Vector bundles

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Assumptions

Let X be a Hausdorff, compact topological space.
We consider complex finitely dimensional vector bundles:

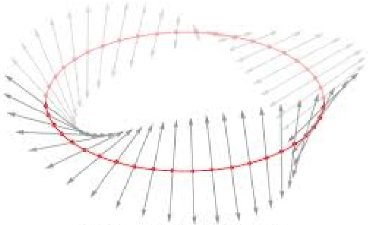
$$\begin{array}{c} E \\ \downarrow p \\ X \end{array}$$

Our goal is to describe vector bundles over fixes space X up to isomorphism.

For reference: K-theory (M.F.Atiyah)

General philosophy

The general idea of vector bundles is the following one. A vector bundle is a collection of vector spaces parametrized by a base space X in some geometric (e.g. continuous, smooth, analytical) way.



Bundle with Möbius strip topology

$Vect(X)$

Definition 1.

$Vect(X)$ is a set of isomorphism classes of vector bundles over X together with abelian monoid structure induced by $\oplus$. Identity element is the class of trivial bundle $X \times \{0\}$.

Definition 2.

$Vect_n(X)$ is a subset of $Vect(X)$ given by bundles of dimension n.

To show the basic relation between a topological structure of X and $Vect(X)$, we need the following lemma:

Lemma 1.

Let $f : Y \times I \to X$ be a homotopy and E a vector bundle over X. Then:

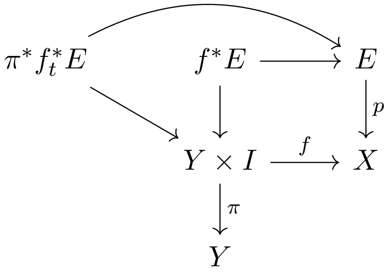
$$f_0^*(E) \cong f_1^*(E)$$

Softness

Before we prove the lemma let's observe that for a closed subset $Y \subset X$, sections of restricted bundle $E|_Y$ can be extended to global ones. This easily follows from normality of X (Tietze theorem) and partition of unity.

Let's apply this observation to the vector bundle $Hom(E, F)$. If section $s \in HOM(E|Y, F|Y)$ is an isomorphism, then its extension $\tilde{s} \in HOM(E, F)$ is an isomorphism on some open neighbourhood U of Y.

Proof of lemma



Vector bundles $\pi^* f_t^* E$ and $f^* E$, restricted to subspace $Y \times \{t\}$ are isomorphic. So there is an isomorphism on some open subset of $Y \times I$, by compactness of Y on some strip $Y \times (t - \epsilon, t + \epsilon)$. From that we now function:

$$I \to Vect(Y)$$

$$t \to f_t^* E$$

is locally constant, so constant by connectedness of the interval. $\square$

Lemma 2.

A homotopy equivalence: $f : X \to Y$ induces an isomorphism of monoids:

$$f^* : Vect(Y) \to Vect(X)$$

In particular, for X contractible, every vector bundle is trivial and $Vect(X)$ is isomorphic to natural numbers.

Lemma 2 follows from previous one (lemma 1) and fact that:

$$f^*(E \oplus F) \cong f^*E \oplus f^*F$$

Pullback of bundle $Y \times \{0\}$ is obviously $X \times \{0\}$.

By $\Gamma(E)$, for vector bundle E, we denote the vector space of sections of E.

Ample subspace

Definition 3.

A subspace $V \subset \Gamma(E)$ *we call ample if it induces an epimorphism of vector bundles:* $\phi : X \times V \to E \to 0$*, defined by:* $\phi(x, s) = s(x)$

Lemma 3.

Any vector bundle E*, there exists a finitely dimensional ample subspace* $V \subset \Gamma(E)$

Proof: On a trivial bundle $X \times W$ it can be easily done, by associating $w \in W$ with section $s_w : x \to (x, w)$. To prove general case, we take any finite trivializing cover U_α (from compactness). For every U_α we have an ample subspace V_α. We can extend it to subspace of $\Gamma(E)$ by partition of unity to $\tilde{V}_\alpha$.
$\Pi_\alpha \tilde{V}_\alpha$ is an ample, finitely dimensional subspace of $\Gamma(E)$.

