Quick reminder:

CW-complex is a space $K = \bigcup_{n=0}^{\infty} K^n$

where K^0 is a discrete set of points

$K^n \subseteq K^{n+1}$ and K^{n+1} is obtained

by attaching disks D^{n+1} to K^n along maps

$\varphi : S^n \longrightarrow K^n$

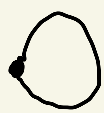
$$\begin{tikzcd} \coprod S^n \arrow[r] \arrow[d] & \coprod D^{n+1} \arrow[d] \\ K^n \arrow[r] & K^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

This is pushout

Each of the resulting maps $D^{n+1} \longrightarrow K^{n+1}$ is a called a <u>cell</u>

Example $\mathbb{RP}^2$

K^0 $\cdot$

K^1 

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "\text{double cover}"] & D^2 \arrow[d] \\ K^1 \arrow[r] & \mathbb{RP}^2 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

G - complex, G - finite group

Def 1.

G-complex is a CW-complex together with $G \curvearrowright K$ by cellular maps $\left(g(K^n) \subseteq K^n\right)$, s.t.

$\forall_{g \in G} \; \{x \in K \mid g(x) = x\}$ is a subcomplex of K

Def 2

K is a G-complex if it is a CW-complex obtained by attaching cells of the form $G/H \times D^n$ along maps $G/H \times S^{n-1}$

$$\begin{tikzcd} G/H \times S^{n-1} \arrow[r] \arrow[d] & G/H \times D^n \arrow[d] \\ K^n \arrow[r] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Pushout

Def 1 and Def 2 are equivalent

this is non-trivial

Equivariant map

$f : X \longrightarrow Y$ continuous map of G-spaces

f is equivariant if $\forall_{g \in G}$ $\underset{\substack{\uparrow \\ \text{acts on } Y}}{g} \circ f = f \circ \underset{\substack{\uparrow \\ \text{acts on } X}}{g}$

Equivariant homotopy

$\alpha, \beta : X \longrightarrow Y$ equiv. maps of G-spaces

we have a natural G-action on $X \times I$

Eq. homotopy between α and β

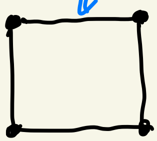
$H : X \times I \longrightarrow Y$ equiv. map

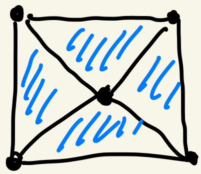
s.t. $H(x,0) = \alpha(x)$ $H(x,1) = \beta(x)$

$$g(H(x,t)) = H(g(x),t)$$

Example of equiv. homotopy.

this is empty inside

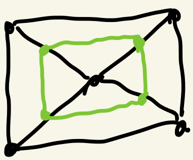
$X =$ 

$Y =$ 

$\mathbb{Z}_4$ action on those spaces

$\alpha : X \longrightarrow Y$ "natural" injection

p taking everything to center



$H(x, t_0)$ for some fixed t_0

this is equivariant

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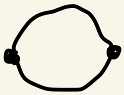
Examples of G-complexes

$\mathbb{Z}_2$ action on sphere

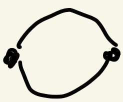
a) K^1  $\mathbb{Z}_2$ acts on K^1 by antipodal map

$K^0 \simeq \mathbb{Z}_2$ (as discrete G-space)

K^1 is obtained by attaching $\mathbb{Z}_2 \times D^1$ to K^0

$K^2 = K$  $\mathbb{Z}_2 \times D^2$ (upper and lower semi-sphere)

$\mathbb{Z}_2$ acts on K by antipodal map

b) K^1  $\mathbb{Z}_2$ acts trivially

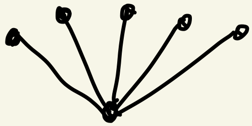
$K^0 = \mathbb{Z}_2/\mathbb{Z}_2 \sqcup \mathbb{Z}_2/\mathbb{Z}_2$ $\qquad$ K^1 also

$\mathbb{Z}_2 \times D^2$ – attach this to K^1 s.t.

(upper and lower semi-sphere)

$K^2 = K$  $\mathbb{Z}_2$ to act by symetry by the plane containing K^1.

Example



of course I have S_5 action on it.

