POWER SUM DECOMPOSITION AND APOLARITY, A GEOMETRIC APPROACH. WARING PROBLEMS, SECANT VARIETIES AND SYLVESTER ALGORITHM. LECTURE NOTES

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Abstract. This is a preliminary version of the Lecture notes on "Waring problems, Secant varieties and Sylvester algorithm" for the 36th Autumn School in Algebraic Geometry "Power sum decomposition and apolarity, a geometric approach" September 1st-7th, 2013 Lukecin, Poland

On dimensions of secant varieties

1. Lecture 1

1.1. **On the Waring problem.** The story begins with a number theory question: in 1770 E. Waring in [33] stated (without proofs) that:

> "Every natural number is sum of at most 9 positive cubes."
> "Every natural number is sum of at most 19 biquadratics."

Moreover, he believed that:

> "for all integers $d \geq 2$ there exists a positive integer $g(d)$ such that each $n \in \mathbb{Z}^+$ can be written as $n = a_1^d + \cdots + a_{g(d)}^d$ with $a_i \geq 0$, $i = 1, \ldots, g(d)$."

Waring belief was showed to be true by Hilbert in 1909 who proved that such a $g(d)$ exists for every $d \geq 2$ and he computed it.

An analogous problem can be formulated for homogeneous polynomials.

Let K be an algebraically closed field of characteristic zero. We will work on the projective space $\mathbb{P}^n = \mathbb{P}(V)$. The polynomial ring $S := K[x_0, \ldots, x_n]$ is a graduated ring and so we can write it as $K[x_0, \ldots, x_n] = \bigoplus_{d\geq 0} S_d$ where
$S_d =< x_0^d, x_0^{d-1}x_1, \ldots, x_n^d >= S^dV$ is the vector space of homogeneous forms of degree d (or the space of symmetric tensors of order d over a vector space of dimension $n+1$). It is a well known fact that $\dim_K(S_d) = \begin{pmatrix} d+n \\ n \end{pmatrix}$. In a geometric language those vector spaces S_d are called *Complete Linear Systems of hypersurfaces of degree d in $\mathbb{P}^n$*.
Sometimes we will write $\mathbb{P}(S_d)$ in order to mean the projectivization of S_d, therefore $\mathbb{P}(S_d)$ will be a $\mathbb{P}^{\binom{n+d}{d}-1}$ whose elements will be classes of forms of degree d: $[F] \in \mathbb{P}(S_d)$ with $F \in S_d$.

The analogous of Waring Problem for polynomials is the so called *Little Waring Problem*:

"Find the minimum $s \in \mathbb{Z}$ such that all forms $F \in S_d$ are sum of at most s d-th powers of linear forms."

The problem we are interested in is a slightly different form of the little Waring problem, it is called the *Big Waring Problem* and it is formulated as follows:

"Which is the minimum $s \in \mathbb{Z}$ such that the *generic* form $F \in S_d$ is a sum of at most s d-th powers of linear forms?"

$$F = L_1^d + \cdots + L_s^d$$

In order to know which elements of S_d can be written as sum of s d-th powers of linear forms, we study the image of the map

$$\phi : \underbrace{S_1 \times \cdots \times S_1}_{s} \longrightarrow S_d, \;\; \phi(L_1, \dots, L_s) = L_1^d + \cdots + L_s^d. \tag{1}$$

The Big Waring problem asks to find the smallest s such that $\overline{Im(\phi_d)} = S_d$ (we just observe that if we require $\dim(\phi_d) = S_d$ we would solve the little Waring problem).

The map ϕ can be viewed as a polynomial map between affine spaces:

$$\phi : \mathbb{A}^{s(n+1)} \longrightarrow \mathbb{A}^{N=\binom{n+d}{n}}.$$

In order to know the dimension of the image of such a map we look at its differential

$$d\phi|_P : T_P(\mathbb{A}^{s(n+1)}) \longrightarrow \mathbb{A}^N.$$

Let $P = (L_1, \dots, L_s) \in \mathbb{A}^{s(n+1)}$ and $v = (M_1, \dots, M_s) \in T_P(\mathbb{A}^{s(n+1)}) \simeq \mathbb{A}^{s(n+1)}$ where $L_i, M_i \in S_1$ for $i = 1, \dots, s$. Let us consider the following parameterizations $t \longmapsto (L_1 + M_1 t, L_2 + M_2 t, \dots, L_s + M_s t)$ of a line $\mathcal{C}$ passing through P whose tangent vector at P is M. The image of $\mathcal{C}$ via ϕ is $\phi(L_1 + M_1 t, L_2 + M_2 t, \dots, L_s + M_s t) = \sum_{i=1}^s (L_i + M_i t)^d$. The tangent vector to $\phi(\mathcal{C})$ in $\phi(P)$ is $\lim_{t\to 0} \frac{d}{dt}\left(\sum_{i=1}^s (L_i + M_i t)^d\right) = \lim_{t \to 0} \sum_{i=1}^s d(L_i + M_i t)^{d-1} M_i = \sum_{i=1}^s dL_i^{d-1} M_i$. Now, as $v = (M_1, \dots, M_s)$ varies in $\mathbb{A}^{s(n+1)}$, the tangent vectors we get span $< L_1^{d-1} S_1, \dots, L_s^{d-1} S_1 >$.

