TWO LECTURES ON APOLARITY AND THE VARIETY OF SUMS OF POWERS

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1. Apolarity, Artinian Gorenstein rings and Arithmetic Gorenstein Varieties

1.1. **Motivating question.**

Given a homogeneous form

$$F \in \mathbb{C}[x_0, \dots, x_n]_d$$

of degree d. A presentation

$$F = l_1^d + \dots + l_r^d, \text{ with linear forms } l_i,$$

is called a power sum decomposition of F of length r.

Question 1.1. *How many distinct decompositions of length r does F have, or if infinite what is the structure of the set of power sum decompositions? (especially, when r is minimal, the so called* rank $r(F)$ *of F?)*

Let us be more precise: Think of the hyperplanes in $\mathbb{P}^n$ defined by the linear forms l_i as points

$$([l_1], ..., [l_r]) \subset (\mathbb{P}^{n-1})^*$$

in the dual space.

Definition **1.2.** *The Variety of Sums of Powers ,*

$$VSP(F, r) \subset \mathrm{Hilb}_r(\mathbb{P}^{n-1})^*,$$

is the closure in the Hilbert Scheme *of $(\mathbb{P}^{n-1})^*$ of the set of r-tuples $[l_1], ..., [l_n]$, such that $F = l_1^d + \dots + l_r^d$.*

Question 1.3. *What are the global properties of $VSP(F, r)$?*

1.2. **Apolarity.**

A natural geometric setting for this question is the d-uple embedding

$$(\mathbb{P}^{n-1})^* \to \mathbb{P}^{\binom{n+d}{d}-1}; \quad [l] \mapsto [l^d].$$

In fact F represents a point $[F] \in \mathbb{P}^{\binom{n+d}{d}-1}$ and

$$F = \lambda_1 l_1^d + \dots + \lambda_r l_r^d$$

for some $\lambda_i \in \mathbb{C}$ if and only if

$$[F] \in \langle [l_1^d], ..., [l_r^d] \rangle.$$

In particular, over $\mathbb{C}$ we may assume that each λ_i is 0 or 1.

This condition is studied using *apolarity.* Sylvester et al. introduced apolarity to find power sum decompositions of forms.

Let $T = \mathbb{C}[y_0, \dots, y_{n-1}]$ act on $S = \mathbb{C}[x_0, \dots, x_{n-1}]$ by differentiation:

$$y_i \in T, \quad F \in S \qquad \Rightarrow \qquad y_i(F) = \frac{\partial}{\partial x_i} F.$$

Then

$$l = \sum a_i x_i \text{ and } g \in T_d \Rightarrow \; g(l^d) = \lambda g(a_0, \dots, a_n),$$

for some $\lambda \neq 0$.

Therefore

$$[F] \in \langle [l_1^d], ..., [l_r^d] \rangle \subset \mathbb{P}(S_d) = \mathbb{P}^{\binom{n+d}{d}-1}$$

if and only if

$$\{g \in T_d | g(l_1^d) = ... = g(l_r^d) = 0\} \subset \{g \in T_d | g(F) = 0\}$$

Definition **1.4.** *$g \in T$ is* apolar *to $F \in S$ if* $\deg g \leq \deg F$ *and $g(F) = 0$.*

Apolarity defines a duality:

$$\mathbb{P}(T_1) = \mathbb{P}(S_1)^*, \quad \mathbb{P}(T_d) = \mathbb{P}(S_d)^*.$$

The ideal

$$F^\perp := \text{Ann } F = \{g \in T | \; g(F) = 0\}.$$

is called the *apolar ideal* of F. It is generated by the forms in T that are apolar to F.

Lemma **1.5.** *(Apolarity) Let $\Gamma = \{[l_1], \dots, [l_r]\} \subset \mathbb{P}(S_1)$, be a collection of r points. Then*

$$F = \lambda_1 l_1^d + \ldots + \lambda_r l_r^d \quad \text{with } \lambda_i \in \mathbb{C}$$

if and only if

$$I_\Gamma \subset F^\perp \subset T.$$

Proof. It remains to show that $I_{\Gamma,d} \subset F^\perp$ only if $I_\Gamma \subset F^\perp$. Assume therefore that $g \in I_{\Gamma,e}$. Then $T_{d-e} g \subset I_{\Gamma,d}$, so $T_{d-e} g(F) = 0$. But then $g(F) = 0$. □

This lemma motivates the definition of an apolar subscheme.

