n, \alpha]$ *homogeneous with respect to the grading by* α. *Suppose that for some natural* c *such that* $0 < c < o$ *we have*

$$\mathrm{ann}_{T^*/J}(\alpha^o) \subseteq \mathrm{ann}_{T^*/J}(\alpha^c) \ \text{and} \ \mathrm{ann}_{T^*/J}(\alpha^o) \subseteq \mathrm{ann}_{T^*/J}(q),$$

then $\mathrm{Spec}\, A$ *is a degeneration of reducible schemes. More precisely the morphism*

$$\varphi : \mathrm{Spec}\, \frac{k[y_1, \ldots, y_n, \alpha, t]}{(\alpha^o - t \cdot \alpha^c - q) + (J)} \rightarrow \mathrm{Spec}\, k[t]$$

is flat, with a general fiber reducible and the fiber over t *isomorphic to* $\mathrm{Spec}\, A$.

Proof. Denote by $I_t = (\alpha^o - t \cdot \alpha^c - q) + (J) \lhd k[y_1, \ldots, y_n, \alpha, t]$. For an invertible $\lambda \in k$, the fiber of φ over $(t - \lambda)$ is supported at the origin and at $(0, 0, \ldots, 0, \sqrt[o-c]{\lambda})$ thus it is a reducible scheme. It remains to prove that φ is flat. We will use Theorem 2.3.9. Define a filtration on $k[y_1, \ldots, y_n, \alpha, t]$ by gradation with respect to α:

$$k[y_1, \ldots, y_n, \alpha, t]_m = \{f \in k[y_1, \ldots, y_n, \alpha, t] \mid \deg_\alpha f \leqslant m\}.$$

Since J is homogeneous with respect to the gradation by α we can calculate $\mathrm{gr}\, I_t$ as the preimage of $\mathrm{gr}\,(\alpha^o - t \cdot \alpha^c - q) \subseteq \mathrm{gr}\, k[y_1, \ldots, y_n, \alpha, t]/(J)$. For clarity let us denote $R := k[y_1, \ldots, y_n, \alpha, t]/(J)$. Now, since (J) is generated by elements of $k[y_1, \ldots, y_n, \alpha]$, we have

$$\mathrm{ann}_R(\alpha^o) = k[t] \cdot \left(\mathrm{ann}_{k[y_1, \ldots, y_n, \alpha]/J}(\alpha^o)\right), \ \text{so} \ \mathrm{ann}_R(\alpha^o) \subseteq \mathrm{ann}_R(\alpha^o - t \cdot \alpha^c - q)$$

and we use Proposition 2.3.7 to deduce that $\mathrm{gr}\,(\alpha^o - t \cdot \alpha^c - q) \subseteq \mathrm{gr}\, R$ is generated by α^o. Now the claim follows from Proposition 2.3.6 and Theorem 2.3.9. $\qquad\square$

Corollary 5.2.3. *Let* $S = k[x_1, \cdots, x_n]$ *and* $T = S[X]$. *Let* $f \in S$ *and* $\partial \in S^*$ *be such that* $\partial^2 \lrcorner f = 0$. *Set* $g = f + X^m \cdot \partial \lrcorner f$, *where* $m \geqslant 2$ *is a natural number. The apolar algebra of* g *is a fiber of a deformation over* $\mathrm{Spec}\, k[t]$, *whose general fiber is isomorphic to a disjoint sum of the apolar algebra of* f *and* $m - 1$ *copies of the apolar algebra of* $\partial \lrcorner f$.

Proof. Let, $S^* = k[\![y_1, \cdots, y_n]\!]$ and $T^* = k[\![y_1, \ldots, y_n, \alpha]\!]$. Clearly ∂ may be taken to be a polynomial in $y_1, \ldots, y_n$. From Proposition 5.2.1 it follows that

$$\mathrm{ann}_{T^*}(g) = T^* \cdot \mathrm{ann}_{S^*}(f) + T^* \cdot \alpha \cdot \mathrm{ann}_{S^*}(\partial \lrcorner f) + (\alpha^m - m! \cdot \partial),$$

Let $J = T^* \cdot \mathrm{ann}_{S^*}(f) + T^* \cdot \alpha \cdot \mathrm{ann}_{S^*}(\partial \lrcorner f)$, then

$$\mathrm{ann}_{T^*/J}(\alpha^m) = \mathrm{ann}_{T^*/J}(\alpha) = \mathrm{ann}_{T^*/J}(\partial) = (\mathrm{ann}_{S^*}(\partial \lrcorner f))/J.$$

It follows that $m! \cdot \partial, m, 1, J$ satisfy the assumptions of Theorem 5.2.2 for q, o, c, J respectively. The required deformation $X \rightarrow \mathrm{Spec}\, k[t]$ comes from this theorem. We will now check that the fiber of X over $(t - \lambda)$, where $0 \neq \lambda \in k$, is isomorphic to a disjoint sum of the apolar algebra to f and $m - 1$ copies of the apolar algebra of $\partial \lrcorner f$. This fiber is isomorphic to

$$\mathrm{Spec}\, \frac{k[y_1, \ldots, y_n, \alpha]}{(\alpha^m - \lambda \cdot \alpha - m! \cdot \partial) + J},$$

33

and has support equal to the union of $(0,\ldots,0)$ and $(0,\ldots,0,\omega)$, where ω runs through the $(m-1)$'st roots of λ. Note that since $\alpha \cdot \partial \in J$ the element $\alpha^{m+1} - \lambda \cdot \alpha^2 = \alpha \cdot ((\alpha^m - \lambda \cdot \alpha - m! \cdot \partial)) + m! \cdot \alpha \cdot \partial$ is in the ideal $I(\lambda)$ defining the fiber.

