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Oświadczenie kierującego pracą

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Abstract

We generalize methods used to compute various kinds of rank (apolarity and the catalecticant bound) to the case of a toric variety X embedded into projective space using a very ample line bundle $\mathcal{L}$. We use this to compute cactus rank, rank and border rank of monomials in $H^0(X, \mathcal{L})^*$ when X is the Hirzebruch surface $\mathbb{F}_1$ or a fake projective plane (a quotient of $\mathbb{P}^2$ by an action of $\mathbb{Z}/3$).

Słowa kluczowe

secant variety, Waring rank, cactus rank, border rank, toric variety, apolarity, catalecticant

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.1 Matematyka

Klasyfikacja tematyczna

14-xx Algebraic geometry
14Mxx Special varieties
14M25 Toric varieties

Tytuł pracy w języku angielskim

Multigraded apolarity

Contents

Introduction

Let $\mathbb{C}[y_0,\dots,y_n]$ be the polynomial ring in $n+1$ variables over the field of complex numbers $\mathbb{C}$. Let $\mathbb{C}[y_0,\dots,y_n]_d$ denote its d-th graded piece. Suppose $F \in \mathbb{C}[y_0,\dots,y_n]_d$. One can define the Waring rank (also known as the symmetric rank) of F by

$$\mathrm{r}(F) = \min\{r \in \mathbb{Z}_{\geq 0} : F = f_1^d + \dots f_r^d \text{ for some } f_i \in \mathbb{C}[y_0,\dots,y_n]_1\}.$$

We may also think of this in a geometrical way. Let V be an $(n+1)$-dimensional vector space. Let $\mathbb{P}V$ denote the naive projectivization of V, i.e. the space of lines in V. Let $\operatorname{Sym}^d V$ be the d-th graded piece of the symmetric algebra. This is canonically identified with the subspace of symmetric tensors in $V^{\otimes d}$, because we are over a field of characteristic zero. We define the Veronese embedding

$$v_d : \mathbb{P}V \to \mathbb{P}\operatorname{Sym}^d V$$

by $v_d(u) = u^d$. Now, choosing a basis $y_0,\dots y_n$ of V gives rise to a graded isomorphism $\operatorname{Sym}^\bullet V \cong \mathbb{C}[y_0,\dots,y_n]$. In this language, the Veronese embedding is raising linear forms to d-th power. Now we can reformulate the definition above. Define

$$\sigma_r^0(\mathbb{P}V) = \{[F] \in \mathbb{P}\operatorname{Sym}^d V : [F] \in \langle v_d(p_1),\dots,v_d(p_r)\rangle \text{ for some } \ p_i \in \mathbb{P}V\}.$$

Here $[F]$ denotes the class in the projective space and $\langle\rangle$ denotes the (projective) linear span. Then

$$\mathrm{r}(F) = \min\{r \in \mathbb{Z}_{\geq 0} : [F] \in \sigma_r^0(\mathbb{P}V)\}.$$

We also define

$$\sigma_r(\mathbb{P}V) = \overline{\sigma_r^0(\mathbb{P}V)}.$$

Here the bar denotes the Zariski closure of a set in $\mathbb{P}\operatorname{Sym}^d V$. This variety is called the r-th secant variety of the d-th Veronese embedding. Similarly, we define

$$\underline{\mathrm{r}}(F) = \min\{r \in \mathbb{Z}_{\geq 0} : F \in \sigma_r(\mathbb{P}V)\},$$

which is called the (symmetric) border rank of F.

We may use the so-called apolarity action to help us to calculate rank. Let $\mathbb{C}[x_0,\dots,x_n]$ be the coordinate ring of $\mathbb{P}V$ (so $x_0,\dots,x_n$ is a basis of V^* dual to $y_0,\dots,y_n$). We will think of x_i as differential operators acting on $\mathbb{C}[y_0,\dots,y_n]$. More precisely, let

$$x_0^{a_0}\cdot\dots\cdot x_n^{a_n} \circ y_0^{b_0}\cdot\dots\cdot y_n^{b_n} = \begin{cases} \frac{b_0!\cdot\ldots\cdot b_n!}{(b_0-a_0)!\cdot\ldots\cdot(b_n-a_n)!}\cdot y_0^{b_0-a_0}\cdot\dots\cdot y_n^{b_n-a_n} & \text{if } b_i \geq a_i \text{ for all i,} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We extend it by $\mathbb{C}$-linearity to the rings $\mathbb{C}[x_0,\dots,x_n]$ and $\mathbb{C}[y_0,\dots,y_n]$. This makes $\mathbb{C}[y_0,\dots,y_n]$ into a $\mathbb{C}[x_0,\dots,x_n]$-module.

For $F \in \mathbb{C}[y_0,\dots,y_n]_d$, let $F^\perp$ denote its annihilator, which is a homogeneous ideal of $\mathbb{C}[x_0,\dots,x_n]$.

This can help us to calculate rank by means of the following fact.

Fact 0.1 (Classical Apolarity Lemma)**.** *Let $F \in \mathbb{C}[y_0, \dots, y_n]_d$. For any closed subset $Y \subseteq \mathbb{P}(\mathbb{C}[y_0, \dots y_n]_1)$ let $I(Y)$ denote its radical ideal in $\mathbb{C}[x_0, \dots, x_n]$. Then for any set of one-forms $Z = \{l_1, \dots, l_r\} \subseteq \mathbb{C}[y_0, \dots, y_0]_1$ we have*

$$F \in \langle l_1, \dots, l_r \rangle \iff I(Z) \subseteq F^{\perp}.$$

To see how this works, let us look at the following

Example 0.2. Look at the d-th Veronese embedding $\mathbb{C}[y_0, y_1]_1 \hookrightarrow \mathbb{C}[y_0, y_1]_d$ (where $d \geq 2$), and let $F = y_0^{d-1} y_1$. Here $F^{\perp} = (x_0^d, x_1^2)$. Then $I = (x_0^d - x_1^d) \subset F^{\perp}$ is radical ideal of d points, so $\mathrm{r}(F) \leq d$. For a proof that $\mathrm{r}(F) = d$, see [CCG12] (where the authors determine the rank of any monomial).

This way of thinking about rank can be generalized. Let W be any finite-dimensional vector space, and X any subvariety of $\mathbb{P}W$. We define

$$\sigma_r^0(X) = \{F \in \mathbb{P}W : F \in \langle p_1, \dots, p_r \rangle\}$$

and $\sigma_r(X) = \overline{\sigma_r^0(X)}$. Analogously, we define the X-rank and the X-border rank of F:

$$\mathrm{r}_X(F) = \min\{r \in \mathbb{Z}_{\geq 0} : F \in \sigma_r^0(X)\}$$

and

$$\underline{\mathrm{r}}_X(F) = \min\{r \in \mathbb{Z}_{\geq 0} : F \in \sigma_r(X)\}.$$

Often, if X is fixed, we omit the prefix and call them rank and border rank, respectively. This generalization has important special cases. For instance, if $X = \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k \hookrightarrow \mathbb{P}(V_1 \otimes \cdots \otimes V_k)$ is the Segre embedding (given by $[v_1, \dots, v_k] \mapsto [v_1 \otimes \cdots \otimes v_k]$), then the X-rank becomes the tensor rank.

For a zero-dimensional scheme R (of finite type over $\mathbb{C}$), let $\mathrm{lh}\, R$ denote its length, i.e. $\dim_{\mathbb{C}} H^0(R, \mathcal{O}_R)$. This is equal to the degree of R in any embedding into projective space.

Let us also define the cactus X-rank:

$$\mathrm{cr}_X(F) = \min\{\mathrm{lh}\, R : R \hookrightarrow X, \dim R = 0, F \in \langle R \rangle\}.$$

We have the following inequalities:

$$\mathrm{cr}(F) \leq \mathrm{r}(F),$$
$$\underline{\mathrm{r}}(F) \leq \mathrm{r}(F).$$

Suppose we go back to the case of the Veronese embedding. Then we can make Fact 0.1 work for the cactus rank:

Fact 0.3. *Let $F \in \mathbb{C}[y_0, \dots, y_n]_d$. For any closed subscheme $R \subseteq \mathbb{P}(\mathbb{C}[y_0, \dots y_n]_1)$ let $I(R)$ denote its saturated ideal in $\mathbb{C}[x_0, \dots, x_n]$. Then for any zero-dimensional closed subscheme $R \subseteq \mathbb{P}(\mathbb{C}[y_0, \dots, y_0]_1)$ we have*

$$F \in \langle R \rangle \iff I(R) \subseteq F^{\perp}.$$

Let us go back to Example 0.2. We can see that $(x_1^2) \subseteq F^{\perp}$, and the length of the subscheme defined by x_1^2 is two. This means that the cactus rank is at most two. In fact, it is two, because $y_0^{d-1} y_1$ is not a d-th power.

0.1. Background

The topic goes back to works of Sylvester on apolarity in the 19th century. For introductions to this subject, see [Lan12] and [IK99]. For a short introduction to the concept of rank for many different subvarieties $X \subseteq \mathbb{P}^N$ and many ways to give lower bounds for rank, see [Tei14] (see also many references there). For a short review of the apolarity action in the case of the Veronese embedding, see [BB14, Section 3].

In this paper, we see what happens when X is a toric variety. For an introduction to this subject, see the newer [CLS11] and the older [Ful93]. For toric varieties, many invariants can be computed quite easily. This can be used to study ranks and secant varieties. In [CS07], the authors investigate the second secant variety $\sigma_2(X)$, where X is a toric variety embedded into some projective space. As they write there, "Many classical varieties whose secant varieties has been studied are toric". Here we take a slightly different approach. We generalize apolarity to toric varieties, and then, as an application, we compute the rank, cactus rank and border rank of some polynomials.