Corollary 1.

Every finitely dimensional vector bundle is a direct summand of trivial bundle $X \times \mathbb{C}^n$.

Let's take an inclusion: $f : E \to X \times V$.
Using our general point of view, this means E is a collection of subspaces of fixed "ambient" vector space V. Naturally a need arises for a topological space of fixed dimensional subspaces of V.

Definition 4.

Let's fix a vector space V, $GL(V)$ acts transitively on n-dimensional subspaces of V. Take H the stabilizer of any n dimensional subspace. Then we define the Grassmannian as quotient of Lie groups:

$$Gr_n(V) = GL(V)/H$$

It is as a set, a collection of all n dimensional subspaces of V.

Inclusion into trivial bundle induces a continuous map to the Grassmannian just as described above:

$$\tilde{f} : X \to im(f_x)$$

The Grassmanian comes with obvious vector bundle U_n, where fiber over subspace $g \subset V$ is just g.

Definition 5.

Vector bundle $U_n \subset Gr_n(V) \times V$ consisting of all points:

$$(g, v) \in Gr_n(V) \times V \ \ s.t.\ v \in g$$

we call classifying bundle over $Gr_n(V)$.

An important property of U_n is that $\tilde{f}^* U_n = E$ (as subset of $X \times V$)

Universal bundle pt.2

$$\begin{array}{ccc} \tilde{f}^*U_n & \longrightarrow & U_n \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tilde{f}} & Gr_n(V) \end{array}$$

Points of $\tilde{f}^*U_n$ are $(x, g, v) \in X \times Gr_n \times V,\ s.t.\ v \in g$, that is exactly $(x, im(f_x), v \in im(f_x))$. $\square$

Let's consider the tautological exact sequence:

$$0 \to U_n \to Gr_n(V) \times V \to Q_n \to 0$$

Any map $f : X \to Gr_n(V)$ induces an exact sequence of vector bundles over X:

$$0 \to f^*U_n \to X \times V \to f^*Q_n \to 0$$

Universal bundle pt.3

We have shown that any short exact sequence:

$$0 \to E \to X \times V \to F \to 0$$

is in the form as on the previous frame.
Actually we have just shown that $Gr_n(V)$ is representing space for certain functor. Category of compact T2 topological spaces with continuous functions as morphisms, we denote by $\mathfrak{F}$. Let's define a functor:

$$\mathcal{F}^n : \mathfrak{F}^{op} \to Set$$

$$X \to \{\text{set of codim}=n \text{ quotient bundles of } X \times V\}$$

$$(X \xrightarrow{\phi} Y) \to \{\phi^* : \mathcal{G}^n(Y) \to \mathcal{G}^n(X)\}$$

We have established a natural isomorphism of functors:

$$\eta : Hom(\_, Gr_n(V)) \to \mathcal{G}^n$$

Direct system

Let's take a projection map:

$$\pi_m : \mathbb{C}^{m+1} \to \mathbb{C}^m$$

$$(z_1, \dots z_{n-1}, z_n) \to (z_1, \dots z_n)$$

it induces a continuous map between the Grassmannians:

$$i_m : Gr_{m-n}(\mathbb{C}^m) \to Gr_{m+1-n}(\mathbb{C}^{m+1})$$

Map i_m is injection. Denote an universal bundle on $Gr_{k-n}(\mathbb{C}^k)$ as E_k, then:

$$i_m^*(E_m) \cong E_{m-1}$$

We build direct system ($[X, Y]$ denotes a set of homotopy classes of continuous maps)

$$\dots \to [X, Gr_{m-n}(\mathbb{C}^m)] \xrightarrow{i_m} [X, Gr_{m+1-n}(\mathbb{C}^{m+1})] \to \dots$$

Main theorem

We treat $Vect_n(X)$ as functor from $\mathfrak{F}_h^{op}$ (morphisms up to homotopy) to Set.

Theorem 1.

