K-theory

Aim: Make $\oplus$ in $Vect(X)$ into a group operation.

We will assume that our base space X is compact Hausdorff and vector bundles will be over $\mathbb{C}$.

Prop. If $p: E \to X$ is a vector bundle over compact space X, then there exists bundle $E' \to X$ such that
$$E \oplus E' \to X \text{ is trivial.}$$

Some notations: • trivial n-dim. vector bundle will be denoted by ε^n

• stably isomorphic bundles we denote: $E_1 \approx_s E_2$ iff $E_1 \oplus \varepsilon^n \simeq E_2 \oplus \varepsilon^n$ for some n.

• $E_1 \sim E_2$, iff $E_1 \oplus \varepsilon^n \simeq E_2 \oplus \varepsilon^m$ for some m, n.

Note that $\approx_s$, $\sim$ are equivalence relations.

We can „add" equivalence classes of either relations.

<u>Proposition</u>: If X is compact, then the set of $\sim$-equivalence classes of vect. bundles over X is an abelian group.

$\tilde{K}(X)$ - reduced K-theory of X

$E \to X \quad , \quad E' \to X \quad : \quad E \oplus E' \cong \varepsilon^n$

$E \oplus E' \sim 0$

We want to construct a group on $\approx_s$-equivalence classes.

Construction of $K(X)$ (Grothendieck group)

$K(X)$ consists of differences: $E - E'$ of vect. bundles over X.

$$E_1 - E_1' = E_2 - E_2' \iff E_1 \oplus E_2' \approx_s E_2 \oplus E_1'$$

Cancellation property: $E_1 \oplus E_2 \approx_s E_1 \oplus E_3 \Rightarrow E_2 \approx_s E_3$.

Define addition in $K(X)$:

$$(E_1 - E_1') + (E_2 - E_2') = E_1 \oplus E_2 - E_1' \oplus E_2'.$$

The kernel consists of form. differences:

$$\varepsilon^n - \varepsilon^m$$

$$\langle \varepsilon^n - \varepsilon^m \rangle \subseteq K(X) \text{ is isomorphic to } \mathbb{Z}$$

$$K(X) \longrightarrow K(x_0) \simeq \mathbb{Z} \qquad x_0 \in X \text{ - basepoint}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & K(X) \arrow[l, bend right] \arrow[r] & \tilde{K}(X) \arrow[r] & 0 \\
& K(x_0) \arrow[u, phantom, "\simeq" rotate=90] & & &
\end{tikzcd}$$

$$K(X) \simeq \tilde{K}(X) \oplus \mathbb{Z}$$

(requires choosing a basepoint)

Ring structure in $K(X)$

For any $E_1, E_2 \in K(X)$ we define:

$$E_1 \cdot E_2 = E_1 \otimes E_2$$

The identity el. is $\mathbb{C}^1$. Note that if we choose a basepoint we have that $\tilde{K}(X)$ is also a ring (not necessarily with identity).

$K(X)$ is a functor from homotopy category of top. spaces into ring category

$$X \xrightarrow{f} Y \mapsto f^*: K(Y) \to K(X)$$

$$E - E' \mapsto f^*(E) - f^*(E')$$

$$f^*(E_1 \oplus E_2) \simeq f^*(E_1) \oplus f^*(E_2)$$

$$f^*(E_1 \otimes E_2) \simeq f^*(E_1) \otimes f^*(E_2).$$

If $f \simeq g$ then $f^* = g^*$. (similarly for $\tilde{K}(X)$ but we need a basepoint).

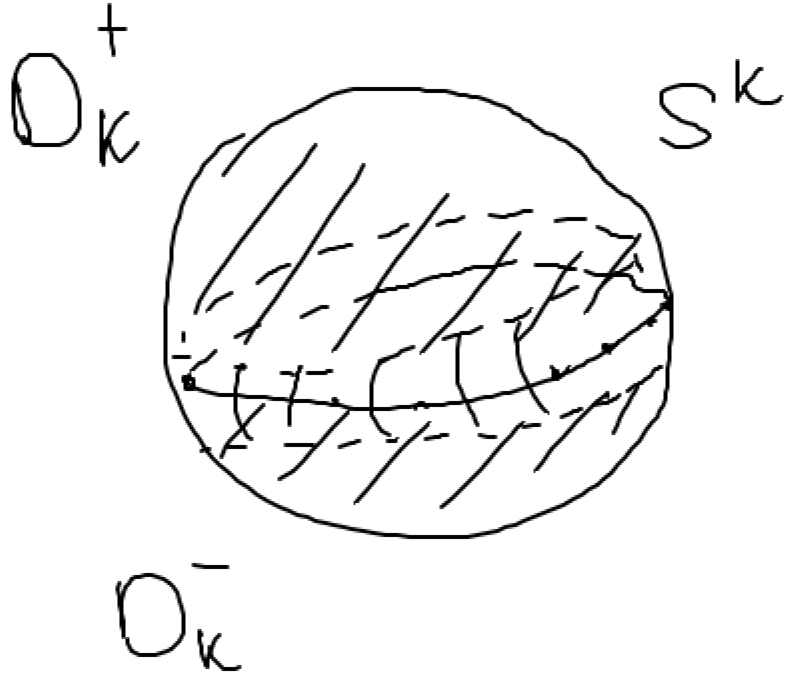
We define an external product:

$$\mu : K(X) \otimes K(Y) \longrightarrow K(X \times Y)$$

by $\quad \mu(a \otimes b) = p_1^*(a)\, p_2^*(b)$

μ is ring homomorphism

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_1"'] \arrow[dr] & \\ X & & Y \end{tikzcd}$$



$D_k^+ \cap D_k^- = S^{k-1}$

$$D_k^+ \times \mathbb{C}^n \sqcup D_k^- \times \mathbb{C}^n \,/\, \sim \;= E_f$$

$(x, v) \sim (x, f(x)v)$ for any $x \in S^{k-1}$ and $v \in \mathbb{C}^n$.