0.2. Main result

We need to introduce some notions to state the main result. We revisit these ideas in a different language in the following chapters. Suppose X is a smooth projective toric variety. Then $\operatorname{Cl} X = \operatorname{Pic} X$ is free of finite rank. Fix a basis $\mathcal{L}_1, \dots, \mathcal{L}_l$ of $\operatorname{Cl} X$ ($\mathcal{L}_i$ are line bundles). Define the Cox ring of X:

$$S = \bigoplus_{m_1,\dots,m_l \in \mathbb{Z}} H^0(X, \mathcal{L}_1^{\otimes m_1} \otimes \dots \otimes \mathcal{L}_l^{\otimes m_r}),$$

where the multiplication is the tensor product of sections. By definition, it is graded by $\operatorname{Cl} X$. Since X is a toric variety, S is a polynomial ring with finitely many variables, so we may write $S \cong \mathbb{C}[x_1, \dots, x_r]$. We look at the ring $T = \mathbb{C}[y_1, \dots, y_r]$. We forget that it is a ring, and treat it as an S-module with action $\circ$ defined as in Equation (1). We grade T in an analogous way as S:

$$\deg y_1^{a_1} \cdot \ldots \cdot y_r^{a_r} = \deg x_1^{a_1} \cdot \ldots \cdot x_r^{a_r}.$$

Then we identify $H^0(X, \mathcal{L})^*$ with the graded piece of T of degree $\mathcal{L}$ (this identification is described in Proposition 2.3 in detail), and for a homogeneous $F \in T$ we define $F^\perp$ as its annihilator in S (with respect to the action $\circ$).

Let $R \hookrightarrow X$ be any closed subscheme with ideal sheaf $\mathcal{I}_R$. Then we can define $I(R)$, the ideal of R in S, by

$$I(R) = \bigoplus_{m_1,\dots,m_l \in \mathbb{Z}} H^0(X, \mathcal{I}_R \otimes \mathcal{L}_1^{\otimes m_1} \otimes \dots \otimes \mathcal{L}_l^{\otimes m_r}).$$

Recall that for any line bundle $\mathcal{L}$ the space $H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \subseteq H^0(X, \mathcal{L})$ is the subspace of those sections which are zero when pulled back to R.

The main result of this paper is:

Theorem 0.4 (Multigraded Apolarity Lemma)**.** *Let $\mathcal{L}$ be a very ample line bundle on X and consider the associated morphism $\varphi : X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L})^*)$. Take any $F \in H^0(X, \mathcal{L})^* = T_{\mathcal{L}}$. Then for any subscheme $R \hookrightarrow X$.*

$$I(R) \subseteq F^\perp \iff F \in \langle R \rangle.$$

Here $\langle R \rangle$ denotes the linear span of R in $H^0(X, \mathcal{L})^*$.

This allows us to give upper bounds for the cactus rank (when R is a zero-dimensional subscheme) and rank (when R is a reduced zero-dimensional subscheme).

We prove Theorem 0.4 in Chapter 3 (Proposition 3.2) in a more general form: we allow X to be a $\mathbb{Q}$-factorial projective toric variety (so that $\operatorname{Pic} X$ has finite index in $\operatorname{Cl} X$), in the toric language the corresponding notion is a simplicial fan. Then it may happen that the class group has torsion.

We may think of it from a different perspective: start with $T = \mathbb{C}[y_1, \ldots, y_r]$, pick a finitely generated abelian group G, and assign to each i $(1 \leq i \leq r)$ an element $g_i \in G$, and then we define a grading on T in G by

$$\deg y_i := g_i.$$

We define the grading on S in an analogous way. Then we choose a homogeneous polynomial $F \in T_g$ and look for a toric variety X such that G is the class group of X and g is a very ample class.

To sum up, we look at the polynomial ring $\mathbb{C}[y_1, \ldots, y_r]$ with a grading in a finitely generated group G (such that monomials are homogeneous), and we generalize the apolarity action to this multigraded setting.

Finally, we use this to compute the ranks of polynomials when X is a projective toric surface. The first example is the Hirzebruch surface $\mathbb{F}_1$ (which can be defined as $\mathbb{P}^2$ blown up in one point), and the second one is a fake projective plane — the quotient of $\mathbb{P}^2$ by the action of $\mathbb{Z}/3 = \{1, \varepsilon, \varepsilon^2\}$ (where $\varepsilon^3 = 1$) given by $\varepsilon \cdot [\lambda_0, \lambda_1, \lambda_2] = [\lambda_0, \varepsilon\lambda_1, \varepsilon^2\lambda_2]$.

0.3. Overview

In Chapter 1 we introduce the necessary facts on toric varieties, satisfying ourselves with references on the main points. In Chapter 2 we generalize the notion of apolarity to toric varieties. Chapter 3 contains the proof of the main result, in an even more general form, but not involving the apolarity action, and also the statement and proof in the toric case as a corollary of this. In Chapter 4 we give some examples of how apolarity can be used to calculate ranks for toric varieties embedded into projective space.

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Chapter 1

Toric varieties

Let M and N be dual lattices (abelian groups isomorphic to $\mathbb{Z}^k$ for some $k \geq 1$) and $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ be a duality between them. Let X_Σ be the toric variety of a fan $\Sigma \subseteq N_\mathbb{R} := N \otimes \mathbb{R}$ with no torus factors. "No torus factors" means that the linear span of Σ in $N_\mathbb{R}$ is the whole space. Let $\Sigma(1)$ denote the set of rays of the fan Σ (similarly $\sigma(1)$ will denote the set of rays in the cone σ). Then X_Σ can be obtained as an almost geometric quotient of an action of $G := \mathrm{Hom}(\mathrm{Cl}\, X_\Sigma, \mathbb{C}^*)$ on $\mathbb{C}^{\Sigma(1)} \setminus Z$, where Z is a subvariety of $\mathbb{C}^{\Sigma(1)}$. Let us go briefly through the construction of this quotient. We follow [CLS11, Section 5.1].

Since X_Σ has no torus factors, we have an exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}\, X_\Sigma \to 0.$$

After applying $\mathrm{Hom}(-, \mathbb{C}^*)$, this gives

$$0 \to \mathrm{Hom}(\mathrm{Cl}\, X_\Sigma, \mathbb{C}^*) \to (\mathbb{C}^*)^{\Sigma(1)} \to T_N \to 0,$$

where $T_N = \mathbb{C}^* \otimes N \subseteq X_\Sigma$ is the torus of X_Σ. So $G = \mathrm{Hom}(\mathrm{Cl}\, X_\Sigma, \mathbb{C}^*)$ is a subset of $\mathbb{C}^{\Sigma(1)}$, and the action is given by multiplication on coordinates. Let $S = \mathrm{Spec}\, \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$. In other words, S is the polynomial ring with variables indexed by the rays of the fan Σ. The ring S is the coordinate ring of the affine space $\mathbb{C}^{\Sigma(1)}$. For a cone $\sigma \in \Sigma$, define

$$x^{\widehat{\sigma}} := \prod_{\rho \notin \sigma(1)} x_\rho.$$

Then define an ideal in S:

$$B = B(\Sigma) = (x^{\widehat{\sigma}} : \sigma \in \Sigma) \subseteq S, \tag{1.1}$$

and let $Z = Z(\Sigma) \subseteq \mathbb{C}^{\Sigma(1)}$ be the vanishing set of B. For the definition of the quotient map π

$$\mathbb{C}^{\Sigma(1)} \setminus Z \xrightarrow{\ \pi\ } (\mathbb{C}^{\Sigma(1)} \setminus Z)//G = X_\Sigma$$

see [CLS11, Proposition 5.1.9]. If $(\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r = \mathbb{C}^{\Sigma(1)}$, we will sometimes write $[\lambda_1, \ldots, \lambda_r]$ for $\pi(\lambda_1, \ldots, \lambda_r)$.

Fix an ordering of all the rays of the fan, let $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$. Then S becomes $\mathbb{C}[x_{\rho_1}, \ldots, x_{\rho_r}] =: \mathbb{C}[x_1, \ldots, x_r]$. The ring S is the Cox ring of X_Σ. For more details, see [CLS11, 5.2], where S is called the total coordinate ring. This ring is graded by the class group $\mathrm{Cl}\, X_\Sigma$, where

$$\deg x_i := [D_{\rho_i}],$$

and D_{ρ_i} is the torus-invariant divisor corresponding to ρ_i, see [CLS11, Chapter 4]. For the rest of the paper, if A is a graded ring, A_α will denote the graded piece of A of degree α.

Take any ideals $I, J \subseteq S$. Let $(I :_S J)$ be the set of all $x \in S$ such that $x \cdot J \subseteq I$; it is an ideal of S. Then for any ideals $I, J, K \subseteq S$:

- $I \subseteq (I :_S J)$,
- if $J \subseteq K$, then $(I :_S J) \supseteq (I :_S K)$,
- $(I :_S J \cdot K) = ((I :_S J) :_S K)$.

Recall the ideal $B \subseteq S$ defined in Equation (1.1). This ideal is called the irrelevant ideal. Take any ideal $I \subset S$. We define the B-saturation of I as

$$I^{\mathrm{sat}} := \bigcup_{i \geq 1} (I :_S B^i).$$

Note that this is an increasing union because $B^i \supseteq B^j$ for $i < j$, so I^{sat} is an ideal. Since S is Noetherian, the union stabilizes in a finite number of steps. We always have $I \subseteq I^{\mathrm{sat}}$. If this is an equality, we say that I is B-saturated. In order to show that I is B-saturated, it suffices to find any $i \geq 1$ such that $I = (I :_S B^i)$.