Hence we can say:

Proposition 1. *Let $L_1, \dots, L_s$ be linear forms in $S = K[x_0, \dots, x_n]$, where $L_i = a_{i_0} x_0 + \cdots + a_{i_n} x_n$ and*

$$\phi : \underbrace{S_1 \times \cdots \times S_1}_{s} \longrightarrow S_d, \;\; \phi(L_1, \dots, L_s) = L_1^d + \cdots + L_s^d;$$

then

$$rk(d\phi)|_{(L_1, \dots, L_s)} = \dim_K < L_1^{d-1} S_1, \dots, L_s^{d-1} S_1 > .$$

It is very interesting to have a look at how the problem of determining this dimension has been solved, because the solution involves many algebraic and geometric tools.

1.2. **Veronese variety.** The first geometric object that is related with our problem is the "Veronese variety". We recall that the *Veronese variety* is the image of the following embedding:

$$\begin{array}{rcl} \nu_d : \mathbb{P}^n & \hookrightarrow & \mathbb{P}^{\binom{n+d}{d}-1} \\ (u_0 : \ldots : u_n) & \mapsto & (u_0^d : u_0^{d-1} u_1 : u_0^{d-1} u_2 : \ldots : u_n^d). \end{array}$$

This embedding can also be dually characterized as:

$$\begin{array}{rccll} \nu_d: & \mathbb{P}(S_1)=(\mathbb{P}^n)^* & \hookrightarrow & \mathbb{P}(S_d) & =\left(\mathbb{P}^{\binom{n+d}{d}-1}\right)^* \\ & [L] & \mapsto & [L^d] & \\ & [v] & \mapsto & [v^{\otimes d}]. & \end{array}$$

Therefore we can think to the Veronese variety as the variety that parameterizes d-th powers of linear forms or completely decomposable symmetric tensors.

Example 1. Let $V=\mathbb{C}^2$ and $d=3$, then

$$\begin{array}{rccc} \nu_3: & \mathbb{P}^1 & \hookrightarrow & \mathbb{P}^3 \\ & [a_0,a_1] & \mapsto & [a_0^3, a_0^2a_1, a_0a_1^2, a_3] \end{array} .$$

If we take $\{z_0,\dots,z_3\}$ be the coordinates in $\mathbb{P}^3$, then the equations of the Veronese curve in $\mathbb{P}^3$ are $F_0(\underline{z})=z_0z_2-z_1^2$, $F_1(\underline{z})=z_0z_3-z_1z_2$, $F_2(\underline{z})=z_1z_3-z_2^2$. Observe that those equations can be obtained as maximal minors of the following matrix:

$$\begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \end{pmatrix}. \tag{2}$$

Notice that this matrix can be obtained both as the defining matrix of the following linear map:

$$S^2\mathbb{C}^{2*}\to S^1\mathbb{C}^2,\ \ \partial_{t_i}^2\mapsto \partial_{t_i}^2(f)$$

where $f=\sum_{i=0}^3\binom{d}{i}^{-1}z_it_0^{3-i}t_1^i$; or by flattening a $2\times2\times2$ cube where at the vertex in position ijk there is the element z_{i+j+k} and then removing the repeated column.

The phenomenon described in Example 1 is a general fact: Veronese varieties are always defined by 2×2 minors of matrices constructed as (2) (we will call them Catalecticant matrices).

A hypermatrix (or a tensor) $A=(x_{i_1,\dots,i_t})_{1\le i_j\le n_j\,,\,j=1,\dots,t}$ is said to be a *generic hypermatrix of indeterminates* (or more simply *generic hypermatrix*) of $S:=K[x_{i_1,\dots,i_t}]_{1\le i_j\le n_j\,,\,j=1,\dots,t}$, if the entries of A are the independent variables of S.

The ideal of the 2-minors of a generic hypermatrix $A=(x_{i_1,\dots,i_t})_{1\le i_j\le n_j,\,j=1,\dots,t}$ is

$$I_2(A):=(x_{i_1,\dots,i_l,\dots,i_t}x_{j_1,\dots,j_l,\dots,j_t}-x_{i_1,\dots,j_l,\dots,i_t}x_{j_1,\dots,i_l,\dots,j_t})_{l=1,\dots,t;\;1\le i_k,j_k\le n_j\,,\,k=1,\dots,t}.$$

It is a classical result (see [20]) that a set of equations for a Segre Variety is given by all the 2-minors of a generic hypermatrix. In fact a Segre variety parameterizes decomposable tensors, i.e. all the "rank one" tensors.