S_5 - complex

Definitions

1) Retraction $\quad A \subseteq X$

$f : X \longrightarrow A$ is a retraction if $f|_A = id_A$

2) Deformation retraction $\quad A \subseteq X$

def. retr. is $H : X \times I \longrightarrow X$ s.t. $\forall_{x \in X}\ H(x,0) = x$

$\forall_{x \in X}\ H(x,1) \in A$

$\forall_{a \in A}\ H(a,1) = a$

3) Strong def. retr. $\quad A \subseteq X$

$H : X \times I \longrightarrow X$ $\quad$ H is a def. retr.

$\forall_{a \in A}\ \forall_{t \in I}\ H(a,t) = a$

Homotopy extension property.

$j : A \longrightarrow X$ satisfies homotopy extension property (j is a cofibration) if

$$\begin{tikzcd}
A\times\{0\} \arrow[r, "i_0"] \arrow[d, "j"'] & A\times I \arrow[dl, "H"'] \arrow[d, "j\times id"] \\
Y & \\
X\times\{0\} \arrow[u] \arrow[r, "i_0"'] & X\times I \arrow[ul, dashed]
\end{tikzcd}$$

Fact

$i : S^{n-1} \longrightarrow D^n$ satisfies HEP.

Fact

All maps are equivariant

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left, "eq"] & \\
C \arrow[r] \arrow[drr, bend right, "eq"'] & B\cup_A C \arrow[dr, dashed, "\varphi"] & \\
 & & Y
\end{tikzcd}$$

$\exists!\,\varphi$ which is equiv.

checked easily.

Theorem Let K be a G-complex (Def 2)
L be a G-invariant subcomplex of K
$(\forall_{g \in G} \ \forall_{x \in L} \ g(x) \in L)$ then $j : L \longrightarrow K$
satisfies equivariant homotopy extension property.

$$\begin{tikzcd}
L \times \{0\} \arrow[r, "eq"] \arrow[dd, "eq"'] & L \times I \arrow[dl, "eq"'] \arrow[dd, "eq"] \\
Y & \\
K \times \{0\} \arrow[r, "eq"'] \arrow[u, "eq"] & K \times I \arrow[ul, dashed, "eq"']
\end{tikzcd}$$

Proof: induction on the CW-skeleton of K.

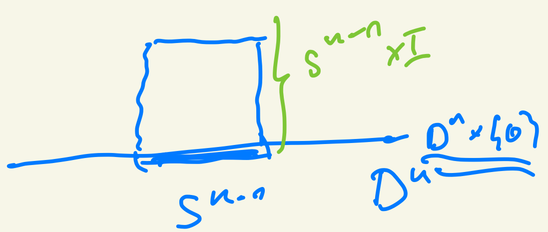
For K^0. For $x \in K^0$
either $x \in L \longrightarrow$ ok
$x \notin L$ $H(x, t) = x$ this equiv.
because it is equiv. on the zero level.

Let's assume that we extended the homotopy to K^{n-1}.
We will take one cell $G/H \times D^n$ and extend it here.

$$\begin{tikzcd}
\coprod G/H \times S^{n-1} \arrow[r] \arrow[d] & \coprod G/H \times D^n \arrow[d] \\
K^{n-1} \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & P
\end{tikzcd}$$

$$\begin{tikzcd}
(G/H \times S^{n-1}) \times \{0\} \arrow[rr] \arrow[dd] & & (G/H \times S^{n-1}) \times I \subset K^{n-1} \times I \arrow[dl, "H"'] \arrow[dd] \arrow[dr] & \\
& Y & & G/H \times (D^n \cup S^{n-1} \times I) \arrow[ll, "eq"'] \\
(G/H \times D^n) \times \{0\} \arrow[ur] \arrow[urrr] \arrow[rr] & & (G/H \times D^n) \times I \arrow[ur, "eq"'] \arrow[ul, dashed, "F", "eq"'] &
\end{tikzcd}$$

$D^n \times I \longrightarrow D^n \cup S^{n-1} \times I$



$D^n \times I \longrightarrow D^n \cup S^{n-1} \times I$

Prop $X = L\times I \cup K\times\{0\}$ $\quad X \subseteq K\times I$

X is a eq strong deformation retract of $K\times I$

Construct $L \subset B \subset K$ ①

B neighborhood of L

there exists eq. str. def. ret $B \to L$

② There exists $X \subset U \subset K\times I$

U is a eq. strong deformation retraction on X

s.t. there exists $L \subset A \subset K$ $\quad U \supseteq A\times I$

A ↳ open

③ There exists eq. str. def. retr. of $K\times I \to U$

$U \to X$

④ Thesis May "A concise course on alg. top".