Moreover, if I and J are homogeneous, then so is $(I :_S J)$. It follows that for I homogeneous the ideal I^{sat} is homogeneous.

As an example, let us look at the projective space $\mathbb{P}^k_{\mathbb{C}}$. See [CLS11, Example 5.1.7]. Here $S = \mathbb{C}[x_0, \dots, x_k]$, $B = (x_0, \dots, x_k) = \bigoplus_{i \geq 1} S_i$ and $Z = \{0\}$. In this case

$$I^{\mathrm{sat}} = \{f \in S : \text{ for all } i = 0, 1, \dots, k \text{ there is } n \text{ such that } x_i^n \cdot f \in I\}.$$

Recall that in this case there is a 1-1 correspondence between closed subschemes of $\mathbb{P}^k_{\mathbb{C}}$ and homogeneous B-saturated ideals of S. Moreover, the ideal given by $\bigoplus_{i \geq 0} H^0(X, \mathcal{I}_R \otimes \mathcal{O}(i))$, where $\mathcal{I}_R$ is the ideal sheaf of R in $\mathbb{P}^k_{\mathbb{C}}$, is B-saturated. For more on this, see [Har77, II, Corollary 5.16 and Exercise 5.10].

For a toric variety the situation is more complicated. We will assume that the fan Σ is simplicial for technical reasons. There can be many B-saturated ideals defining a subscheme R. But they have to agree in the Pic part. See [Cox95, Theorem 3.7 and the following discussion] for more details. Consider the map

$$\bigoplus_{\alpha \in \operatorname{Cl} X_\Sigma} H^0(X, \mathcal{I}_R \otimes \mathcal{O}(\alpha)) \to \bigoplus_{\alpha \in \operatorname{Cl} X_\Sigma} H^0(X, \mathcal{O}(\alpha)) \tag{1.2}$$

induced by $\mathcal{I}_R \hookrightarrow \mathcal{O}_{X_\Sigma}$. We may take I to be the image of this map. This is done in the proof of [CLS11, Proposition 6.A.6]. Note that in this case for any $\alpha \in \operatorname{Pic} X_\Sigma$ the vector space $H^0(X_\Sigma, \mathcal{I}_R \otimes \mathcal{O}(\alpha))$ can be identified with those global sections of $\mathcal{O}(\alpha)$ which vanish on R. So let us make the following

Definition 1.1. *Let X_Σ be a simplicial toric variety. Let $R \hookrightarrow X_\Sigma$ be a closed subscheme. We define $I(R) \subseteq S$, the ideal of R, to be the image of homomorphism* (1.2).

Fact 1.2. *Suppose the fan Σ is simplicial. Let $\alpha \in \operatorname{Pic} X_\Sigma$ be the class of a Cartier divisor. Let $R \hookrightarrow X_\Sigma$ be any closed subscheme. Then $(I(R))_\alpha = (I(R) :_S B)_\alpha$ (i.e. $I(R)$ agrees with $I(R)^{sat}$ in degree α).*

Proof. Take $x \in S_\alpha$ such that $x \cdot B \subseteq I(R)$. It is enough to show that x is zero on R. Take any point $p \in R$. We will show that x is zero on R around that point. Since $V(B) = \varnothing$, we know that some homogeneous element $b \in B$ is non-zero at p. By taking a big enough power, we may assume $b \in B_\beta$ for some $\beta \in \operatorname{Pic} X_\Sigma$ (here we are using that Σ is simplicial!). Because b is non-zero at p, there is an open neighbourhood $p \in U \subseteq X_\Sigma$ such that $\mathcal{O}_{X_\Sigma}(\beta)$ is trivialized on U by b. But then x is zero when pulled back to R on U if and only if $x \cdot b$ is zero when pulled back to R on U. But the latter thing is true as $x \cdot b \in I(R)$. $\square$

We will need the following

Fact 1.3. *Let $\alpha \in \operatorname{Pic} X_\Sigma \subseteq \operatorname{Cl} X_\Sigma$. Recall the isomorphism of $H^0(X_\Sigma, \mathcal{O}(\alpha))$ and $\mathbb{C}[x_1, \ldots, x_r]_\alpha$ given in [CLS11, Proposition 5.3.7]. Take any section $s \in H^0(X_\Sigma, \mathcal{O}(\alpha))$ and the corresponding polynomial $f \in S_\alpha$. Also let p be a point in X_Σ and take any $(\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$ such that $\pi(\lambda_1, \ldots, \lambda_r) = p$. Then*

$$s(p) = 0 \iff f(\lambda_1, \ldots, \lambda_r) = 0.$$

Proof. Take any σ such that $p \in U_\sigma$. We will trivilize the line bundle $\mathcal{O}(\alpha)$ on U_σ in order to move the situation to regular functions on U_σ. We will do it by finding a section that is nowhere zero both as a polynomial and as a section.

We know that $U_\sigma = \operatorname{Spec}(S_{x^{\widehat{\sigma}}})_0$, where $x^{\widehat{\sigma}} = \prod_{\rho \notin \sigma} x_\rho$, the inner subscript refers to localization, and the outer one is taking degree 0. From the definition of $\mathcal{O}(\alpha)$ we have $H^0(U_\sigma, \mathcal{O}(\alpha)) = (S_{x^{\widehat{\sigma}}})_\alpha$. Our goal is to find a monomial in $(S_{x^{\widehat{\sigma}}})_\alpha$ which is nowhere zero as a section. Take any torus-invariant representative $\sum_\rho a_\rho D_\rho$ of class α (here $a_\rho \in \mathbb{Z}$). From [CLS11, Theorem 4.2.8] there exists an $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for $\rho \in \sigma(1)$ (here $\sigma(1)$ is the set of rays of the cone σ, and $u_\sigma \in N$ is the generator of ray ρ). Then

$$\sum_\rho \langle m_\sigma, u_\rho \rangle D_\rho + \sum_\rho a_\rho D_\rho = \sum_{\rho \notin \sigma(1)} (\langle m_\sigma, u_\rho \rangle + a_\rho) D_\rho$$

belongs to the class α as well. This is a direct consequence of the exact sequence [CLS11, Theorem 4.2.1]. The outcome is that the monomial $g := \prod_{\rho \notin \sigma(1)} x_\rho^{\langle m_\sigma, u_\rho \rangle + a_\rho}$ has degree α. Notice that it belongs to $(S_{x^{\widehat{\sigma}}})_\alpha$.

We want to show that g is nowhere zero as a section of $\mathcal{O}(\alpha)$. Polynomial $g \in S_{x^{\widehat{\sigma}}}$ is invertible, with inverse $g^{-1} \in (S_{x^{\widehat{\sigma}}})_{-\alpha}$. But then $g^{-1} \cdot g = 1 \in (S_{x^{\widehat{\sigma}}})_0$. If g were zero at some point $p \in X_\Sigma$, then we would have $0 = g^{-1}(p) \cdot g(p) = 1$, a contradiction.

The fact that $g \in S_{x^{\widehat{\sigma}}}$ is nowhere zero on $\operatorname{Spec} S_{x^{\widehat{\sigma}}}$ (as a polynomial) is a consequence of g being invertible in this ring. Now we can set $\bar{f} := g^{-1} f$ and then $\bar{f}$ is a regular function on $\operatorname{Spec}(S_{x^{\widehat{\sigma}}})_0$. We need to see if $\bar{f}(p) = 0$ is equivalent to $\bar{f}(\lambda_1, \ldots, \lambda_r) = 0$. In fact, even more is true: $\bar{f}(p) = \bar{f}(\lambda_1, \ldots, \lambda_r)$. That follows from the fact that $\bar{f}$ is a function both on $\operatorname{Spec} S_{x^{\widehat{\sigma}}}$ and on $\operatorname{Spec}(S_{x^{\widehat{\sigma}}})_0$ and its evaluation at any point is the same as at its image. $\square$

Corollary 1.4. *Suppose $f_1, f_2 \in S_\alpha$ are polynomials and s_1, s_2 are the corresponding sections of $\mathcal{O}(\alpha)$. Also fix, as above, $p \in X_\Sigma$ and $(\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$ such that $\pi(\lambda_1, \ldots, \lambda_r) = p$. Then if $f_2(\lambda_1, \ldots, \lambda_r)$ and $s_2(p)$ are non-zero, we get*

$$\frac{f_1(\lambda_1, \ldots, \lambda_r)}{f_2(\lambda_1, \ldots, \lambda_r)} = \frac{s_1(p)}{s_2(p)}.$$

Proof. Take $\mu \in \mathbb{C}$ such that $f_1(\lambda_1, \ldots, \lambda_r) = \mu f_2(\lambda_1, \ldots, \lambda_r)$. Then use the previous fact for $f_1 - \mu f_2$ and the corresponding section $s_1 - \mu s_2$. $\square$

Chapter 2

Apolarity

Introduce $T := \mathbb{C}[y_1, \ldots, y_r]$. We will think of T as an S-module, where the multiplication (denoted by $\lrcorner$) is induced by

$$x_i \lrcorner y_1^{b_1} \cdot \ldots \cdot y_r^{b_r} := \begin{cases} y_1^{b_1} \cdot \ldots \cdot y_i^{b_i - 1} \cdot \ldots \cdot y_r^{b_r} & \text{if } b_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

This multiplication is called apolarity and when $g \lrcorner F = 0$ we often say that g is apolar to F. The grading on T is the same as on S:

$$\deg y_i := [D_{\rho_i}].$$

Remark 2.1. Notice that here we changed the notation from $\circ$ to $\lrcorner$, and we gave up multiplying by the constant b_i. So the multiplication is not the same as in the introduction. But it would be the same if we somehow identified y_i^b from the introduction with $b! \cdot y_i^b$ from this chapter. This can be done by taking T to be the ring of divided powers, see [IK99, Appendix A], or [Eis95, Chapter A2.4] for a coordinate free version. For characteristic zero, this amounts to setting $y_i^{(b)} = \frac{y_i^b}{b!}$. But here we do not need T to be a ring, we only need it to be a module. So we might as well write y_i^b instead of $y_i^{(b)}$. It will not matter, provided we do not multiply $y_i^{b_1}$ by $y_i^{b_2}$. This will make some calculations easier.