Definition **1.6.** *$\Gamma \subset \mathbb{P}(S_1)$ is an* apolar subscheme *to F if $\mathcal{I}_\Gamma \subset F^\perp \subset T$.*

1.3. **Apolar ring of a form.** The quotient ring

$$A_F = T/F^\perp$$

is called the *apolar* ring of F. It is a (graded) Artinian Gorenstein ring, i.e. the socle $(0 : m)$ is 1-dimensional:

$$\begin{aligned}(0 : m) &= \{g \in T | g \cdot T_1 \subset F^\perp\}/F^\perp \\ &= \{g \in T | T_1 \cdot g(F) = 0\}/F^\perp = T_d/F^\perp \cong \mathbb{C}\end{aligned}$$

Furthermore, the multiplication

$$A_{F,i} \times A_{F,d-i} \to A_{F,d} \cong \mathbb{C}$$

defines a perfect pairing, so the Hilbert function $H_F(i) = \dim_{\mathbb{C}} A_{F,i}$ is symmetric:

$$H_F(i) = H_F(d-i) \qquad i = 0, ..., d$$

Proposition **1.7.** *(Macaulay correspondence) There is a $1-1$ correspondence between graded Artinian Gorenstein quotients of T and homogeneous forms in S up to scalars.*

Proof. For a graded Artinian Gorenstein quotient A with socle $(0 : m)$ in degree d the linear map

$$T_d \to A_d \cong \mathbb{C}$$

is defined by a linear form on T_d, i.e. an element of $S_d \cong T_d^*$. □

1.4. **Arithmetic Gorenstein Varieties.** A variety $X^n \subset \mathbb{P}^N$ of dimension n is *arithmetic Gorenstein* if its homogeneous coordinate ring $S(X)$ is Gorenstein, i.e. if every quotient of $S(X)$ by a regular sequence of homogeneous forms of length $n+1$ is an Artinian Gorenstein ring.

Equivalently, an arithmetic Gorenstein variety $X \subset \mathbb{P}^N$ is a subcanonical arithmetic Cohen Macaulay variety. Subcanonical here means that the canonical sheaf is $\mathcal{O}_X(n)$ for some n.

The minimal free resolution of the homogeneous ideal I_X is symmetric and of length equal to the codimension of X.

A general linear subspace P of dimension $N-n-1$ does not intersect $X^n \subset \mathbb{P}^N$. But a basis of linear forms in the ideal I_P define a regular sequence of length $n+1$ in $S(X)$ and hence an Artinian Gorenstein quotient $S(X \cap P)$.

By the Macaulay correspondence,

$$S(X \cap P) \cong A_{F_P}$$

for some homogeneous form F_P.

Lemma **1.8.** *The length of A_{F_P} equals the degree of X.*

Example **1.9.** *(Elliptic curves)$E \subset \mathbb{P}^N$ an elliptic normal curve of degree $N+1$ is arithmetic Gorenstein. For any codimension two subspace P such that $P \cap E = \emptyset$, the homogenous from F_P is a quadric of rank $N-1$. In fact, the values of the Hilbert function are $H_{F_P}(0,1,2) = (1, N-1, 1)$.*

Example **1.10.** *(Canonical curves) A canonical curve $C \subset \mathbb{P}^{g-1}$ is arithmetic Gorenstein. For any codimension two subspace P such that $P \cap C = \emptyset$, the homogenous from F_P is a cubic form in $g-2$ variables. In fact, C has degree $2g-2$ and the values of the Hilbert function are $H_{F_P}(0,1,2,3) = (1, g-2, g-2, 1)$.*

Other examples include complete intersections, Grassmannians (linear, Lagrangian, Orthogonal) and $K3$-surfaces.

