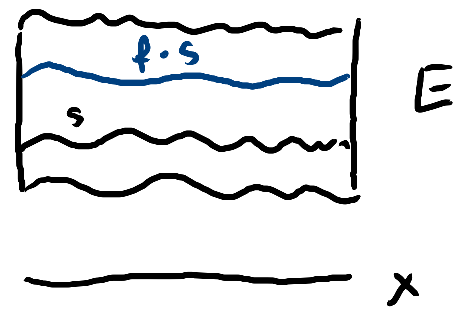
Vector bundles as projective modules

$\{ X \xrightarrow{\text{cont.}} \mathbb{C} \}$

Aim: Show that $\Gamma : \mathrm{Vect}(X) \xrightarrow{\text{equiv}} \{\text{fin. gen. proj. } C(X)\text{-mod}\}$

① $\Gamma(E)$ is a $C(X)$-module



The module structure is given by
$$(f \cdot s)(x) = f(x) \cdot s(x)$$
for $f \in C(X)$, $s \in \Gamma(E)$

Because homomorphisms $\varphi : E \to F$ are linear on fibers

$$\Gamma_\varphi : \Gamma(E) \longrightarrow \Gamma(E)$$
$$s \longmapsto \varphi \circ s$$

are $C(X)$-modules homomorphisms

Thus we constructed a functor $\Gamma : \mathrm{Vect}(X) \longrightarrow C(X)\text{-Mod}$

② $\{\text{trivial bundles}\} \xrightarrow{\simeq} \left\{\begin{array}{l}\text{free } C(X)\text{-mod} \\ \text{with finite rank}\end{array}\right\}$

WLOG $E = X \times V$. Then E has sections $s_i(x) = (x, e_i)$

and $\Gamma(E) = \bigoplus C(X) \cdot s_i$

This map is essentially surjective

Now consider $F = X \times W$

$$HOM(E,F) \overset{\text{ok!}}{\cong} \Gamma(Hom(E,F)) \overset{(*)}{\cong} Hom_{C(X)}(\Gamma(E), \Gamma(F))$$

Let $f \in \Gamma(Hom(F,E))$. Let $\mathfrak{I}_i^j(v_e) = \begin{cases} w_j \ (\text{base vect.}) & : \text{for } e = i \\ 0 & : \text{for } e \neq i \end{cases}$

Then $f = \sum f_{ij} \mathfrak{I}_i^j$, $f_{ij} \in C(X)$, but this is precisely what elements of $Hom_{C(X)}$ look like

Thus Γ is fully faithfull

③ Let $\mathrm{Proj}\,\mathrm{Vect}(X) \subseteq \mathrm{Vect}(X)$: a subcategory whose objects
$\mathrm{Proj}\,C(X) \subseteq C(X)\text{-Mod}$: are images of proj. operators

$$\begin{tikzcd}
\{\text{Trivial bundles}\} \arrow[r, shift left] \arrow[r, phantom, "\simeq"] & \{\text{Free } C(X) \text{ of finite rank}\} \arrow[l, shift left] \arrow[d, "\simeq"] \\
\Gamma : \mathrm{Proj}\,\mathrm{Vect}(X) \arrow[r] & \{\text{projective } C(X)\text{-modules of finite rank}\} \\
\mathrm{Vect}(X) \arrow[u, "\simeq"'] &
\end{tikzcd}$$

Then
$\Gamma : \mathrm{Proj}\,\mathrm{Vect}(X) \longrightarrow \{\text{projective } C(X)\text{-modules of finite rank}\}$

$\uparrow \simeq$ $\mathrm{Vect}(X)$, as for E : bundle, there exist F such that $E \oplus F$ is trivial

Finally $\Gamma : \mathrm{Vect}(X) \xrightarrow{\simeq} \{\text{fin. gen. proj. } C(X)\text{-mod}\}$

Additional structures

Last time, we considered hermitian metric
Some other classic structures (V: vector bundle)

- (non-degenerate) bilinear form on V is an element of $\Gamma(\mathrm{Hom}(V \otimes V, \underline{\mathbb{1}}))$ s.t. for every x, the induced element of $\mathrm{Hom}(\underline{V_x \otimes V_x, \mathbb{C}})$ is non-deg.
  $\equiv \Gamma(\mathrm{Iso}(V, V^*))$
  The pair (V, T), $T \in \mathrm{ISO}(V, V^*)$ is called a self-dual bundle
- For (V, T) as above, T: symmetric, we have an orthogonal bundle („a choice of an inner product for every $x \in X$")

  —— " ——, T: skew-symmetric, we have a symplectic bundle („a choice of a symplectic form ...")