Remark 2.2. Notice that when we take $g \in S_\alpha$ and $F \in T_\beta$, then $g \lrcorner F$ is of degree $\alpha - \beta$ for any α, $\beta \in \operatorname{Cl} X_\Sigma$. That follows from the fact that when we multiply by subsequent x_i's, the degree of F decreases by $[D_{\rho_i}]$. This means that, although T is not a graded S-module, it becomes a graded S-module if we define the grading by

$$\deg y_i := -[D_{\rho_i}].$$

Futhermore, if $F \in T$ is homogeneous, we will denote by $F^\perp$ its annihilator, which is a homogeneous ideal in that case.

From now on assume X_Σ is a proper variety. Then we have $S_0 = T_0 = \mathbb{C}$ and S_α, T_α are finite-dimensional vector spaces for any $\alpha \in \operatorname{Cl} X_\Sigma$.

Proposition 2.3. *The map $S_\alpha \times T_\alpha \to S_0 = \mathbb{C}$ given by $(g, F) \mapsto g \lrcorner F$ is a duality for any $\alpha \in \operatorname{Cl} X_\Sigma$.*

Proof. We will show that the basis

$$\{x_1^{a_1} \cdot \ldots \cdot x_r^{a_r} : [a_1 D_{\rho_1} + \ldots + a_r D_{\rho_r}] = \alpha\}$$

is dual to

$$\{y_1^{b_1} \cdot \ldots \cdot y_r^{b_r} : [b_1 D_{\rho_1} + \ldots + b_r D_{\rho_r}] = \alpha\}.$$

We know that $x_1^{a_1} \cdot \ldots \cdot x_r^{a_r} \lrcorner\, y_1^{a_1} \cdot \ldots \cdot y_r^{a_r} = 1$. Consider the value of $x_1^{a_1} \cdot \ldots \cdot x_r^{a_r} \lrcorner\, y_1^{b_1} \cdot \ldots \cdot y_r^{b_r}$ when $(a_1, \ldots, a_r) \neq (b_1, \ldots, b_r)$. We know that

$$x_1^{a_1} \cdot \ldots \cdot x_r^{a_r} \lrcorner\, y_1^{b_1} \cdot \ldots \cdot y_r^{b_r} = \begin{cases} y_1^{b_1 - a_1} \cdot \ldots \cdot y_r^{b_r - a_r} & \text{if } b_i \geq a_i \text{ for all } i. \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

We want to prove (2.1) is zero, so suppose otherwise. The degree of (2.1) is zero. But the only monomial whose degree is the trivial class is the constant monomial 1 (we are using that X_Σ is proper). This implies that $b_i = a_i$ for all i. But this cannot be true, since we assumed $(a_1, \ldots, a_r) \neq (b_1, \ldots, b_r)$. This contradiction means that $x_1^{a_1} \cdot \ldots \cdot x_r^{a_r} \lrcorner\, y_1^{b_1} \cdot \ldots \cdot y_r^{b_r} = 0$, as desired. □

As a corollary, we see that $T = \bigoplus_{\alpha \in \operatorname{Cl} X_\Sigma} H^0(X, \mathcal{O}(\alpha))^*$.

Combining Proposition 2.3 and Corollary 1.4, we get

Fact 2.4. *For any $\alpha \in \operatorname{Pic} X_\Sigma$ such that $\mathcal{O}(\alpha)$ is basepoint free, the map*

$$\varphi \colon X_\Sigma \to \mathbb{P}(H^0(X_\Sigma, \mathcal{O}(\alpha))^*)$$

is given by

$$\varphi([\lambda_1, \ldots, \lambda_r]) = \left[\sum_{\substack{b_1, \ldots, b_r \in \mathbb{Z}_{\geq 0}: \\ y_1^{b_1} \cdot \ldots \cdot y_r^{b_r} \in T_\alpha}} \lambda_1^{b_1} \cdot \ldots \cdot \lambda_r^{b_r} \cdot y_1^{b_1} \cdot \ldots \cdot y_r^{b_r} \right]. \tag{2.2}$$

Proof. In general, if $\{s_i : i \in I\}$ is a basis of $H^0(X, \mathcal{O}(\alpha))$ (I is some finite index set), and $\{s^i : i \in I\} \subseteq H^0(X, \mathcal{O}(\alpha))^*$ is the dual basis, then

$$\varphi(p) = \left[\sum_{i \in I} s_i(p) \cdot s^i \right],$$

where $s_i(p)$ means evaluating section s_i at point p. Note that it does not make sense to talk about the value of a section, but the quotient $s_i(p)/s_j(p)$ makes sense, and the sum makes sense as a class in the projectivization of $H^0(X, \mathcal{O}(\alpha))^*$.

By the proof of Proposition 2.3, the monomials $y_1^{b_1} \ldots y_r^{b_r} \in T_\alpha$ form a dual basis to $x_1^{b_1} \ldots x_r^{b_r}$. So from Corollary 1.4 we know that for any $i = (b_1, \ldots, b_r)$, $i' = (b'_1, \ldots, b'_r)$ such that $s_{i'}(p)$ is non-zero we have

$$\frac{s_i(p)}{s_{i'}(p)} = \frac{(x_1^{b_1} \cdot \ldots \cdot x_r^{b_r})(p)}{(x_1^{b'_1} \cdot \ldots \cdot x_r^{b'_r})(p)} = \frac{\lambda_1^{b_1} \cdot \ldots \cdot \lambda_r^{b_r}}{\lambda_1^{b'_1} \cdot \ldots \cdot \lambda_r^{b'_r}}.$$

The formula (2.2) follows. □

2.1. Hilbert function

Fix $F \in T_\alpha$. The ring $S/F^\perp$ is called the apolar ring of F. It is graded by the class group of X_Σ. Let us denote it by A_F. Consider its Hilbert function $H : \operatorname{Cl} X_\Sigma \to \mathbb{Z}_{\geq 0}$ given by

$$\beta \mapsto dim_{\mathbb{C}}\left((A_F)_\beta\right).$$

The Hilbert function is symmetric. The proof for projective space also applies to toric varieties:

Fact 2.5. *For any $\beta \in \operatorname{Cl} X_\Sigma$:*

$$dim_{\mathbb{C}}(A_F)_\beta = dim_{\mathbb{C}}(A_F)_{\alpha-\beta}.$$

Proof. We will prove that the bilinear map $(A_F)_\beta \times (A_F)_{\alpha-\beta} \to \mathbb{C} \cong (A_F)_0$ given by $(g,h) \mapsto (g \cdot h) \lrcorner F$ is a duality. Take any $g \in S_\beta$ such that $g \lrcorner F \neq 0$. Then there is $h \in S_{\alpha-\beta}$ such that $h \lrcorner (g \lrcorner F) \neq 0$ (because $\lrcorner$ makes $S_{\alpha-\beta}$ and $T_{\alpha-\beta}$ dual by Proposition 2.3). But this means that $(h \cdot g) \lrcorner F \neq 0$. We have proven that multiplying by any non-zero $g \in (A_F)_\beta$ is non-zero as a map $(A_F)_{\alpha-\beta} \to \mathbb{C}$. Similarly, multiplying by any non-zero $h \in (A_F)_{\alpha-\beta}$ is non-zero as a map $(A_F)_\beta \to \mathbb{C}$. We are done. □

Remark 2.6. The values of the Hilbert function of $S/F^\perp$ are the same as the ranks of the catalecticant homomorphisms. More precisely, let

$$C_F^\beta : S_\beta \to T_{\alpha-\beta}$$

be given by

$$g \mapsto g \lrcorner F.$$

This map is called the catalecticant homomorphism. We have

$$\operatorname{rank} C_F^\beta = \dim_{\mathbb{C}}(A_F)_\beta.$$

This is because the graded piece of $F^\perp$ of degree β is the kernel of C_F^β. For more on catalecticant homomorphisms, see [Tei14, Section 2] or [IK99, Chapter 1].

Chapter 3

Apolarity Lemma

Let us work in a more general setting for a while.

Suppose X is a proper variety over $\mathbb{C}$. Let $\mathcal{L}$ be a very ample line bundle on X, and $\varphi\colon X \to \mathbb{P}(H^0(X,\mathcal{L})^*)$ the associated morphism. For a closed subscheme $i\colon R \hookrightarrow X$, $\langle R\rangle$ denotes its linear span in $\mathbb{P}(H^0(X,\mathcal{L})^*)$, and $\mathcal{I}_R$ denotes its ideal sheaf on X. Recall that for any line bundle on X, the vector subspace $H^0(X,\mathcal{I}_R\otimes\mathcal{L}) \subseteq H^0(X,\mathcal{L})$ consists of the sections which pull back to zero on R.