Let $X^n \subset \mathbb{P}^N$ be arithmetic Gorenstein. Then there is an open set $U \subset \mathbb{G}((,N)-n-1,N)$ parameterizing linear subspaces P of dimension $N-n-1$ that does not intersect X. Let d be the socle degree of $S(X \cap P)$, for $P \in U$. The Macaulay correspondence define a map

$$\alpha_X : U \to S_d//GL(N-n); \quad P \mapsto F_P$$

Question 1.11. *What is the image of α_X?*

1.5. $VSP(F_P, r)$**.** If $X^n \subset Y^{n+1} \subset \mathbb{P}^N$, then

$$I_Y \subset I_X.$$

If both X and Y are arithmetically Cohen Macaulay and P is a general linear subspace that intersects Y properly, then

$$I_{Y\cap P} \subset I_X + I_P.$$

In particular, P is of dimension $N-n-1$, X is arithmetic Gorenstein and I_{Y_P} is the ideal of $Y \cap P$ in $S(P)$, then

$$I_{Y_P} \subset F_P^{\perp}$$

and $Y \cap P$ is apolar to F_P.

Let

$$\mathcal{Y} = \{Y^{n+1} \supset X^n | Y \text{ is aCM}, \ \deg Y = r\}$$

If $P \subset \mathbb{P}^N$ is a general linear subspace of dimension $N-n-1$, then

$$\mathcal{Y} \to Hilb_r P \quad Y \mapsto Y \cap P$$

defines a map

$$\mathcal{Y} \to VSP(F_P, r)$$

Example **1.12.** *(Complete intersections)*

If $X^n \subset \mathbb{P}^N$ is a complete intersection $X = V(g_1, g_2, ..., g_{N-n})$ with $g_i \in T_e$, then any corank one subspace $U \subset \langle g_1, g_2, ..., g_{N-n}\rangle$ defines a variety $Y_U \supset X$ of dimension $n+1$ and degree e^{N-n-1}.

Let $P \subset \mathbb{P}^N$ be a general linear subspace of dimension $N-n-1$, then $Y \cap P$ is an apolar subscheme to F_P of length e^{N-n-1},

In fact F_P has rank e^{N-n-1} (cf. [10] *for the argument presented below) and*

$$VSP(F_P, e^{N-n-1}) \supset \mathbb{P}^{N-n-1}:$$

Assume that F_P has an apolar subscheme Z of length d. Since F_P is generated by forms of degree e and $I_Z \subset F_P^{\perp}$, we can find a form g of degree e such that $I_Z + (g) \subset F_P^{\perp}$ and $Z \cap V(g) = \emptyset$. But then $S(Z)/(g)$ is an Artinian ring of length $\deg Z \cdot e \geq e^{N-n}$, i.e. $\deg Z \geq e^{N-n-1}$. By Bertini a general complete intersection $Y_U \cap P$ is smooth, so F_P has rank e^{N-n-1}. Finally, the set of corank one subspaces U form a $\mathbb{P}^{N-n-1}$.

1.6. **Binary forms.**

Proposition **1.13.** *(Serre) Every arithmetic Gorenstein variety of codimension two is a complete intersection.*

Corollary **1.14.** *The apolar ideal $F^{\perp} \subset \mathbb{C}[y_0, y_1]$ of a binary form $F \in \mathbb{C}[x_0, x_1]$ is a complete intersection $F^{\perp} = (g_1, g_2)$.*

Corollary **1.15.** *The rank of F equals the degree of g_1 or the degree of g_2.*

Proof. Assume that $\deg g_1 \leq \deg g_2$. If g_1 is square free, then (g_1) generates the ideal of a minimal set of distinct points apolar to F. If g_1 is not square free, any element in $F^{\perp}$ of degree less than $\deg g_2$ is a multiple of g_1 and hence not square free. $F^{\perp}$ is generated in degree $\deg g_2$, so a general element in this degree is square free by Bertini. $\square$

2. Exercises

Problem 2.1. *Assume F be a binary form with apolar ideal $F^{\perp}$ generated by g_1 and g_2 of degrees d_1 and d_2 respectively. Assume that $d_1 \leq d_2$. Compute the Hilbert function $H_F(i)$. In particular, show that $d_1 + d_2 = \deg F + 2$.*

Problem 2.2. *Assume $C \subset \mathbb{P}^{g-1}$ is a canonical trigonal curve, and let $P \subset \mathbb{P}^{g-1}$ be a general linear subspace of codimension 2. Compute the rank of the form F_P.*

3. Variety of sums of powers

In small codimension there are general structure theorems for arithmetic Cohen Macaulay and arithmetic Gorenstein varieties. We ended the first lecture with the case of binary forms.