We can even recover a real vector bundle W s.t $V = W \otimes_{\mathbb{R}} \mathbb{C}$
Recall that for a vector space k, a complex conjugate space $\bar{k}$ is the vector space s.t

- $|\bar{k}| = |k|$
- $\alpha \cdot_{\bar{k}} v = \bar{\alpha} \cdot_k v$

- A self-conjugate bundle is a pair (V, T), $T \in \mathrm{ISO}(V, \bar{V})$.
- A real bundle is a self-conjugate (complex) bundle such that $T^2 = id$. Why??

$f(\alpha v) = \bar{\alpha} f(v)$

$T \in \mathrm{ISO}(V, \bar{V})$ can be thought of as an anti-linear isomorphism $V \to V$.
Let $W = \{ v = T(v) \}$. Then W is a real vector bundle
Take $v \in W$. Then $T(iv) - iv = -iT(v) - iv = -2iv$
Thus, as $T^2 = id$, $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}} V$ and $V \cong W \otimes_{\mathbb{R}} \mathbb{C}$

- in similar manner, if $T^2 = -\mathrm{id}$, then (V,T) is a quaternion bundle. We can define a quaternion vector space on each V_x by $j \cdot v = T(v)$

division ring

Generally: Let F, G: continuous functors on vector spaces. By an $F \to G$ bundle we mean a pair (V,T) where $T \in \mathrm{Iso}(F(V), G(V))$

Ex
- self-conjugate bundle: $F = \mathrm{id}$, $G = *$
- an $m \to m$ bundle: $F(V) = G(V) = \underbrace{V \oplus V \oplus \dots \oplus V}_{m}$ (m-bundle)

The m-bundle (V,T) is trivial, if there exists $S \in \mathrm{Aut}(V)$ s.t. $T = mS$

Note that a hermitian metric h gives an iso $\bar{V} \to V^*$

Thus {self-conjugate bundles} <-> {self-dual bundles}

In fact, up to homotopy:

self-conjugate		self-dual
orthogonal	<-->	real
symplectic		quaternion

Then we can apply previous results!

G-Bundles over G-spaces

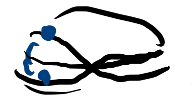
Let G be a topological group.
Then a G-space is a topological space with a continuous action $G \times X \longrightarrow X$.
A G-map between G-spaces is a map commuting with the action of G.
A G-vector bundle E is G-space over the G-space X s.t.
1) E is a vector bundle over X
2) the projection $E \to X$ is a G-map
3) for each $g \in G$, the map $E_x \longrightarrow E_{g(x)}$ is a vector space homomorphism.

For us: G is finite!

1) X is a free G-space if $g \neq e \Rightarrow g(x) \neq x$
2) X is a trivial G-space, if $g(x) = x$ for all x, g.

- Moebius band and the action of $\mathbb{Z}_2 = \{0,1\}$

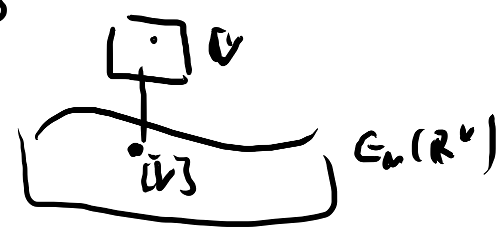
$$1 \cdot (x,v) = (x,-v)$$



Here X is a trivial $\mathbb{Z}_2$-space

- Take the tautological bundle E of a grassmannian. Then GL acts on E

$$g \cdot (V, v) = (g[V], g(v))$$



Here, X ~~is a~~ free GL-space

- Take $X = \{*\}$. Then G-bundles over X are the same as representations of group G

We want to understand G-bundles via G-representations

Quick intro to group representations

Take M : a finite dimensional vector space $/\mathbb{C}$

We have two equivalent definitions of a G-representation (repr. of group G)

$\sum_{g \in G} c_g g$ $\varphi : \mathbb{C}G \longrightarrow \mathrm{End}_{\mathbb{C}}(M)$ (left module)	$\varphi : G \longrightarrow GL_{\mathbb{C}}(V)$ (action)
simple modules (no proper non-zero submodules)	irreducible representation, ie (φ, V) s.t. no $0 \neq W \subsetneq V$ is a sub repr.