Let $(\cdot \lrcorner \cdot) : H^0(X,\mathcal{L}) \otimes H^0(X,\mathcal{L})^* \to \mathbb{C}$ denote the natural pairing (this agrees with the notation introduced in Chapter 2). Now we are ready to formulate the Apolarity Lemma:

Proposition 3.1 (Apolarity Lemma, general version)**.** *Let $F \in H^0(X,\mathcal{L})^*$ be a non-zero element. Then for any closed subscheme $i : R \hookrightarrow X$ we have*

$$F \in \langle R\rangle \iff H^0(X,\mathcal{I}_R\otimes\mathcal{L}) \lrcorner F = 0.$$

Proof. Take any $s \in H^0(X,\mathcal{L})$, let H_s be the corresponding hyperplane in $H^0(X,\mathcal{L})^*$. Then,

$$\langle R\rangle \subseteq H_s \iff i^*(s) = 0 \iff s \in H^0(X,\mathcal{I}_R\otimes\mathcal{L}).$$

Below we identify sections $s \in H^0(X,\mathcal{L})$ with hyperplanes H_s in $H^0(X,\mathcal{L})^*$. Then for any R

$$\begin{aligned} F \in \langle R\rangle &\iff \forall_{s\in H^0(X,\mathcal{L})}(\langle R\rangle \subseteq H_s \implies F \in H_s) \\ &\iff \forall_{s\in H^0(X,\mathcal{L})}(s \in H^0(X,\mathcal{I}_R\otimes\mathcal{L}) \implies F \in H_s) \\ &\iff \forall_{s\in H^0(X,\mathcal{L})}(s \in H^0(X,\mathcal{I}_R\otimes\mathcal{L}) \implies s \lrcorner F = 0) \\ &\iff H^0(X,\mathcal{I}_R\otimes\mathcal{L}) \lrcorner F = 0. \end{aligned}$$

□

Proposition 3.2 (Apolarity Lemma, toric version)**.** *Let Σ be a simplicial fan, and X_Σ the toric variety defined by it. Let $\alpha \in \operatorname{Pic} X_\Sigma$ be a very ample class and $\varphi\colon X \hookrightarrow \mathbb{P}(H^0(X_\Sigma,\mathcal{O}(\alpha))^*)$ be the associated morphism. Fix a non-zero $F \in H^0(X_\Sigma,\mathcal{O}(\alpha))^*$. Then for any closed subscheme $R \hookrightarrow X_\Sigma$ we have*

$$F \in \langle R\rangle \iff I(R) \subseteq F^\perp.$$

Recall that here $I(R)$ is the ideal of R from Definition 1.1.

Proof. From Proposition 3.1 we know that $F \in \langle R\rangle$ if and only if $I(R)_\alpha \subseteq F^\perp_\alpha$. It remains to prove that $I(R)_\alpha \subseteq F^\perp_\alpha$ implies $I(R) \subseteq F^\perp$. Suppose $I(R)_\alpha \subseteq F^\perp_\alpha$. Take any $x \in I(R)_\beta$ for some $\beta \in \operatorname{Cl} X_\Sigma$. We want to show that $x \lrcorner F = 0$. We have $S_{\alpha-\beta}\cdot x \subseteq I_\alpha$, because g

is in the ideal. This means that $(S_{\alpha-\beta} \cdot x) \lrcorner F = 0$, i.e. $S_{\alpha-\beta} \lrcorner (x \lrcorner F) = 0$. Now, $x \lrcorner F$ is an element of $T_{\alpha-\beta}$, which is zero when multiplied by anything from $S_{\alpha-\beta}$, which is equal to $T^*_{\alpha-\beta}$ by Proposition 2.3. It follows that $x \lrcorner F$ is zero. $\square$

Remark 3.3. By Fact 1.2, we might have taken $I(R)^{\text{sat}}$ instead of $I(R)$ in the proposition above. By [Cox95, Theorem 3.7], we might have taken any B-saturated ideal defining R.

Chapter 4

Examples

We use what we proved to look at some examples. Before that, we need some auxiliary propositions:

Proposition 4.1. *Let X be a proper variety and R be a zero-dimensional subscheme of X with ideal sheaf $\mathcal{I}_R$. Then for any line bundle $\mathcal{L}$*

$$\operatorname{lh} R \geq h^0(X, \mathcal{L}) - h^0(X, \mathcal{I}_R \otimes \mathcal{L}).$$

Proof. We have an exact sequence

$$0 \to \mathcal{I}_R \to \mathcal{O}_X \to \mathcal{O}_R \to 0.$$

We tensor it with $\mathcal{L}$:

$$0 \to \mathcal{I}_R \otimes \mathcal{L} \to \mathcal{L} \to \mathcal{L}_{|R} \to 0.$$

After taking global sections (which are left-exact), we get an exact sequence

$$0 \to H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \to H^0(X, \mathcal{L}) \to H^0(R, \mathcal{L}_{|R}).$$

It follows that

$$h^0(R, \mathcal{L}_{|R}) \geq h^0(X, \mathcal{L}) - h^0(X, \mathcal{I}_R \otimes \mathcal{L}).$$

But on a zero-dimensional scheme, every line bundle trivializes. This means $h^0(R, \mathcal{L}_{|R}) = h^0(R, \mathcal{O}_R)$, which is the length of R. $\square$

From now on, assume again that $X = X_\Sigma$ is a projective simplicial toric variety. Moreover, let us fix a very ample class $\alpha \in \operatorname{Pic} X_\Sigma$.

Suppose $\beta \in \operatorname{Pic} X_\Sigma$. The linear map $\lrcorner : S_\beta \otimes T_\alpha \to T_{\alpha-\beta}$ can be seen as coming from the isomorphism

$$\mathcal{O}(\beta) \otimes \mathcal{O}(\alpha - \beta) \to \mathcal{O}(\alpha)$$

by taking multiplication of global sections:

$$H^0(X_\Sigma, \mathcal{O}(\beta)) \otimes H^0(X_\Sigma, \mathcal{O}(\alpha - \beta)) \to H^0(X_\Sigma, \mathcal{O}(\alpha))$$

and rearranging the terms:

$$H^0(X_\Sigma, \mathcal{O}(\beta)) \otimes H^0(X_\Sigma, \mathcal{O}(\alpha))^* \to H^0(X_\Sigma, \mathcal{O}(\alpha - \beta))^*.$$

Remember that for any $\gamma \in \operatorname{Pic} X_\Sigma$ the space $H^0(X_\Sigma, \mathcal{O}(\gamma))$ is S_γ and we identify T_γ with $H^0(X_\Sigma, \mathcal{O}(\gamma))^*$ by Proposition 2.3. Notice that if we fix $F \in H^0(X_\Sigma, \mathcal{O}(\alpha))^*$, then the map above becomes the catalecticant homomorphism C_F^β from Remark 2.6.

As a corollary of Proposition 4.1 and the Apolarity Lemma (Proposition 3.2), we get the catalecticant bound in the special case of line bundles.

Corollary 4.2. *For any $F \in H^0(X_\Sigma, \mathcal{O}(\alpha))^*$, and any $\beta \in \operatorname{Pic} X_\Sigma$ we have*

$$\operatorname{cr}(F) \geq \operatorname{rank} C_F^\beta.$$

Proof. Take any zero-dimensional scheme $R \hookrightarrow X_\Sigma$ such that $F \in \langle R \rangle$. Let I be any B-saturated ideal defining R. We have

$$\begin{aligned} \operatorname{lh} R \geq h^0(X_\Sigma, \mathcal{O}(\beta)) - h^0(X_\Sigma, \mathcal{I}_R \otimes \mathcal{O}(\beta)) = \dim_{\mathbb{C}}(S/I)_\beta \\ \geq \dim_{\mathbb{C}}(S/F^\perp)_\beta = \dim_{\mathbb{C}} \operatorname{im} C_F^\beta, \end{aligned}$$

where the first inequality follows from Proposition 4.1, and the second from Proposition 3.2. We also used that $I(R)$ agrees with any saturated ideal defining R, see Remark 3.3, and the fact that values of the Hilbert function are ranks of catalecticant homomorphisms (Remark 2.6). □

Proposition 4.3. *Fix $\beta \in \operatorname{Pic} X_\Sigma$. Then for any $l \in \mathbb{Z}_+$ the set of points $F \in \mathbb{P}(H^0(X_\Sigma, \mathcal{O}(\alpha))^*)$ such that* $\operatorname{rank}(C_F^\beta) \leq l$ *is Zariski-closed.*

Proof. Pick a basis of $H^0(X_\Sigma, \mathcal{O}(\beta))$ and a basis of $H^0(X_\Sigma, \mathcal{O}(\alpha - \beta))$. Then $\lrcorner$ becomes a matrix with entries in $H^0(X_\Sigma, \mathcal{O}(\alpha))$. In order to get the rank of the map $C_F^\beta = \cdot \lrcorner F$, we evaluate the matrix at $F \in H^0(X_\Sigma, \mathcal{O}(\alpha))^*$. Hence the set of those F's such that the rank of $\cdot \lrcorner F$ is at most l is given by the vanishing of the $(l+1)$-th minors of the matrix. These minors are polynomials from $\operatorname{Sym}^\bullet H^0(X_\Sigma, \mathcal{O}(\alpha))$. We are done. □

Corollary 4.4. *For any $\beta \in \operatorname{Pic} X_\Sigma$ and any $F \in H^0(X_\Sigma, \mathcal{O}(\alpha))^*$ we have*

$$\underline{\mathrm{r}}(F) \geq \operatorname{rank} C_F^\beta.$$

Proof. From Proposition 4.3 we know that for $l := \operatorname{rank} C_F^\beta$ the set of $F' \in \mathbb{P}(H^0(X_\Sigma, \mathcal{O}(\alpha))^*)$ such that $\operatorname{rank} C_{F'}^\beta \geq \operatorname{rank} C_F^\beta$ is Zariski-open. But from Corollary 4.2 we know that $\mathrm{r}(F') \geq \operatorname{rank} C_{F'}^\beta$. Hence for any F' in some Zariski-open neighbourhood of F

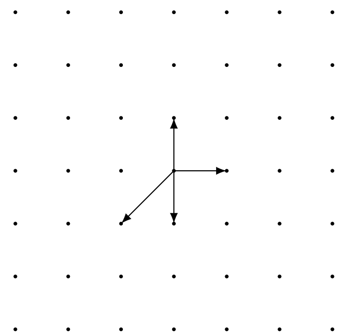
$$\mathrm{r}(F') \geq \operatorname{rank} C_{F'}^\beta \geq \operatorname{rank} C_F^\beta.$$

It follows that $\underline{\mathrm{r}}(F) \geq \operatorname{rank} C_F^\beta$. □

In the rest of the paper, we will denote the coordinates of the ring S by Greek letters $\alpha, \beta, \dots$ and the corresponding coordinates in T by $x, y, \dots$ (possibly with subscripts).