We will now look near $(0,\ldots,0,0)$, i.e. localise the fiber at the ideal $\mathfrak{m} = (y_1,\ldots,y_n,\alpha)$. In this localisation $\alpha^{m-1} - \lambda$ is invertible, thus $\alpha^2 = (\alpha^{m+1} - \lambda \cdot \alpha^2) \cdot (\alpha^{m-1} - \lambda)^{-1}$ belongs to the localised ideal $I(\lambda)_{\mathfrak{m}}$. Consequently $\lambda\alpha - m! \cdot \partial$ belongs to this ideal and $I(\lambda)_{\mathfrak{m}} = ((\lambda\alpha - m! \cdot \partial) + J)_{\mathfrak{m}}$. This proves that

$$\left(\frac{k[y_1,\ldots,y_n,\alpha]}{I(\lambda)}\right)_{\mathfrak{m}} \simeq \left(\frac{k[y_1,\ldots,y_n,\alpha]}{(\lambda\alpha - m! \cdot \partial) + J}\right)_{\mathfrak{m}} \simeq \frac{k[\![y_1,\cdots,y_n]\!]}{\operatorname{ann}_{S^*}(f)}.$$

Next we look near $\mathfrak{m} = (0,\ldots,\omega)$. Here α^2 is invertible, thus $\alpha^{m-1} - \lambda$ belongs to $I(\lambda)_{\mathfrak{m}}$, then $I(\lambda)_{\mathfrak{m}} = (\alpha^{m-1} - \lambda, \operatorname{ann}_{S^*}(\partial\lrcorner f))$ and the localised algebra is isomorphic to the apolar algebra of $\partial\lrcorner f$. $\qquad\square$

5.3. Smoothability of algebras with Hilbert function $(1,5,4,1)$.

For simplicity and brevity of presentation we will use the fact that algebras of length at most 10 are smoothable, see [CN11]. In fact it is known that algebras of length 11 are smoothable, so the sceptic reader should replace $(1,5,4,1)$ with e.g. $(1,7,4,1)$. The crucial part of the proof is Proposition 5.1.1, allowing us to compute the apolar ideal and use Corollary 5.2.3.

The proof is straightforward once we use the following inductive lemma:

Lemma 5.3.1. *Let $A \simeq k[\![y_1,\cdots,y_n]\!]/I$ be a Gorenstein algebra of socle degree $j \geqslant 3$ with Hilbert function decomposition containing the term $\Delta_{A,j-2} = (0,e,0)$, where $e > 0$.*

The scheme $\operatorname{Spec} A$ is a flat degeneration of schemes isomorphic to $\operatorname{Spec} k \sqcup \operatorname{Spec} B$, where B is a local Gorenstein k-algebra of socle degree j, which has the same Hilbert function decomposition as A except for the term $\Delta_{B,j-2} = (0,e-1,0)$.

Proof. Let $g \in k[x_1,\ldots,x_n]$ be the dual generator from Proposition 5.1.1, then we can write $g = f + x_1^2$, where $f \in k[x_2,\ldots,x_n]$. Clearly there exists $\partial \in k[\![y_1,\cdots,y_n]\!]$ such that $\partial\lrcorner f = 1$, then $\partial^2\lrcorner f = 0$ and we may apply Corollary 5.2.3 to $g = f + x_1^2 \cdot \partial\lrcorner f$. It follows that the apolar algebra of g is a degeneration of union of the apolar algebra of $\partial\lrcorner f = 1$ – which is isomorphic to $\operatorname{Spec} k$ – and the apolar algebra B of f. The assumptions on B except for the Hilbert function decomposition are straightforward. For the Hilbert function decomposition, see Proposition 4.5.1. $\qquad\square$

5.4. Smoothability of algebras with Hilbert function $(1,4,4,3,1)$.

In this subsection we have two main points: proving irreducibility, which is easy once we apply secant varieties, and proving the existence of a smooth, smoothable point of the Hilbert scheme.