There is a finite set $V_1 \dots V_k$ of irr. reps of G

$\Uparrow$ Maschke + Wedderburn	$\Uparrow$ [Fulton] direct calculation

Consequently every G-repr $V = \bigoplus n_i V_i$.

Note that for V_i, V_j irreducible,

$$\mathrm{Hom}_G(V_i, V_j) = \begin{cases} 0, & \text{for } i \neq j \\ \mathbb{C}, & \text{for } i = j \end{cases}$$

Sketch: by irr. such homomorphism is linear iso or 0 (ker, im are G-inv)

So nonzero $\mathrm{Hom}_G(V_i, V_i)$ is an automorphism of repr. Take $f, g \in \mathrm{Hom}_G(V_i, V_i)$. By $\mathbb{C} \in ACF$,

$\det(f - \lambda g) = 0$ for some λ

$\overset{\text{irr}}{\Longrightarrow} \quad f = \lambda g$

Thus $^{(*)}$ $\sum V_i \otimes \mathrm{Hom}_G(V_i, V) \longrightarrow V$ is a G-isomorphism

$$v \otimes f \longmapsto f(v)$$

$$(g \cdot v) \otimes g \circ f g^{-1} \longmapsto g(f(v))$$

Now we will prove the following theorem:

Let X be a trivial G-space; $W_1 \dots W_n$: a complete set of irr. G-repr, $V_i = X \times W_i$: corr. G-bundle

(+) Then every G-bundle F over X is isomorphic to a direct sum $\sum V_i \otimes E_i$, where E_i are vector bundles with trivial G-bundle

(‡) Moreover, E_i are unique up to $\cong$ and given by $E_i = \mathrm{Hom}_G(V_i, F)$

<u>Proof</u> We shall extend (*) to

$$\sum V_i \otimes \mathrm{Hom}_G(V_i, F) \xrightarrow{\cong} F$$

Thus to prove (+), it is enough to show that G acts trivially on $\mathrm{Hom}_G(V_i, F)$ and that $\mathrm{Hom}_G(V_i, F)$ is a vector bundle.

Define an endomorphism A_V of E:

$$A_V(e) = 1/|G| \sum_{g \in G} g(e)$$

Note that this is projection and its image is a vector bundle.

$\operatorname{im} A_V$ = „invariant subspace of E" $=: E^G$.

Consequently, for F: G-vector bundle, $\operatorname{Hom}_G(E,F) = \operatorname{Hom}(E,F)^G$
So G acts trivially on $\operatorname{Hom}_G(E,F)$.

$g \cdot f(v) = f(gv)$

$gfg^{-1} = f$

To show $(\ddagger)$ compute

$$\begin{aligned} \operatorname{Hom}_G(V_i, F) &\cong \sum \operatorname{Hom}_G(V_i, V_j \otimes E_j) \\ &\cong \sum \operatorname{Hom}_G(V_i, V_j) \otimes E_j \\ &\cong E_i \end{aligned}$$

Now, back to general, compact G-space X. We want to extend some previous results.

Let X be a compact G-space, $Y \subseteq X$ a closed, G-inv subspace, E : G-bundle over X. Then any inv. section $s : Y \longrightarrow E|Y$ extends to an inv section over X.

We can extend s to some section t of E over X.
Take $Av(t) = \frac{1}{|G|} \sum_{g \in G} g \cdot t = \frac{1}{|G|} \sum_{g \in G} g \circ t \circ g^{-1}$

To prove following corollaries, note that if E, F : G-Bundles, then $Hom(E, F)$: G-bundle ($g \cdot f = g \circ f \circ g^{-1}$) and

$$\Gamma(Hom(E,F))^G \cong HOM_G(E,F)$$

- Let Y be a G-inv, closed subset of a compact G-space X, E, F : G-vector bundles over X. If $E|_Y \to F|_Y$: G-iso, then there exist a G-inv open set $U \supseteq Y$ and extension $f: E|_U \to F|_U$, which is G-iso

<u>Idea</u> extend $f \in \Gamma(\mathrm{Hom}_G(E|Y, F|Y))$ to $\Gamma(\mathrm{Hom}_G(E, Y))$, take good subset.

- Let Y be a compact G-space, X : a G-space, $f_t : Y \to X$: a G-homotopy and E a G-vector bundle over X. Then $f_0^* E \cong f_1^* E$ as G-bundles.