4.1. Hirzebruch surface $\mathbb{F}_1$

Consider the set $\{\rho_{\alpha,0} = (1,0), \rho_{\alpha,1} = (-1,-1), \rho_{\beta,0} = (0,1), \rho_{\beta,1} = (0,-1)\}$. Let Σ be the only complete fan such that this set is the set of rays of Σ. The example in [CLS11, Example 3.1.16] is the same, only with a different ray arrangement. Then X_Σ is called the Hirzebruch surface $\mathbb{F}_1$.



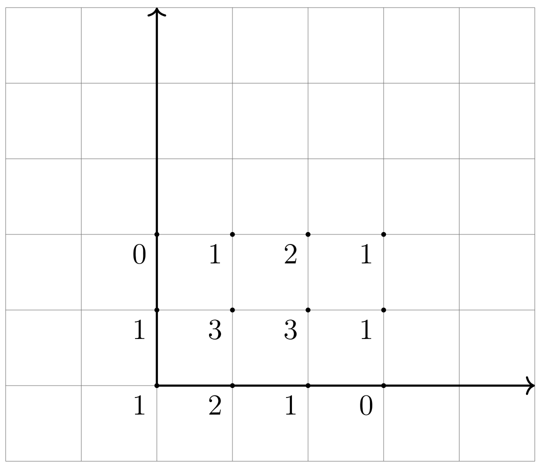
Its class group is the free abelian group on two generators $D_{\rho_{\alpha,0}} \sim D_{\rho_{\alpha,1}}$ and $D_{\rho_{\beta,1}}$. Moreover, $D_{\rho_{\beta,0}} \sim D_{\rho_{\beta,1}} + D_{\rho_{\alpha,0}}$. Here $\sim$ means the linear equivalence. Let $\alpha_0, \alpha_1, \beta_0, \beta_1$ be the variables corresponding to $\rho_{\alpha,0}$, $\rho_{\alpha,1}$, $\rho_{\beta,0}$, $\rho_{\beta,1}$. As a result, we may think of S as the polynomial ring $\mathbb{C}[\alpha_0, \alpha_1, \beta_0, \beta_1]$ graded by $\mathbb{Z}^2$, where the grading is given by

f	α_0	α_1	β_0	β_1
$\deg f$	1	1	1	0
	0	0	1	1

The nef cone in $(\mathrm{Cl}\, X_\Sigma)_\mathbb{R}$ is generated by $D_{\rho_{\alpha,0}}$ and $D_{\rho_{\beta,0}} \sim D_{\rho_{\alpha,0}} + D_{\rho_{\beta,0}}$. In this chapter, we will denote the value of $\pi : \mathbb{C}^4 \setminus Z \to X_\Sigma$ by $[\lambda_0, \lambda_1; \mu_0, \mu_1]$.

Example 4.5. Consider the monomial $F := x_0 x_1 y_0 y_1$, where x_0, x_1, y_0, y_1 is the basis dual to $\alpha_0, \alpha_1, \beta_0, \beta_1$. It has degree $(3, 2)$, so it is in the interior of the nef cone, so the corresponding line bundle is very ample. We claim that the rank and cactus rank are four, and the border rank is three.

Let us compute the Hilbert function of the apolar algebra of F.



Notice that it can only be non-zero in the first quadrant. Hence, the symmetry implies it can only be non-zero in the rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 2)$, $(0, 2)$. Computing each value of the Hilbert function is just computing the kernel of a linear map. For instance, for degree $(1, 0)$, we have

$$(a_0\alpha_0 + a_1\alpha_1) \lrcorner\, x_0x_1y_0y_1 = a_0x_1y_0y_1 + a_1x_0y_0y_1,$$

which is zero if and only if $a_0 = 0$ and $a_1 = 0$. Hence, the Hilbert function is

$$dim_\mathbb{C}(S/F^\perp)_{(1,0)} = dim_\mathbb{C} S_{(1,0)} - dim_\mathbb{C} F^\perp_{(1,0)} = 2 - 0 = 2.$$

For degree $(2,1)$, we get

$$(a\alpha_0^2\beta_1 + b\alpha_0\alpha_1\beta_1 + c\alpha_1^2\beta_1 + d\alpha_0\beta_0 + e\alpha_1\beta_0) \lrcorner x_0x_1y_0y_1 = by_0 + dx_1y_1 + ex_0y_1.$$

So the result is zero precisely for vectors of the form $(a,0,c,0,0)$, where $a,c\in\mathbb{C}$. Then

$$dim_{\mathbb{C}}(S/F^{\perp})_{(2,1)} = 5-2 = 3.$$

The apolar ideal $F^{\perp}$ is $(\alpha_0^2,\alpha_1^2,\beta_0^2,\beta_1^2)$. (It is independent of the grading, so we can just copy the result from the Waring rank case, see [RS11].)

Firstly, we will show that the rank is at most four. By the Apolarity Lemma, toric version (Proposition 3.2), it is enough to find a reduced zero-dimensional subscheme of length four R of X_Σ (i.e. a set of finitely many points in X_Σ) such that $I(R)\subseteq F^{\perp}$. The subscheme defined by $I = (\alpha_0^2-\alpha_1^2, \beta_0^2-\alpha_1^2\beta_1^2)\subseteq F^{\perp}$ satisfies these requirements. This scheme is a reduced union of four points: $[1,1;1,1],[1,1;1,-1],[1,-1;1,1],[1,-1;1,-1]$. As a consequence, we may write

$$x_0x_1y_0y_1 = \frac{1}{4}\left(\varphi(1,1;1,1)-\varphi(1,1;1,-1)-\varphi(1,-1;1,1)+\varphi(1,-1;1,-1)\right).$$

We will show that the cactus rank is at least four. Suppose it is at most three. Then there is a B-saturated homogeneous ideal $I\subseteq F^{\perp}$ defining a zero-dimensional subscheme R of length at most three. From the calculation of the Hilbert function, we know that $\dim_{\mathbb{C}} F^{\perp}_{(2,1)} = 2$. Let us calculate $\dim_{\mathbb{C}} I_{(2,1)}$. Since I is B-saturated, by Fact 1.2, the vector subspace $I_{(2,1)}\subseteq S_{(2,1)}$ are the sections which are zero on R. But from Proposition 4.1

$$3 \geq \text{length of}\, R \geq \dim_{\mathbb{C}} S_{(2,1)} - \dim_{\mathbb{C}} I_{(2,1)} = 5 - \dim_{\mathbb{C}} I_{(2,1)},$$

so

$$\dim_{\mathbb{C}} I_{(2,1)} \geq 2.$$

By the Apolarity Lemma (Proposition 3.2), we have $I_{(2,1)}\subseteq (F^{\perp})_{(2,1)}$. As the dimensions are equal, it follows that $I_{(2,1)} = (F^{\perp})_{(2,1)}$. But this means $\alpha_0^2\beta_1,\alpha_1^2\beta_1\in I$. But I is B-saturated, so $\alpha_0\alpha_1\beta_1\in I$. But $\alpha_0\alpha_1\beta_1 \lrcorner x_0x_1y_0y_1\neq 0$, a contradiction.

Let us show that border rank of F is at most three. Take $p=[\lambda,1;1,\mu]\in\mathbb{F}_1$. Then from Fact 2.4, we know that

$$\begin{aligned}[\lambda,1;1,\mu]\mapsto \lambda\mu\cdot\Big(&\lambda^2\mu x_0^3y_1^2+\lambda\mu x_0^2x_1y_1^2+\mu x_0x_1^2y_1^2+\frac{\mu}{\lambda}x_1^3y_1^2\\ &+\lambda x_0^2y_0y_1+x_0x_1y_0y_1+\frac{1}{\lambda}x_1^2y_0y_1\\ &+\frac{1}{\mu}x_0y_0^2+\frac{1}{\mu\lambda}x_1y_0^2\Big).\end{aligned}$$

But

$$[0,1;1,\mu]\mapsto \mu\cdot\left(\mu x_1^3y_1^2+x_1^2y_0y_1+\frac{1}{\mu}x_1y_0^2\right),$$

and

$$[1,0;1,0]\mapsto x_0y_0^2.$$

Hence,

$$\begin{aligned}&-x_0x_1y_0y_1+\frac{1}{\lambda\mu}\varphi([\lambda,1;1,\mu])-\frac{1}{\lambda\mu}\varphi([0,1;1,\mu])-\frac{1}{\mu}\varphi([1,0;1,0])\\ &\qquad=\lambda^2\mu x_0^3y_1^2+\lambda\mu x_0^2x_1y_1^2+\mu x_0x_1^2y_1^2+\lambda x_0y_0^2\xrightarrow{\lambda,\mu\to 0}0.\end{aligned}$$

It follows that $x_0x_1y_0y_1$ is expressible as a limit of linear combinations of three points on X_Σ, so the border rank is at most three.