3.1. **Ternary forms.**

Proposition **3.1.** *(Buchsbaum-Eisenbud) Every arithmetic Gorenstein variety of codimension three is defined by the $2r$-dimensional pfaffians of a $(2r+1 \times 2r+1)$-dimensional skew symmetric matrix of homogeneous forms for some r.*

Corollary **3.2.** *The apolar ideal $F^{\perp} \subset \mathbb{C}[y_0, y_1, y-2]$ of a ternary form $F \in \mathbb{C}[x_0, x_1, x_2]$ is a the rank $2r-2$-locus of a $(2r+1 \times 2r+1)$-dimensional skew symmetric matrix of homogeneous forms for some r.*

We call the skew symmetric matrix the Buchsbaum-Eisenbud matrix of the apolar ideal $F^{\perp}$.

Now, any finite subscheme of $\mathbb{P}^2$ is arithmetically Cohen Macaulay.

Proposition **3.3.** *(Hilbert-Burch) The ideal of an arithmetic Cohen Macaulay variety of codimension two is defined by the maximal minors of a $(n \times n+1)$-dimensional matrix of homogeneous forms for some n.*

The matrix is the matrix of syzygies between the generators of the ideal, and is called the Hilbert-Burch matrix of the variety (or ideal).

Example **3.4.** *The general cubic ternary form F has Betti numbers*

$$\begin{matrix} 1 & - & - & - \\ - & 3 & - & - \\ - & - & 3 & - \\ - & - & - & 1 \end{matrix}.$$

for $F^{\perp}$. The pfaffians of the skew symmetric 3×3 matrix is simply the three nonzero quadratic entries in the matrix. So $F^{\perp}$ is a complete intersection. The rank of F is therefore 4, any apolar subscheme of length 4 has a Hilbert-Burch matrix of dimension 1×2 with entries two quadrics, so $VSP(F, 4) = \mathbb{P}^2$.

For a general $2r+1 \times 2r+1$-dimensional skew symmetric matrix with linear entries the ideal generated by the $2r$-dimensional pfaffians have Betti numbers

$$\begin{matrix} 1 & - & - & - \\ - & - & - & - \\ \vdots & \vdots & \vdots & \vdots \\ - & 2r+1 & 2r+1 & - \\ \vdots & \vdots & \vdots & \vdots \\ - & - & - & - \\ - & - & - & 1 \end{matrix}.$$

The Hilbert function H of the Artinian Gorenstein quotient A must have $H(r) = \binom{r+2}{2} - 2r - 1 = \binom{r}{2}$, so

$$H(0,1,2,3,..,r-2,r-1,r,..,2r-3,2r-2) = (1,3,6,...,\binom{r}{2},\binom{r+1}{2},\binom{r}{2},...,3,1)$$

In particular, $A = A_F$ for a form of degree $2r-2$. In fact, this Hilbert function is clearly maximal for such forms, and a general ternary form F of degree $2r-2$ appear this way.

Problem 3.5. *Show that a ternary form F of degree $2r-2$ has an apolar ideal generated by the pfaffians of a $2r+1 \times 2r+1$ skew symmetric matrix of linear forms if and only if its Catalecticant $(\binom{r+1}{2} \times \binom{r+1}{2})$-matrix is nonsingular.*

The minimal length of a finite subscheme in $\mathbb{P}^2$ not contained in any curves of degree $r-1$ is $\binom{r+1}{2}$, and such a scheme has Hilbert Burch matrix of dimensions $(r) \times (r+1)$ with linear entries.