$p : E_f \longrightarrow S^k$ – this is a vector bundle

Note that homotopic maps induce isomorphic bundles

Theorem: (classifying vect. bundles over sphere)

$$[S^{k-1}, GL_n(\mathbb{C})] \longrightarrow Vect^n_{\mathbb{C}}(S^k)$$
$$[f] \longmapsto E_f$$

above map is bijection.

$E \longrightarrow S^k$ $\quad$ $- E|_{D_k^+}, E|_{D_k^-}$ - trivial bundles

h_1, h_2 - trivializations

$$S^{k-1} \longrightarrow GL_n(\mathbb{C}) \qquad x \mapsto h_2 h_1^{-1}(x)$$

Example: Any vector bundle over S^1 is trivial.

$$K(S^1) = \mathbb{Z}$$

H be a tautological line bundle over $S^2 \simeq \mathbb{C}P^1$

$$\mathbb{C}P \simeq \mathbb{C}^2 \setminus \{0\} / \mathbb{C}^*$$

$$H : \{(x, v) : v \in x\}$$

We want to see that clutching function of H is

$S^1 \to \mathbb{C}^* \qquad z \mapsto (z)$

$\mathbb{CP}^1 \simeq \mathbb{C} \cup \{\infty\} \simeq S^2$

$[z_0 : z_1] \mapsto \frac{z_0}{z_1}$

$D_0 D_\infty \subset \mathbb{CP}^1$

$D_0 = \{ |z| \leq 1 \}$

$D_\infty = \{ |z| \geq 1 \}$

On D_0 we have section:

$$z \mapsto (z, 1) \quad , \quad |z| \leq 1$$

On D_∞ we have section: $z \mapsto (1, z^{-1})$, $|z^{-1}| \leq 1$

We need to identify on S^1 vect. $(z,1) \sim (1, z^{-1})$

Clutching function is (z) or (z^{-1})

α induces a homotopy:

$$(f\oplus\mathbb{1})\alpha_t(\mathbb{1}\oplus g)\alpha_t$$

between $f\oplus g$ and $fg\oplus\mathbb{1}$. Therefore:

$$E_f\oplus E_g\simeq E_{fg}\oplus\varepsilon^n$$

In particular:

$$(H\otimes H)\oplus\mathbb{1}\simeq H\oplus H$$

In $K(S^2)$: $H^2+1=2H \Rightarrow (H-1)^2=0.$

We have ring homom.:

$$\mathbb{Z}[H]/(H-1)^2 \longrightarrow K(S^2)$$

turns out to be an isomorphism. More generally:

$$K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \xrightarrow{\simeq} K(X \times S^2)$$

<u>Bott periodicity theorem</u>. $\tilde{K}(X) \simeq \tilde{K}(S^2 X)$

Exact sequences

X - compact

Proposition: If we have $A \subseteq X$ - closed, then:

$$A \xrightarrow{i} X \xrightarrow{q} X/A \qquad A/A$$

induce:

$$\tilde{K}(X/A) \xrightarrow{q^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(A)$$

sequence is exact.

Obs. X/A is also compact Hausdorff

Proof: $q\,i : A \to A/A$, $\tilde{K}(A/A) = 0$

$i^* q^* = 0$

The only thing to prove is $\ker i^* \subseteq \operatorname{im} q^*$.

$p : E \to X$ such that $p|_A$ is trivial bundle.

$p^{-1}(A) \xrightarrow{h} A \times \mathbb{C}^n$ and let : A/A

E/h : quotient space under it.

$h^{-1}(x,v) \sim h^{-1}(y,v)$

for any $x, y \in A$.

(φ_j, φ) - partition of unity sub. to cover:
$\{V_j, X \setminus A\}$

$\tau_i := \sum_j \varphi_j s_{ij}$ - extension of section s_i defined on X

τ_i forms a basis in fiber over A
τ_i form a basis in some op. neighb. of A

$E/h \longrightarrow X/A$ is a bundle

is isomorphism on fibers

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "p"'] & E/h \arrow[d] \\
X \arrow[r, "q"'] & X/A
\end{tikzcd}$$

$E \longrightarrow X$ is in $\mathrm{im}\, q^*$

__Lemma:__ If $A \subseteq X$ closed subset, contractible, then:

$$X \xrightarrow{q} X/A \rightsquigarrow q^*: \mathrm{Vect}^n(X/A) \xrightarrow{\cong} \mathrm{Vect}(X)$$

$$\cdots \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A)$$