But there is another proof that the border rank of F is at most three. We will show that the third secact variety $\sigma_3(X) = \mathbb{P}^8$. It suffices to show that $\dim \sigma_3(X_\Sigma)$ is eight. The expected dimension is eight, so we may use Terracini's Lemma. Let us recall it.

Fact 4.6 (Terracini's Lemma)**.** *Suppose $\varphi : X \hookrightarrow \mathbb{P}^N$ is a subvariety. Let r be a positive integer. Then for r general points $p_1, \dots, p_r \in X$ and a general point $q \in \langle p_1, \dots, p_r\rangle$ we have*

$$T_q\sigma_r(X) = \langle T_{p_1}X, \dots, T_{p_r}X\rangle.$$

Here T_qX denotes the tangent space of X embedded in the projective space at point q.

For a proof, see [Lan12, Section 5.3].

Since $X_\Sigma \to \mathbb{P}(H^0(X_\Sigma, \mathcal{O}(\alpha))^*)$ is given by a parametrization, we can calculate the tangent space. Take points of the form $[1, \lambda; \mu, 1]$, where $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. Then

$$\varphi([1, \lambda; \mu, 1]) = [1, \lambda, \lambda^2, \lambda^3, \mu, \mu\lambda, \mu\lambda^2, \mu^2, \mu^2\lambda].$$

The affine tangent space at $\varphi([1, \lambda; \mu, 1])$ is spanned by the vector

$$v = [1, \lambda, \lambda^2, \lambda^3, \mu, \mu\lambda, \mu\lambda^2, \mu^2, \mu^2\lambda]$$

and its two derivatives with respect to λ and μ:

$$\begin{aligned}\frac{\partial v}{\partial \lambda} &= [0, 1, 2\lambda, 3\lambda^2, 0, \mu, 2\mu\lambda, 0, \mu^2],\\ \frac{\partial v}{\partial \mu} &= [0, 0, 0, 0, 1, \lambda, \lambda^2, 2\mu, 2\mu\lambda],\end{aligned}$$

If we take three general points, say $[1, x, y, 1], [1, s, t, 1], [1, u, v, 1]$, we can take a look at the space spanned by the three tangent spaces. This will be the space spanned by the rows of the following matrix:

$$M = \begin{pmatrix} 1 & x & x^2 & x^3 & y & yx & yx^2 & y^2 & y^2x \\ 0 & 1 & 2x & 3x^2 & 0 & y & 2yx & 0 & y^2 \\ 0 & 0 & 0 & 0 & 1 & x & x^2 & 2y & 2yx \\ 1 & s & s^2 & s^3 & t & ts & ts^2 & t^2 & t^2s \\ 0 & 1 & 2s & 3s^2 & 0 & t & 2ts & 0 & t^2 \\ 0 & 0 & 0 & 0 & 1 & s & s^2 & 2t & 2ts \\ 1 & u & u^2 & u^3 & v & vu & vu^2 & v^2 & v^2u \\ 0 & 1 & 2u & 3u^2 & 0 & v & 2vu & 0 & v^2 \\ 0 & 0 & 0 & 0 & 1 & u & u^2 & 2v & 2vu \end{pmatrix}$$

We can calculate the determinant using for instance Macaulay2

$$\det M = (s-u)(u-x)(s-x)(ys - xt - yu + tu + xv - sv)^4.$$

This is non-zero for general points on the variety. This means that the tangent space of the third secant variety at a general point has affine dimension nine, so $\dim \sigma_3(X_\Sigma) = 8$, hence $\sigma_3(X)$ fills the whole space.

Finally, the border rank is at least three by Corollary 4.4. We are using it for the class $(2, 1)$, recall that $\dim_{\mathbb{C}}(S/F^\perp)_\beta = C_F^\beta$.

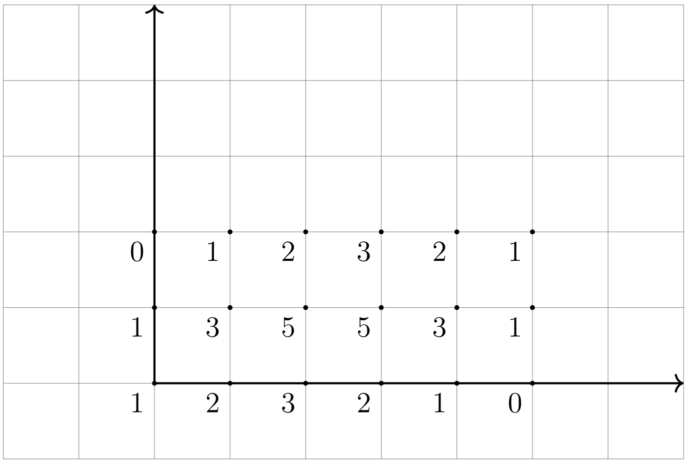
Remark 4.7. We could also define the smoothable X-rank:

$$\mathrm{sr}_X(F) = \min\{\mathrm{lh}\, R : R \hookrightarrow X, \dim R = 0, F \in \langle R \rangle, R \text{ smoothable}\}.$$

For the definition of a smoothable scheme, see [IK99, Definition 5.16]. For more on the smoothable rank, see [BB13]. We always have $\mathrm{cr}(F) \leq \mathrm{sr}(F) \leq \mathrm{r}(F)$, so in the case of $\mathbb{F}_1$ and $F = x_0x_1y_0y_1$ we get $\mathrm{sr}(F) = 4$. In particular, we obtain what the authors in [BB13] call a "wild" case, i.e. the border rank is strictly less than the smoothable rank.

Example 4.8. For a similar case on the same variety, let $F = x_0^2x_1^2y_0y_1$, then $\deg F = (5, 2)$. Here the line bundle $\mathcal{O}(5, 2)$ gives an embedding of X_Σ into $\mathbb{P}^{14}$. We will show that here the rank and the cactus rank are six, and that the border rank is five.

The apolar ideal is $F^\perp = (\alpha_0^3, \alpha_0^3, \beta_0^2, \beta_1^2)$. The Hilbert function of $S/F^\perp$ is the following:



The ideal $I = (\alpha_0^3 - \alpha_1^3, \beta_0^2 - \beta_1^2\alpha_1^2) \subseteq F^\perp$ is a B-saturated radical homogeneous ideal defining a subscheme of length six, so the rank is at most six.

Suppose there is a homogeneous B-saturated ideal $I \subseteq F^\perp$ defining a subscheme of length five. We have

$$\begin{aligned} S_{(3,1)} &= \langle \alpha_0^2\beta_0, \alpha_0\alpha_1\beta_0, \alpha_1^2\beta_0, \alpha_0^3\beta_1, \alpha_0^2\alpha_1\beta_1, \alpha_0\alpha_1^2\beta_1, \alpha_1^3\beta_1 \rangle, \text{ and} \\ (F^\perp)_{(3,1)} &= \langle \alpha_0^3\beta_1, \alpha_1^3\beta_1 \rangle. \end{aligned}$$

From Propostion 4.1 we have $\dim_{\mathbb{C}}(S/I)_{(3,1)} \leq 5$, so $\dim_{\mathbb{C}} I_{(3,1)} \geq 7 - 5 = 2$. But $I_{(3,1)} \subseteq (F^\perp)_{(3,1)}$ from the Apolarity Lemma (Proposition 3.2), and also $\dim_{\mathbb{C}}(F^\perp)_{(3,1)} = 2$. This means that $I_{(3,1)} = (F^\perp)_{(3,1)}$.

Hence, $\alpha_0^3\beta_1, \alpha_1^3\beta_1 \in I$. As I is B-saturated, we get $\alpha_0^2\alpha_1^2\beta_1 \in I \subseteq F^\perp$, but this is a contradiction since $\alpha_0^2\alpha_1^2\beta_1 \lrcorner F \neq 0$.

The border rank is at least five because of Corollary 4.4. Similarly to what we did before, we show that fifth secant variety fills the whole space, so the border rank of any polynomial is at most five. Here $\varphi = \varphi_{|\mathcal{O}(5,2)|}$ is given by

$$[1, \lambda; \mu, 1] \mapsto [1, \lambda, \lambda^2, \lambda^3, \lambda^4, \lambda^5, \mu, \lambda\mu, \lambda^2\mu, \lambda^3\mu, \lambda^4\mu, \mu^2, \lambda\mu^2, \lambda^2\mu^2, \lambda^3\mu^2].$$

The tangent space is spanned by $v = \varphi(1, \lambda; \mu, 1)$ and the two derivatives

$$\begin{aligned} \frac{\partial v}{\partial \lambda} &= [0, 1, 2\lambda, 3\lambda^2, 4\lambda^3, 5\lambda^4, 0, \mu, 2\lambda\mu, 3\lambda^2\mu, 4\lambda^3\mu, 0, \mu^2, 2\lambda\mu^2, 3\lambda^2\mu^2], \\ \frac{\partial v}{\partial \mu} &= [0, 0, 0, 0, 0, 0, 1, \lambda, \lambda^2, \lambda^3, \lambda^4, 2\mu, 2\lambda\mu, 2\lambda^2\mu, 2\lambda^3\mu]. \end{aligned}$$

