Extracts of the research programme of the project "Secant varieties, computational complexity, and toric degenerations"

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The research within this project will be divided into two areas.

Computational complexity, generalised Waring problem, and secant varieties

Consider a complex vector space W and a projective algebraic variety $X \subset \mathbb{P}W$ with its affine cone $\hat{X} \subset W$. Suppose X is not contained in any hyperplane. For $p \in W$ we define *X-rank* of p to be the minimal integer $r = R_X(p)$, such that

$$p = \hat{x}_1 + \hat{x}_2 + \cdots + \hat{x}_r \text{ for some } \hat{x}_i \in \hat{X}. \qquad (*)$$

Equivalently, r is the minimal integer such that $[p] \in \langle x_1, x_2, \ldots, x_r \rangle$, where $[p] \in \mathbb{P}W$ denotes the underlying point in projective space, $\langle \ldots \rangle$ denote the linear span, and $x_i \in X$.

It is a classical problem of 19th century geometers to determine X-ranks of points in certain specific situations, see for instance [Salm65], [Salm60], [Sylv51]. The problem is often referred to as a *generalised Waring problem* in tribute to E. Waring, who in 18th century asked about presentation of integers as sums of powers, see [VW02] for a survey. Recently, the problem has also become significant for applications. Roughly, X is the set of *"simple"* elements and the X-rank is a measure how *"complicated"* p is with respect to X. The expression of the form $(*)$ is a *"minimal decomposition"*. In typical applications X can be one of Veronese variety $v_d(\mathbb{P}^n)$, Segre product of projective spaces $Seg(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$, a Grassmannian $Gr(k, n)$, or a combination of such, for instance $Seg(\mathbb{P}^k \times v_d(\mathbb{P}^n))$. We begin with briefly explaining two of these application, which are the simplest to present.

Matrix multiplication

The multiplication of matrices is a bilinear map $\mathbb{C}^{fg} \times \mathbb{C}^{gh} \to \mathbb{C}^{fh}$. Thus it can be viewed as a tensor $M_{f,g,h} \in (\mathbb{C}^{fg})^* \otimes (\mathbb{C}^{gh})^* \otimes \mathbb{C}^{fh} = A \otimes B \otimes C = W$. The naive algorithm to perform such matrix multiplication uses fgh multiplications of complex numbers. As a tensor $M_{f,g,h} = \sum_{i,j,k} a_{ij} \otimes b_{jk} \otimes c_{ik}$, i.e., in this presentation, the decomposition consists of fgh *simple tensors* $a \otimes b \otimes c$. Let $X := Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) \subset \mathbb{P}W$ be the set of classes of simple tensors. The presentation above implies $R_X(M_{f,g,h}) \leq efg$.

Strassen proved that 2×2 matrices can be multiplied using only 7 multiplications of numbers [Stra69]. This statement is translated in the language of tensors as $R_X(M_{2,2,2}) \leq 7$ (in fact, the X-rank is equal to 7). Iteratively, the statement gives an algorithm to multiply two large $f \times f$ square matrices using approximately $f^{\log_2 7} \simeq f^{2.81}$ multiplications of numbers.

Evaluating polynomials

Consider $W := S^d\mathbb{C}^{n+1}$ the vector space of homogeneous degree d polynomials in $n+1$ variables. We denote by $X := v_d(\mathbb{P}^n) \subset \mathbb{P}W$ the set of classes of l^d for some $l \in \mathbb{C}^{n+1}$, that is the set of d-th power of a linear polynomial. Note that the linear polynomials are relatively cheap to evaluate. Given $p = l^d$ and an $(n+1)$-tuple of complex numbers $a = (a_0, \dots, a_n)$, we can evaluate $l(a)$ and iteratively calculate $l(a)^d = l(a)^{\lfloor d/2 \rfloor} \cdot l(a)^{\lceil d/2 \rceil}$. Therefore the order of complexity of evaluation is at most $\sim (n+1)\log_2 d$ in this case. Thus any polynomial p has a complexity of evaluation of order at most $\sim R_X(p) \cdot (n+1)\log_2 d$. In this sense, the X-rank measures the complexity of the evaluation of polynomials.

Other applications and generalisations

Other applications of X-rank and finding a minimal decompositions are discussed in [Land11, §1.3] and references therein. These applications include *fluorescence spectroscopy*, where different chemicals diluted at different concentrations are compared and analysed in order to find the number r of different chemicals present. This r is a rank of an appropriate tensor. More generally, the decomposition $(*)$ is a tool to analyse statistical data, and extract meaningful information from this data.

