|TiGR| G – finite group $\quad \mathcal{C}$ – cat. of G-complexes
$\mathcal{C}_0$ – cat. of G-complexes with base point
$\mathcal{C}^2$ – cat. of pairs (K,L) with $L \subset K$ subcpx.

Def.: An equivariant cohomology theory (generalised) on $\mathcal{C}$ is a collection of functors
$\mathcal{H}^n : \mathcal{C}^2 \to Ab$ together with $\delta^n : \mathcal{H}^n(L,\emptyset) \to \mathcal{H}^{n+1}(K,L)$
with axioms:
1° if f_0 is equiv. homotopic to f_1 then $\mathcal{H}^n(f_0) = \mathcal{H}^n(f_1)$
2° $(K, K\cap L) \subseteq (K\cup L, L)$ induces an isomorphism
$$\mathcal{H}^n(K, K\cap L) \xrightarrow{\simeq} \mathcal{H}^n(K\cup L, L)$$
3° $(K,L) \in \mathcal{C}^2$ $\quad \dots \to \mathcal{H}^n(K,L) \xrightarrow{j^*} \underset{\mathcal{H}^n(K)}{\mathcal{H}^n(K,\emptyset)} \xrightarrow{i^*} \underset{\mathcal{H}^n(L)}{\mathcal{H}^n(L,\emptyset)} \xrightarrow{\delta^n} \mathcal{H}^{n+1}(K,L) \to$
is exact.

Remark: G is abelian $\quad \mathcal{H}^n(K,L)$ has a natural structure of a G-module.

$\mathcal{C}^2 \to \mathcal{C}_0 \qquad \mathcal{C}_0 \to \mathcal{C}^2$
$(K,L) \mapsto K/L \qquad K \mapsto (K, x_0)$ – base point of K

L/L is a base point in K/L (if $L = \emptyset$ then base point of $K/\emptyset =: K^+$ is disjoint pt.)

$K \in \mathcal{C}_0$, ΣK – reduced suspension of K. Eq. coh. theory on $\mathcal{C}_0$ is:
$$\tilde{\mathcal{H}}^n : \mathcal{C}_0 \to Ab, \quad \sigma^n(K) : \tilde{\mathcal{H}}^n(K) \to \tilde{\mathcal{H}}^{n+1}(\Sigma K)$$
1° $f_0 \sim f_1 \quad \tilde{\mathcal{H}}^n(f_0) = \tilde{\mathcal{H}}^n(f_1)$
2° $\sigma^n(K)$ is iso for $\forall n, \forall K$
3° $\tilde{\mathcal{H}}^n(K/L) \to \tilde{\mathcal{H}}^n(K) \to \tilde{\mathcal{H}}^n(L)$

Coef.: $H^0(pt)$.

In equiv. case our building blocks are coset spaces of G
$\{G/H : H \leq G\}$
"contain"
$$G/H_0 \to G/H_1$$
$$\mathcal{H}^*(G/H_1) \to \mathcal{H}^*(G/H_0)$$

A theory on $\mathcal{C}$ (or $\mathcal{C}_0$) is called classical when it satisfies an additional axiom:
4° $\mathcal{H}^n(G/H) = 0 \quad \forall n \neq 0 \ \forall H \leq G$
(4° $\tilde{\mathcal{H}}^n((G/H)^+) = 0$ — " —)

Category of canonical orbits $\mathcal{O}_G$:
Objects – left coset spaces G/H
Morphisms – equiv. maps.

Classification of morphisms in $\mathcal{O}_G$:
$f : G/H \to G/K$ – any map, $f(H) = aK$ for some $a \in G$
f equivariant $\iff f(gH) = gaK \ \forall g \in G$.
$f(gH) \overset{def}{=} gaK$ (equiv.) s.t. $f(ghH) = f(gH)$. So $ghaK = gaK \ \forall h \in H$
$\iff haK = aK \implies a^{-1}Ha \subseteq K$

Let $a \in G$ s.t. $a^{-1}Ha \leq K$ $\quad \hat{a} : G/H \to G/K$ as $\hat{a}(gH) = gaK$

Then $\hat{a}$ is equiv and every equiv map is of this form.
$\hat{a} = \hat{b} \iff aK = bK \iff a^{-1}b \in K$

Suppose that $a^{-1}Ha \subset K$. This induces $G/a^{-1}Ha \to G/K$ (equiv)
and $H \in aKa^{-1} \implies G/H \to G/aKa^{-1}$

$R_a : G/H \to G/a^{-1}Ha$ $\quad (gH \mapsto gHa = ga(a^{-1}Ha))$ and $R_a : G/aKa^{-1} \to G/K$
$gaKa^{-1} \mapsto gaKa^{-1}a = gaK$

$$\begin{tikzcd} G/H \arrow[r, "pr"] \arrow[d, "R_a"'] \arrow[dr, dashed, "\hat{a}"] & G/aKa^{-1} \arrow[d, "R_a"] \\ G/a^{-1}Ha \arrow[r, "pr"] & G/K \end{tikzcd}$$
This commutes.