There exists a natural equivalence:

$$\eta : \varinjlim_{m} [\_, Gr_{m-n}(\mathbb{C}^m)] \to Vect_n(\_)$$

Proof: We construct natural a transformation as before. From previous considerations it is surjective. Let's take an epimorphism $\phi : X \times \mathbb{C}^m \to E$ it induces $f : X \to Gr_{m-n}(\mathbb{C}^n)$, we need to check that the homotopy class of f (in $Gr_{m'-n}(\mathbb{C}^{m'})$ for m' big enough) doesn't depend on choice of ϕ.

Let's take ϕ_0 (from $X \times \mathbb{C}^{m_0}$), ϕ_1 (from $X \times \mathbb{C}^{m_1}$) inducing the same vector bundle E. Define:

$$\xi_t : X \times \mathbb{C}^{m_0} \times \mathbb{C}^{m_1} \to E$$

$$\xi_t(x, v_0, v_1) = (1-t)\phi_0(x, v_0) + t\phi_1(x, v_1)$$

ξ_t is epimorphic for every t, it induces

$$f_t : X \to Gr_{m_1+m_2-n}(\mathbb{C}^{m_1+m_2})$$

We get a homotopy $f : X \times I \to Gr_{m_1+m_2-n}(\mathbb{C}^{m_1+m_2})$.
Class of f_0 in direct limit is obviously the same, as map induced by ϕ_0. On the other hand class of map induced by ϕ_1, on the level $m_1 + m_2 - 1$ differs by permutation of coordinates, which is homotopic to identity.

Functor $Vect_n(X)$ is kind of representable. ($\varinjlim_m Gr_{m-n}(\mathbb{C}^m)$ is not compact!!!)

Corollary 2.

$$\varinjlim_m [X, Gr_{m-n}(\mathbb{C}^m)] = [X, \varinjlim_m Gr_{m-n}(\mathbb{C}^m)]$$

We consider only compact spaces. Fix space X. Image of compact space X is compact, therefore contained in one of $Gr_{m-n}(\mathbb{C}^m)$, so we have surjective function from left to right. For space $X \times I$ function is also surjective, so every homotopy on the right comes from homotopy on the left.

Let's take a vector bundle E over X, and suppose $Y \subset X$ closed, such that there exists an isomorphism:

$$\alpha : E|_Y \to Y \times V$$

Such α we call **trivialization** of E over Y.

We would like to define vector bundle over quotient space X/Y induced by α. On $E|Y$ we can define equivalence relation $\sim$

$$\alpha(e) = (\alpha_0(e), \alpha_1(e))$$

$$e \sim e' \iff \text{ for } \alpha_1(e) = \alpha_1(e')$$

We extend this relation by identity on the complement of Y.
By E/α we denote quotient space of E by relation $\sim$.

We claim that E/α is a vector bundle. It is enough to check it on some open neighbourhood of Y/Y. On Y we have trivialization it extends to some open set U, and it induces local triviality on U/Y.

Lemma 1.

Let E be vector bundle over X, Y closed subest of X, α trivialization, then isomorphism class of E/α depends only on homotopy class of α.

Proof: Homotopy between α_0 and α_1 determines trivialization β of $Y \times I$ (as closed subset of $X \times I$) of $E \times I$.

$$\begin{array}{ccc} f^*(E \times I/\beta) & \longrightarrow & (E \times I)/\beta \\ \downarrow & & \downarrow \\ (X/Y) \times I & \xrightarrow{f} & (X \times I)/(Y \times I) \end{array}$$

Restriction of the pullback bundle to $(X/Y) \times \{i\}$ is exactly E/α_i

Lemma 2.

Let Y be closed contractible subset of X, then quotient $f : X \to X/Y$ induces an isomorphism:

$$f^* : Vect(Y/X) \to Vect(X)$$

Proof:

Let's take a vector bundle E over X and $\alpha : E|Y \to Y \times V$ trivialization of Y. Any two such trivializations differ by automorphism of $Y \times V$(contractible set is connected so rank is constant), so by map $Y \to Gl_n(V)$. Group $Gl_n(V)$ (over complex numbers) is connected contractible space, so by previous lemma, isomorphism class of E/α is determined by that of E.