Proposition **3.6.** *(Mukai) When F is a general ternary form of even degree $2r-2$ less than 10, then F has rank $\binom{r+2}{2}$, and the Hilbert-Burch matrix of any apolar subscheme of length $\binom{r+1}{2}$ is a submatrix of the Buchsbaum -Eisenbud matrix of $F^\perp$ complementary to a $r \times r$-square of zeros.*

Furthermore Mukai describes $VSP(F, \mathrm{rank}F)$ in these cases:

Corollary **3.7.** *(Mukai)*

1. *If F is a ternary quadric of rank 3, then $VSP(F,3)$ is a Fano 3-fold of degree 5.*
2. *If F is a general ternary quartic of rank 6, then $VSP(F,6)$ is a Fano 3-fold of degree 22.*
3. *If F is a general ternary sextic of rank 10, then $VSP(F,10)$ is a $K3$-surface of degree 38.*
4. *If F is a general ternary octic of rank 15, then $VSP(F,15)$ is a set of 16 points.*

3.2. **Projection from partials.** We know that apolar subschemes by their span in $\mathbb{P}(S_d)$ define linear subspaces through $[F]$. This property may be extended to partials of F, in which case the apolar subschemes may be easier to detect.

Let $e \leq d$. Then $F_e^\perp$ define a rational map

$$a_{F,e} : \mathbb{P}(S_1) \to \mathbb{P}((F_e^\perp)^*)$$

Since

$F_e^{\perp} = \{g \in T_e | g(F) = 0\} = \{g \in T_e | T_{d-e}(g(f)) = g(T_{d-e}(f)) = 0\} = (T_{d-e}(f))^{\perp} \subset T_e$

the rational map a_F is the composition of the e-uple embedding $\mathbb{P}(S_1) \to \mathbb{P}(S_e)$ and the projection from the space spanned by the partials

$$\langle T_{d-e} \rangle \subset \mathbb{P}(S_e).$$

Therefore we call the map $a_{F,e}$ the projection from the partials of degree e. If $\Gamma \subset \mathbb{P}(S_1)$ is apolar to F, then $I_{\Gamma,e} \subset F_e^{\perp}$, which means that the span of $a_{F,e}(\Gamma)$ has codimension equal to the codimension of the span of Γ in $\mathbb{P}(S_e)$.

Example **3.8.** *Let $(n,d) = (2,3)$, and F is general, i.e. $V(F)$ is a general plane cubic curve. Then $F^{\perp}$ is a complete intersection of three quadrics (g_1, g_2, g_3). The projection of the partials of degree 2 is then the morphism*

$$a_{F,2} : \mathbb{P}^2 \to \mathbb{P}^2; \quad (a_0 : a_1 : a_2) \mapsto (g_1(a_0, a_1, a_2) : g_2(a_0, a_1, a_2) : g_3(a_0, a_1, a_2))$$

A subscheme of length 4 in $\mathbb{P}(S_1)$ spans a $\mathbb{P}^4$ in $\mathbb{P}(S_2)$ unless it is contained in a plane, so the span of every apolar scheme Γ of length 5 has codimension 5, so the same is true for the codimension of the span of $a_{F,2}(\Gamma) \subset \mathbb{P}^5$. So $a_{F,2}(\Gamma)$ is a point, and Γ is the complete intersection of two quadrics in (g_1, g_2, g_3).

Example **3.9.** *Let $(n,d) = (3,3)$, and F is general, i.e. $V(F)$ is a general plane cubic surface. Then $F^{\perp}$ is generated by six quadrics $(g_1, ..., g_6)$. The projection of the partials of degree 2 is then the morphism*

$$a_{F,2} : \mathbb{P}^3 \to \mathbb{P}^5; \quad (a_0 : a_1 : a_2 : a_3) \mapsto (g_1(a_0, a_1, a_2, a_3) : ... : g_6(a_0, a_1, a_2, a_3))$$

A subscheme of length 5 in $\mathbb{P}(S_1)$ spans a $\mathbb{P}^4$ in $\mathbb{P}(S_2)$ unless it is contained in a line, so the span of every apolar scheme Γ of length 4 has codimension 2, so the same is true for the codimension of the span of $a_{F,2}(\Gamma) \subset \mathbb{P}^2$. So $a_{F,2}(\Gamma)$ is a point, and Γ is contained in five independant quadrics in $(g_1, ..., g_6)$, i.e. Γ is the fiber over a 5-tuple point of the map $a_{F,2}$. (In fact there is a unique such Γ).