We will also consider a slightly more general problem of finding a simultaneous decomposition of a collection of points. Specifically, let $Z \subset W$ be any subset and define the X-rank of Z as the minimal integer $r = R_X(Z)$ such that there exists $x_1, x_2, \dots, x_r \in X$ such that for each $p \in Z$ we have $[p] \in \langle x_1, x_2, \dots, x_r \rangle$. It is clear, that in the definition Z can be replaced by its linear span, but in applications one often considers Z to be a finite set. For example, given polynomials $f_1, \dots, f_k \in S^d\mathbb{C}^{n+1}$, we want to find a relatively small number of linear forms $l_1, \dots, l_r$, such that each f_i is a linear combination of ${l_j}^d$. In practise, often $R_{v_d(\mathbb{P}^n)}(\{f_1, \dots, f_k\}) < R_{v_d(\mathbb{P}^n)}(f_1) + \dots + R_{v_d(\mathbb{P}^n)}(f_k)$. The study of the X-rank of a linear space is equivalent to the study of $Seg(\mathbb{P}^k \times X)$-rank of a single point, see [BL09].

In applications it is sometimes sufficient to obtain an approximate result. Thus we also consider X-border rank of point p or linear subspace $Z \subset W$, which is the minimal integer $r = \underline{R}_X(P)$ or $\underline{R}_X(Z)$, such that p or Z is a limit of points or spaces of X-rank r. The set of all points of X-border rank at most r is the r-th secant variety of X, denoted $\sigma_r(X)$. One of the problems in the subject is to describe explicitly the set of all equations vanishing on secant varieties $\sigma_r(X)$ for meaningful X. Such equations, especially if presented in a nice compact form, provide effective tests for X-border rank.

Objectives of the project, part 1

In this part of the project we will address the following general problems:

(i) Determine X-ranks and X-border ranks of points and linear spaces.

(ii) Find the minimal decompositions.

(iii) Find equations describing secant varieties.

Although these challenging problems are too wide to hope to solve in general, many partial results for specific X are known — the literature is vast on this subject, see [IK99], [BL09], [BGL10], [AB09], [BB10] for a sample. In the project we will work in collaboration with Kristian Ranestad on more specific problems, which are both significant for applications and lead to better understanding of the general picture.

Computer supported experiments are critical in all of these objectives. Even though in many cases we are able to obtain purely theoretical proofs and results, at an early stage of research conjectures and questions need to be tested on numerous examples.

Toric degenerations

The second part of the project is to be executed in collaboration with Gavin Brown and also with support of experience and knowledge of Kristian Ranestad.

A *toric degeneration* is a family of varieties, that as the central fibre has a union of toric varieties, glued along toric strata, such that the geometry of the total space is (locally, analytically) expressible in terms of toric geometry.

Toric degenerations are expected to be an effective technique to construct mirrors to Calabi-Yau varieties and intensive studies are in progress to understand this technique, see for instance [Baty04], [GS08]. However, very few explicit examples are known beside some hypersurfaces and complete intersections in toric varieties, and even in these cases the calculations are non-trivial, or not known, especially if the intersection is not Gorenstein.

The problem we consider is to actually construct the toric degeneration starting from any degeneration to a union of toric varieties. Some careful birational operations must be performed in such a way that the singularities of the family are sufficiently mild (i.e., toric), but avoiding too many blow-ups, which may generate new strata of the central fibre that are not toric varieties.

Already in dimension 2 (the case of $K3$ surfaces), the problem of finding the toric degeneration is not completely understood, although the existence of toric degeneration is known. In higher dimension, there are obstructions (coming from a *monodromy* of an affine manifold with singularities) to the existence of toric degeneration in some cases.

Another problem is to construct explicitly examples of (mildly singular) Fano 3-folds (and higher dimensional Fano varieties), using our method of calculating the toric degeneration. Typically, such Fano variety is a member of a family of subvarieties of a weighted projective space, with prescribed Hilbert series. The Hilbert series encodes the degrees of generators of the coordinate ring (i.e., weights of the projective space), the degrees of minimal generators of the ideal defining the variety and the degrees of syzygies. In principal there might be reductions, for instance a syzygy might have the same degree as a generator of the ideal making them “invisible” on the level of Hilbert series. A list of all possible Hilbert series of Fano varieties in small codimension is known, however already in codimension 6 not all of the cases from the list have been proved to exist.

Objectives of the project, part 2

We summarise the objectives of the second part of the project:

(iv) to find explicit examples of toric degenerations;

(v) to construct the missing examples of Fano varieties in low codimension.

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