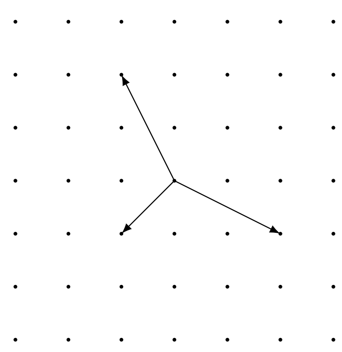
If we take five points, say $[1,x;y,1],[1,u;v,1],[1,s;t,1],[1,a,b,1],[1,c,d,1]$, we get that the tangent space of $\sigma_5(X)$ is spanned by the rows of the following matrix:

$$\begin{pmatrix}
1 & x & x^2 & x^3 & x^4 & x^5 & y & xy & x^2y & x^3y & x^4y & y^2 & xy^2 & x^2y^2 & x^3y^2 \\
0 & 1 & 2x & 3x^2 & 4x^3 & 5x^4 & 0 & y & 2xy & 3x^2y & 4x^3y & 0 & y^2 & 2xy^2 & 3x^2y^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & x & x^2 & x^3 & x^4 & 2y & 2xy & 2x^2y & 2x^3y \\
1 & s & s^2 & s^3 & s^4 & s^5 & t & st & s^2t & s^3t & s^4t & t^2 & st^2 & s^2t^2 & s^3t^2 \\
0 & 1 & 2s & 3s^2 & 4s^3 & 5s^4 & 0 & t & 2st & 3s^2t & 4s^3t & 0 & t^2 & 2st^2 & 3s^2t^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & s & s^2 & s^3 & s^4 & 2t & 2st & 2s^2t & 2s^3t \\
1 & u & u^2 & u^3 & u^4 & u^5 & v & uv & u^2v & u^3v & u^4v & v^2 & uv^2 & u^2v^2 & u^3v^2 \\
0 & 1 & 2u & 3u^2 & 4u^3 & 5u^4 & 0 & v & 2uv & 3u^2v & 4u^3v & 0 & v^2 & 2uv^2 & 3u^2v^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & u & u^2 & u^3 & u^4 & 2v & 2uv & 2u^2v & 2u^3v \\
1 & a & a^2 & a^3 & a^4 & a^5 & b & ab & a^2b & a^3b & a^4b & b^2 & ab^2 & a^2b^2 & a^3b^2 \\
0 & 1 & 2a & 3a^2 & 4a^3 & 5a^4 & 0 & b & 2ab & 3a^2b & 4a^3b & 0 & b^2 & 2ab^2 & 3a^2b^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a^2 & a^3 & a^4 & 2b & 2ab & 2a^2b & 2a^3b \\
1 & c & c^2 & c^3 & c^4 & c^5 & d & cd & c^2d & c^3d & c^4d & d^2 & cd^2 & c^2d^2 & c^3d^2 \\
0 & 1 & 2c & 3c^2 & 4c^3 & 5c^4 & 0 & d & 2cd & 3c^2d & 4c^3d & 0 & d^2 & 2cd^2 & 3c^2d^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & c & c^2 & c^3 & c^4 & 2d & 2cd & 2c^2d & 2c^3d
\end{pmatrix}$$

If we set $(x,y,s,t,u,v,a,b,c,d) = (1,2,3,4,5,6,7,9,0,2)$ and calculate the determinant in the field $\mathbb{Z}/101$, we get 34, something non-zero. This means that the determinant calculated in $\mathbb{C}$ is also non-zero at this point, so it is non-zero on a dense open subset. Hence by Terracini's lemma (Fact 4.6) the affine dimension of $\sigma_5(X)$ is fifteen. It follows that $\sigma_5(X) = \mathbb{P}^{14}$, so the border rank of F is five.

4.2. Fake projective plane

Consider a set of rays $\{\rho_0 = (-1,-1), \rho_1 = (2,-1), \rho_2 = (-1,2)\}$. Let Σ be the only complete fan uniquely determined by these rays. Then X_Σ is is an example of a fake weighted projective space, see [Buc08, 6.2].



Let $\alpha_0, \alpha_1, \alpha_2$ be the corresponding coordinates in S. The class group is generated by $D_{\rho_0}, D_{\rho_1}, D_{\rho_2}$ with relations $D_{\rho_0} \sim 2D_{\rho_1} - D_{\rho_2} \sim 2D_{\rho_2} - D_{\rho_1}$. This is the same as a group with two generators D_{ρ_0} and $D_{\rho_2} - D_{\rho_1}$ with the relation $3(D_{\rho_2} - D_{\rho_1})$. This choice gives an isomorphism with $\mathbb{Z} \times \mathbb{Z}/3$ sending D_{ρ_0} to $(1,0)$ and $D_{\rho_2} - D_{\rho_1}$ to $(0,1)$. The Picard group is the subgroup generated by $3D_{\rho_0}$. It is free.

As a result, $S = \mathbb{C}[\alpha_0, \alpha_1, \alpha_2]$ is graded by $\operatorname{Cl} X_\Sigma = \mathbb{Z} \times \mathbb{Z}/3$, where

$$\begin{aligned}\deg \alpha_0 &= (1,0),\\ \deg \alpha_1 &= (1,1),\\ \deg \alpha_2 &= (1,-1) = (1,2),\end{aligned}$$

and $\operatorname{Pic} X_\Sigma$ is generated by $(3,0)$.

Example 4.9. Consider the line bundle $\mathcal{O}(6,0)$. It is ample, because by [CLS11, Proposition 6.3.25] every proper toric surface is projective, and the line bundles $\mathcal{O}(-3m,0)$ for $m < 0$ have no non-zero sections. By [CLS11, Proposition 6.1.19, (b)] it is very ample. It gives an embedding $\varphi : X_\Sigma \hookrightarrow \mathbb{P}^9$. We denote the dual coordinates by x_0, x_1, x_2. Let $F = x_0^4 x_1 x_2$. The apolar ideal is $(\alpha_0^5, \alpha_1^2, \alpha_2^2)$. We claim that the cactus rank is two, and the rank is at most five. Note that F is not in the image of φ, so the cactus rank and rank are at least two.

We will show that the cactus rank is two. Consider the ideal $I = (\alpha_1^2, \alpha_2^2) \subseteq F^\perp$. It is saturated, since B in this case is $(\alpha_0, \alpha_1, \alpha_2)$, so it is the same as in the case of $\mathbb{P}^2$. We will show that the length of the subscheme given by I is two. Since the support of the scheme is the point $[1,0,0]$, we check it on the set U_σ, where $\sigma = \operatorname{Cone}(\rho_1, \rho_2)$. We localize with respect to α_0, take degree zero, and get the ring

$$\mathbb{C}\left[\frac{\alpha_1^3}{\alpha_0^3}, \frac{\alpha_2^3}{\alpha_0^3}, \frac{\alpha_1\alpha_2}{\alpha_0^2}\right] \cong \mathbb{C}[u,v,w]/(w^3 - uv). \tag{4.1}$$

If we factor out by the ideal generated by α_1^2 and α_2^2, we get

$$\mathbb{C}[u,v,w]/(w^3 - uv, u, v, w^2) \cong \mathbb{C}[w]/(w^2),$$

so the length of the scheme defined by I is two.

Now we show that the rank is at most five. Take a homogeneous ideal $I = (\alpha_0^5 - \alpha_1^4\alpha_2, \alpha_1^3 - \alpha_2^3) \subseteq F^\perp$. We show that the length of the subscheme defined by I is five. From these equations we know that no coordinate can be zero, so we can check the length on the open subset U_σ, where $\sigma = \operatorname{Cone}(\rho_1, \rho_2)$. We get the same ring as in Equation 4.1, and we want to factor it out by the ideal generated by $\alpha_0^5 - \alpha_1^4\alpha_2$ and $\alpha_1^3 - \alpha_2^3$. The second generator gives the relation $u - v$, and the first one the relation $1 - vw$. So we get the ring

$$\mathbb{C}[v,w]/(w^3 - v^2, 1 - vw).$$

But notice that $1 = vw$ implies that w is non-zero. Hence

$$\begin{aligned}\mathbb{C}[v,w]/(w^3 - v^2, 1 - vw) &\cong \mathbb{C}[v,w,w^{-1}]/(w^3 - v^2, 1 - vw)\\ &\cong \mathbb{C}[v,w,w^{-1}]/(w^5 - 1, w^{-1} - v) \cong \mathbb{C}[w,w^{-1}]/(w^5 - 1).\end{aligned}$$

We get a reduced scheme of length five, so the rank is at most five.

Example 4.10. Now take $F = x_0^2 x_1^2 x_2^2$. Here the apolar ideal is $F^\perp = (\alpha_0^3, \alpha_1^3, \alpha_2^3)$. The cactus rank is at least two (because F is not in the image of φ). Let $I = (\alpha_0^3 - \alpha_1^3, \alpha_1^3 - \alpha_2^3)$. In this case also no coordinate can be zero, so we may calculate the length on U_σ (where σ is as before). We get the ring as in Equation 4.1 and the two generators become $1 - u$ and $u - v$. So here the quotient ring is

$$\mathbb{C}[w]/(w^3 - 1).$$

This means that the rank is at most three (notice that we get a reduced scheme).

Remark 4.11. We can calculate the Hilbert function of $A_F = S/F^{\perp}$ (where $F = x_0^2 x_1^2 x_2^2$):

	0	1	2	3	4	5	6
$1 \mod 3$	0	1	2	3	2	1	0
$0 \mod 3$	1	1	2	1	2	1	1
$-1 \mod 3$	0	1	2	3	2	1	0

We have $dim_{\mathbb{C}}(A_F)_{(3,1)} = 3$, so this would give that the cactus rank and rank of F are three, if the catalecticant bound worked for classes that are not in $\operatorname{Pic} X_\Sigma$ (i.e. reflexive sheaves of rank one that are not line bundles). But the proof of Proposition 4.1 does not work in this case (or at least we would have to divide by the supremum of the ranks of fibres of the sheaf, which would make the bound weaker).

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