Here by G-homotopy we mean G-map $F : Y \times \underline{I} \longrightarrow X$ with the trivial G-action on $\underline{I}$.
By trivial G-bundle we mean a G-bundle iso to $X \times V$

„1.4.7 – 1.4.11"

- $\mathrm{Vect}(S(X)) \cong [X, GL(n,\mathbb{C})]$
- Existance of hermitian metric
- ses splits

- If E : G-bundle over a compact G-space X, then $\Gamma(E)$ contains a fin. dim. G-inv ample space V.

<u>Proof</u> If V: ample, then $\sum gV$: ample and invariant.

- To prove 1.4.15 we need to define a
$$\lim [X, G_n(\mathbb{C}^m)] \xrightarrow{\cong} \mathrm{Vect}_n(X)$$
"G-grassmanian" of G-subspaces.

Let $A = C(X)$, X : G-space G acts on A via $g \cdot f = f \circ g^{-1}$.
Recall, that if E is a G-vector bundle, then $\Gamma(E)$ is a proj A-module and G acts on $\Gamma(E)$

$$g(as) = (g \cdot a) \cdot (g \cdot s)$$

Other perspective: $B = A \underset{\mathbb{C}G}{G} = \{ \sum a_g g \}$, $(ag)(a'g') = (aga')gg'$
Then

$$\Gamma : \{ G\text{-bundles}/X \} \xrightarrow{\simeq} \left\{ \begin{array}{c} B\text{-modules that are} \\ \text{proj. finite rank } A \text{ module} \end{array} \right\}$$

Suppose that X is a G-free space, X/G : orbit space
Then $\pi : X \to X/G$: finite covering.

So if E : G-vector bundle over X, E is a G-free space
Note that E/G has a str. of vector bundle over X/G
(locally isomorphic to $E \to X$)

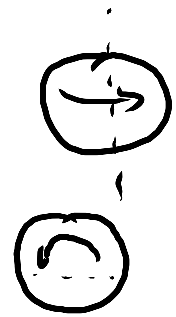
If X is G-free:

$\{G\text{-vect. bund. over } X\} \longleftrightarrow \mathrm{Vect}(X/G)$

S^1, $\mathbb{Z}_2 \curvearrowright S^1$

over S^1 we have
trivial bundle
Moebius band
} 1-dim real vector bundles

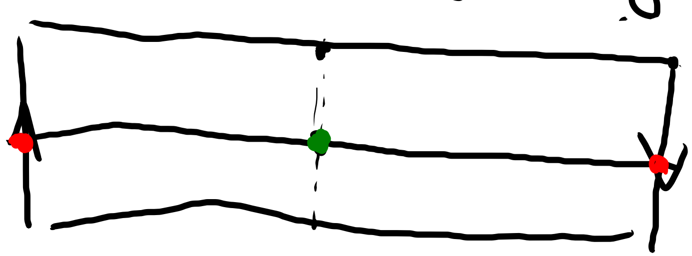
$\mathbb{Z}_2 \curvearrowright S^1$ {
trivial
symmetries
rotation



We can always ext. this trivially $(x, v) \mapsto (g\cdot x, v)$ as long as my bundle is trivial

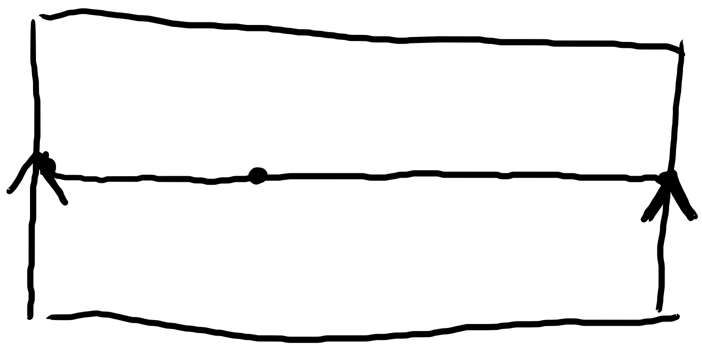
For Moebius band: trivial action S_1 → {
trivial action on Moebius band
to "-1" action (as before)

symmetry → we have action on ,



(Perhaps) action on one fixed point determines action on second one

Rotation: free, so $S^1/\mathbb{Z}_2 = S^1$ two str $\begin{cases} \text{trivial} \\ \text{Moebius band} \end{cases}$



$\begin{cases} \text{rotation „without charge"} \\ \text{rotation with „-1"-action} \end{cases}$

$\underset{\sim}{\mathbb{Z}_2} \times \underset{\sim}{\mathbb{Z}_2} \curvearrowright S^1 \times \mathbb{R}$