Proposition 5.4.1. *The projective set of quartics in three variables, whose apolar algebras have the Hilbert function $(1,n,m,n,1)$ with $m \leqslant 3$, is the third secant variety to the fourth Veronese embedding of $\mathbb{P}^2$. In particular it is irreducible.*

Proof. See e.g. [LO11], which refers to other sources. $\qquad\square$

Proposition 5.4.2. *The Gorenstein algebras with Hilbert function $(1, 4, 4, 3, 1)$ are parametrised, up to isomorphism, by an irreducible Zariski-constructible subset $V \subseteq S^{\leqslant 3} = k[x_1, \cdots, x_n]^{\leqslant 3}$, where we view $S^{\leqslant 3}$ as an affine space.*

Proof. The only possible decomposition of the Hilbert function is $\Delta_0 = (1, 3, 3, 3, 1); \Delta_1 = (0, 1, 1, 0)$. Let us fix a dual socle generator in the standard form $f = f_4 + f_3 + f_2$. Corollary 4.5.6 implies that the apolar algebra of f_4 has Hilbert function Δ_0, thus Δ_0 depends only on f_4. By Proposition 5.4.1 forms f_4 whose apolar algebras have Hilbert function $(1, 3, 3, 3, 1)$ constitute a Zariski locally closed irreducible subset $Z \subseteq S^4$.

Once we know that $h(1) = 4$ and that the first row of the decomposition is $(1, 3, 3, 3, 1)$, then $h = (1, 4, 4, 3, 1)$ is maximal among possible Hilbert functions. Proposition 4.6.2 applied to $Z \times S^{\leqslant 3}$ proves that the set of dual socle generators of algebras of the form $(1, 4, 4, 3, 1)$ is open in $Z \times S^{\leqslant 3}$, thus irreducible. $\qquad\square$

Proposition 5.4.3. *Let $n = 4$. The subset $V \subseteq S^{\leqslant 3}$ defined in Proposition 5.4.2 induces a morphism $V \to \mathrm{HilbGor}^r_{\mathbb{P}^n}$, whose image is irreducible. The image of the point p corresponding to the apolar algebra of*

$$x_1^4 + x_2^4 + x_3^4 + x_4^2 \cdot (x_1 + x_2)$$

is a smooth point of $\mathrm{HilbGor}^r_{\mathbb{P}^n}$ lying in $\mathrm{HilbGor}^r_{\mathbb{P}^n}{}^\circ$.

Proof. The first claim follows from Proposition 4.6.1 and the fact that the image of an irreducible set is irreducible. The Hilbert function computation is most conveniently conducted using Proposition 4.5.1, we leave it to the reader.

Next, one should check that $x_1^4 + x_2^4 + x_3^4 + x_4^2 \cdot (x_1 + x_2)$ satisfies assumptions of Corollary 5.2.3 with respect to the variable $X := x_4$. The resulting flat family presents the apolar algebra of $x_1^4 + x_2^4 + x_3^4 + x_4^2 \cdot (x_1 + x_2)$ as a flat limit of schemes isomorphic to W, where W is a disjoint union of a double point and an apolar algebra of $x_1^4 + x_2^4 + x_3^4$. Then one may continue by applying Corollary 5.2.3 trice or using the fact that every finite Gorenstein algebra A with $h_A(1) \leqslant 3$ is smoothable, see [CN11] and references therein. To calculate that $p \in \mathrm{HilbGor}^r_{\mathbb{P}^n}{}^\circ$ is smooth it suffices to prove that the tangent space at p has k-rank $4 \cdot 13 = 52$. The most straightforward way to do this is to calculate the apolar ideal of $x_1^4 + x_2^4 + x_3^4 + x_4^2 \cdot (x_1 + x_2)$ together with its square and use Proposition 4.1.3. Since this is pure computation (but see Remark 5.4.4 below), we leave it to the reader. $\qquad\square$

Remark 5.4.4. *If $k[t] \to S^*[t]/I_t$ is the flat family from Theorem 5.2.2, then one may analyze flatness of $k[t] \to S^*[t]/I_t^2$ using the same theorem, obtaining a criterion for flatness in terms of I_t. Unfortunately in general its check seems as difficult as making a direct computation.*

5.5. Smoothability of graded algebras with Hilbert function $(1, 3, 3, 3, 3, 1)$

As mentioned in the proof of Proposition 5.4.3, every finite Gorenstein algebra A satisfying $h_A(1) \leqslant 3$ is smoothable; the proof depends on a structure theorem on the resolutions of such A. As an example of a more general method, we present a different approach to this smoothability, which shows a tight connection between finding equations of secant varieties and proving smoothability of certain Gorenstein algebras.

Since we are analyzing graded algebras, we may take homogeneous forms of degree five as the dual socle generators. The following proposition shows that the set of such forms is irreducible, in fact describing a general element.

Proposition 5.5.1. *The projective set of quartics in three variables, whose apolar algebras have the Hilbert function $(1, n, m, m, n, 1)$ with $m \leqslant 3$, is the third secant variety to the fifth Veronese embedding of $\mathbb{P}^2$. In particular it is irreducible.*

Proof. See [LO11, Thm 3.2.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A general form of this set is equal, up to a linear change of coordinates, to the form $x_1^5 + x_2^5 + x_3^5 \in k[x_1, x_2, x_3]$. The apolar algebra of such form is smoothable by Corollary 5.2.3.

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