3.3. **Cubic forms in at most 6 variables.** We have seen that codimension 2 sections of a canonical curve of genus g define via the Macaulay Correspondence cubic forms in $g-2$ variables.. It is natural to ask, which cubic forms appear this way. By simple dimension count, the space of canonical curves has dimension $3g-3$, the Grassmannian of codimension 2 subspaces in $\mathbb{P}^{g-1}$ has dimension $2g-4$, so altogether we get at most a $5g-7$ dimensional family of cubic forms in $g-2$ variables. The space of cubic forms modulo change of variables has dimension $\binom{g}{3} - (g-2)^2$, which is larger than $5g-7$ when $g \geq 10$. Mukai shows that a general canonical curve is a complete intersection on a himogeneous variety for $g \leq 9$, so this count is wrong, the estimate in $g = 8, 9$ is not good enough. In fact you get the general cubic when $g \leq 7$. In genus 8 you get a codimension one family of cubics, and in genus 9 you get a set of cubics of larger codimension.

Let us enumerate the cases $3 \leq g \leq 8$

(1) $g = 3$. C is a plane quartic, and any cubic F in one variable is a a cube with Hilbert function $(1,1,1)$ for the apolar ring $VSP(F,1) = pt$.
(2) $g = 4$. C is a complete intersection $(2,3)$ as is the apolar ideal of a cubic binary form F. $VSP(F,2) = pt$.
(3) $g = 5$. C is a complete intersection $(2,2,2)$ as is the apolar ideal of a cubic binary form F. $VSP(F,4) = \mathbb{P}^2$.

(4) $g = 6$. C is a complete intersection of $\mathbb{G}(2,5)$ with four hyperplanes and a quadric. The four hyperplanes define unique ACM surface of degree 5 containg C. $VSP(F,5) = pt$.
(5) $g = 7$. C is a complete intersection of S_10, the 10-dimensional spinor variety with nine hyperplanes. $VSP(F,8)$ is a Fano 5-fold.
(6) $g = 8$. C is a complete intersection of $G(2,8)$, with eight hyperplanes. $VSP(F,10)$ is a Hyperkähler 4-fold.

We now discuss the two last ones in more detail.

3.4. **Cubic 3-folds.** The spinor 10-fold in $\mathbb{P}^{15}$ is defined by 10 quadrics that define surjective map

$$\alpha : \mathbb{P}^{15} \cdots > Q \subset \mathbb{P}^9$$

where Q is smooth quadric hypersurface. The spinor variety parameterizes one of the two families of $\mathbb{P}^4$'s in Q, in fact the rational map is regular on the blowup of $\mathbb{P}^{15}$ along S_{10} and over each point on S_{10} there is a $\mathbb{P}^4$ in the exceptional divisor which is mapped to a $\mathbb{P}^4$ in Q.

Let P be a general $\mathbb{P}^4$ in $\mathbb{P}^{15}$, and let $F = F_P$ the associated cubic form. The projection from partials $a_{F,2}$ is then nothing but the restriction of the map α to P.

Now, F is general, so by the Alexander Hirschowitz theorem the rank of F is 8. Let $\Gamma \subset P$ be a general set of 8 points apolar to P. Then the span of Γ has codimension 7 in the 2-uple embedding, therefore $\alpha(\Gamma)$ spans only a plane. This plane L must be contained in the quadric $Q \subset \mathbb{P}^9$, so Γ is the intersection of $Y_L = \alpha^{-1}(L)$ with P. In fact one can show that Y_L is an arithmetic CM variety of degree 8 and dimension 9 inside a pencil of tangent hyperplanes to S. Now, the pencils of tangent hyperplanes that contain P coincide with the lines inside the dual variety of S_{10} intersected with the $\mathbb{P}^{10}$ of hyperplanes that contain P. Now, S_{10} is isomorphic to its dual, so these lines are lines inside a fivefold linear section of S_10. In fact, in [9] we prove