$\mathcal{O}_G$ for $\mathbb{Z}_p$ $\quad$ Objects $\underset{P}{G/G}$, $\underset{G}{G/\{e\}}$ $\quad P \to P$, $G \to P$, $G \to G : R_a$

$\mathcal{O}_G$, $G = S_3$. Subgroup of S_3
Object: $\underset{G}{S_3}$, $\underset{P}{\{e\}}$, S_3/A_3, S_3/S_2.
Morphisms:
$P \to P$
$G \to P$
$G \to G \quad R_a$

$S_3 \to S_3$, $\{e\}$, (S_3/A_3), $($

$S_3 \to \{e\}$

$\{e\} \to \{e\}$

S_3/A_3
S_3/S_2 because
$a^{-1}S_2a \nsubseteq A_3$

$\{e\}$ ✓
$Aut(S_3) \subset Mor_{\mathcal{O}_G}(S_3 \to S_3)$

Weyl Group $= N(A_3)/A_3 \cong S_3/A_3 \to S_3/A_3 \qquad a \in N(A_3) \qquad aA_3 \sim A_3$ gives the same automorph.

$1 = N(S_2)/S_2 \cong S_3/S_2 \to S_3/S_2$
$\quad Hom_{eq}(G/H, G/H) \simeq N(H)/H$.
$N(S_2) = S_2 \qquad \sigma \in S_3$ s.t. $\sigma S_2 = S_2\sigma$

$Aut(e)$, $Aut(S_3/S_2)$ ✗
✓ $Aut(S_3)$ $\qquad S_3 \xrightarrow{\simeq} S_3 \to \{e\}, S_3/S_2, S_3/A_3, S_3$

$S_3 \to \{e\}$ – 1 arrow
$Aut(S_3) = S_3$
$G \curvearrowright S_3 \to S_3/A_3$ ∂
1. $G \curvearrowright S_3 \to S_3/S_2$

$G\, S_3 \xrightarrow{\pi} S_3/S_2$

$$\left\{ \begin{array}{l} Mor(G/H, G/K) \simeq N(H,K)/K \\ N(H,K) = \{a : a^{-1}Ha \subset K\} \end{array} \right\}$$

$Mor(S_3, S_3/S_2) = S_3/S_2$
$N(e, S_2) = S_3$
$Mor(S_3, S_3/A_3) = S_3/A_3$
$N(e, A_3) = S_3$

$\boxed{\mathcal{O}_{S_3}}$

Generic coefficient systems:
Def: A g. coef. sys. for G is a contravariant functor $\mathcal{O}_G \to Ab$
A morphism between $\underset{g.c.s.}{M, N}$ is a nat. transformation $T : M \to N$
G. coef. sys. form a category $\mathcal{C}_G$ (abelian).

Examples: 1° $\mathcal{H}$ – eq. coh. theory and $q \in \mathbb{Z}$
$$h^q : \mathcal{O}_G \to Ab \qquad h^q(G/H) = \mathcal{H}^q(G/H)$$
$f : G/H \to G/K$ equiv. $\implies h^q(f) = \mathcal{H}^q(f)$.
2° A – G-module $\quad M : \mathcal{O}_G \to Ab$
$$M(G/H) = A^H$$
For $g \in G$ s.t. $H \subset gKg^{-1}$ we see that $g : A \to A$
takes $A^K \to A^H$ $\quad (a \in A^K \implies Hga \subseteq gKg^{-1}ga = gKa = ga)$
Denote $g_{H,K} : A^K \to A^H$
If $\hat{g} = \hat{g'}$ $\quad (g^{-1}g' \in K)$ then $g_{H,K} = g'_{H,K}$.
$$\hat{g} : G/H \to G/K \qquad M(\hat{g}) = g_{H,K}.$$
3° Y – G space with a base point y_0
Def. $\hat{\omega}_q(Y) : \mathcal{O}_G \to Ab$
$$\hat{\omega}_q(Y)(G/H) = \pi_q(Y^H, y_0) \qquad \pi_1(Y^H, y_0) \text{ – abelian}$$
$$\hat{\omega}_q(Y)(\hat{g}) = g_\# : \pi_q(Y^K, y_0) \to \pi_q(Y^H, y_0)$$

Remark: $Mor(G/H, G/H) \simeq N(H)/H \quad \forall M \in \mathcal{C}_G \quad M(G/H)$ has a structure of $N(H)/H$ – module

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$M \in \mathcal{C}_G \qquad \mathcal{O}_G$ "contains" $\underset{P}{G/G}$, $\underset{G}{G/\{e\}}$, $P \xrightarrow{1} P$, $r : G \to P$, $\hat{a} : G \to G$

$M(P)$, $M(G)$, homomorphisms $M(1) = 1$
$$\varepsilon = M(r) : M(P) \to M(G)$$
$$a_* = M(\hat{a}) : M(G) \to M(G)$$
$M(\widehat{ab}) = M(\hat{a})M(\hat{b}) \qquad M(\hat{a})M(r) = M(r)$ ✓

$M(G)$ may be considered as a G-module $(a, m) \to a_*(m)$
and $M(P)$ — " — a trivial G-module.
ε is an equiv. homomorphism $\varepsilon : M(P) \to M(G)^G$

$\underline{\underline{\mathbb{Z}_p}}$: any gen. coef system is just M_0, M_1 – G-module
with homomorphism : $M_0 \to M_1^G$
$Mo(M, M')$
$$\begin{tikzcd} M_0 \arrow[r, "\varepsilon"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & M_1 \arrow[d] \\ M_0' \arrow[r, "\varepsilon'"] & M_1' \end{tikzcd}$$

Example. $\underline{\hat{\omega}_q(Y)}$, $G = \mathbb{Z}_p$, Y – G-sp. w. b.pt. y_0
$\hat{\omega}_q(Y)$ consists of $\pi_q(Y^G)$, $\pi_q(Y)$ – action by induced $g_\# : \pi_q(Y) \to \pi_q(Y)$
and $\varepsilon : \pi_q(Y^G) \to \pi_q(Y)^G \subset \pi_q(Y)$ induced by $Y^G \subset Y$.