Proposition **3.10.** *$VSP(F;8)$ is isomorphic to the Fano variety of lines in the fivefold proper intersection of S_{10} with a $\mathbb{P}^{10}$.*

3.5. **Cubic 4-folds.** The ideal of the Grassmannian $\mathbb{G}((,2),6) \subset \mathbb{P}^{14}$ is generated by 15 quadrics that define a Cremona transformation

$$\alpha : \mathbb{P}^{14} \cdots > \mathbb{P}^{14}.$$

The secant variety of $\mathbb{G}((,2),6) \subset \mathbb{P}^{14}$ is a Pfaffian cubic hypersurface that is contract by α to the dual $\mathbb{G}((,2),6)$.

Let P be a general $\mathbb{P}^5$ in $\mathbb{P}^{14}$, and let $F = F_P$ the associated cubic form. The projection from partials $a_{F,2}$ is then nothing but the restriction of the map α to P.

Now, the cubics F we obtain this way form a hyperspace in the space of cubic forms. By a simple dimension count, the cubic forms of rank at most 9 form a variety of codimension at least 2, so a general F has rank 10.

Take a general set Γ of 10 points apolar to F. Its span in the 2-uple embedding has codimension 11, so the span of $\alpha(\Gamma)$ is a $\mathbb{P}_3$, called P_Γ.

It is shown in [5] that $\alpha(\Gamma) \subset G(2,6) \cap P_\Gamma$ and that $G(2,6) \cap P_\Gamma$ is a quadric surface. Furthermore,

$$Y = \alpha^{-1}(P_\Gamma) = C_{p_1} \cap C_{p_2} \subset HT_{p_1} \cap HT_{p_2}$$

where p_1 and p_2 are points in the strict dual variety of $G(2,6)$, the hyperplanes HT_{p_1}, HT_{p_2} are the corresponding special tangent hyperplanes, and C_{p_1}, C_{p_2} are the tangent cones to $G(2,6)$ inside these hyperplanes. Each C_{p_i} is isomorphic to a cone over $\mathbb{P}^1 \times \mathbb{P}^3$, and Y is a 7-dimensional aCM variety of degree 10. This variety Y intersects P in Γ, since HT_{p_1}, HT_{p_2} both contain P. Such pair of hyperplanes are parametrized by pairs of points on the variety $S(P) = \mathbb{P}^8 \cap G(2,6)$ of special hyperplanes that contain P. In [5] this argument is completed to give a proof that

Proposition **3.11.** *$VSP(F;10)$ is isomorphic to $Hilb_2(S(P))$. In particular, since $S(P)$ is a $K3$-surface, $VSP(F;10)$ is a hyperkähler 4-fold.*

In fact by deformation, the $VSP(F,10)$ of a general cubic fourfold F is a hyperkähler fourfold.

There is another interesting codimension one family of cubic fourfolds, namely those that are apolar to a Veronese surface. In that case the restriction of the cubic form F to the Veronese surface is a ternary sextic form G. The $VSP(G,10)$ embeds naturally in $VSP(F;10)$. The former is a $K3$ surface of genus 10 as we saw above. In fact $VSP(F;10)$ is singular along this surface. This follows from the following general criterion for singularities on a $VSP(F, \text{rank } F$

Proposition **3.12.** *Assume that Γ is an apolar subscheme of rank equal to the minimal rank of F, and F has generic rank. If some hypersurface of degree equal to $degF$ is singular along Γ, then Γ is a singular point on $VSP(F, \text{rank } F)$*

Proof. From the incidence
$$\{(F,\Gamma)|I_\Gamma \subset F^\perp\}$$
the first projection is ramified, by Terracini's lemma, at any Γ satisfying the condition of the lemma. □

4. Exercises

Problem 4.1. *Assume $F^\perp$ is a complete intersection of r forms of degree e. Find the degree of F.*

Problem 4.2. *Compute the rank of the cubic form $x(xy - z^2)$.*

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