In [21] (Theorem 1.5) it is proved that, if A is a generic hypermatrix of a polynomial ring S of size $n_1\times\cdots\times n_t$, then $I_2(A)$ is a prime ideal in S, therefore:

$$I(Seg(V_1\otimes\cdots\otimes V_t))=I_2(A)\subset S.$$

Definition 1. A hypermatrix $A=(a_{i_1,\dots,i_d})_{1\le i_j\le n,\,j=1,\dots,d}$ is said to be "symmetric" (or completely symmetric) if $a_{i_1,\dots,i_d}=a_{i_{\sigma(1)},\dots,i_{\sigma(d)}}$ for all $\sigma\in\mathfrak{S}_d$ where $\mathfrak{S}_d$ is the permutation group of $\{1,\dots,d\}$.

With an abuse of notation we will say that a tensor $T\in V^{\otimes d}$ is symmetric if it can be represented by a symmetric hypermatrix.

Definition 2. Let $H \subset V^{\otimes d}$ be the $\binom{n+d-1}{d}$-dimensional subspace of the symmetric tensors of $V^{\otimes d}$, i.e. H is isomorphic to the symmetric algebra $Sym_d(V)$. Let $\tilde{S}$ be a ring of coordinates on $\mathbb{P}^{\binom{n+d-1}{d}-1} = \mathbb{P}(H)$ obtained as the quotient $\tilde{S} = S/I$ where $S = K[x_{i_1,\dots,i_d}]_{1\leq i_j \leq n,\, j=1,\dots,d}$ and I is the ideal generated by all

$$x_{i_1,\dots,i_d} - x_{i_{\sigma(1)},\dots,i_{\sigma(d)}}, \forall\, \sigma \in \mathfrak{S}_d.$$

The hypermatrix $(\overline{x}_{i_1,\dots,i_d})_{1\leq i_j \leq n,\, j=1,\dots,d}$ whose entries are the indeterminates of $\tilde{S}$, is said to be a "generic symmetric hypermatrix".

The Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d-1}{d}-1}$ can be viewed as $Seg(V^{\otimes d}) \cap \mathbb{P}(H) \subset \mathbb{P}(H)$.
Let $A = (x_{i_1,\dots,i_d})_{1\leq i_j \leq n,\, j=1,\dots,d}$ be a generic symmetric hypermatrix, then it is a known result that:

$$I(\nu_d(\mathbb{P}^n)) = I_2(A) \subset \tilde{S}. \tag{3}$$

See [32] for set theoretical point of view. In [29] the author proved that $I(Y_{n-1,d})$ is generated by the 2-minors of a particular catalecticant matrix (for a definition of "Catalecticant matrices" see e.g. either [29] or [19]). A. Parolin, in his PhD thesis ([28]), proved that the ideal generated by the 2-minors of that catalecticant matrix is actually $I_2(A)$, where A is a generic symmetric hypermatrix.

The analogous can be done for Segre-Veronese varieties: they are generated by 2×2 minors of a generic hypermatrix with partial symmetric symmetries (cfr [9]).

1.3. **Secant varieties.**

Definition 3. Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n; we define $\sigma_s(X)$ the s-th secant variety of X as follows:

$$\sigma_s(X) := \overline{\bigcup_{P_1,\dots,P_s \in X} < P_1, \dots, P_s >}$$

where $< P_1, \dots, P_s >$ is the $(s-1)$-projective space containing $P_1, \dots, P_s \in X$.

The generic element of $\sigma_s(X)$ is sum of s elements of X.

Example 2. Let $L_1, L_2 \in S^1V$ be tow homogeneous linear forms. The polynomial $L_1^{d-1}L_2$ is clearly in $\sigma_2(\nu_d(\mathbb{P}(V)))$ since $L_1^{d-1}L_2 = \lim_{t\to 0} 1/t((L_1 + tL_2)^d - L_2^d)$ but there do not exist $M_1, M_2 \in S^1V$ such that $L_1^{d-1}L_2 = M_1^d + M_2^d$.

Obviously

$$n = \dim(X) < \dim(\sigma_2(X)) < \dim(\sigma_3(X)) < \cdots < \dim(\sigma_s(X)) = N.$$

Definition 4. The smallest $s \in \mathbb{Z}$ such that $\sigma_s(X) = \mathbb{P}^N$ is the Generic Rank of X.

The generic rank of X is an invariant of the embedded variety X.

If we consider the d-uple Veronese embedding of $\mathbb{P}^n$ it can be viewed as the subset of $K[x_0, \dots, x_n]_d$ made by all forms which can be written as d-powers of linear forms. From this point of view the generic rank s of the Veronese variety is the minimum integer such that the generic form of degree d in $n+1$ variables is a linear combination of s powers of linear forms in the same number of variables. I.e. the generic rank of polynomials of given degree in certain numbers of variables is the solution to the Big Waring Problem.

We want to study the problem of determining the dimension of s-th secant varieties of an n-dimensional projective variety $X \subset \mathbb{P}^N$.

Let $X^s := \underbrace{X \times \cdots \times X}_{s}$, $X_0 \subset X$ be the open subset of regular points of X and $U_s(X)$ be the subset of X^s defined as

$$U_s(X) = \{(P_1, \dots, P_s) \in X^s \mid P_i \in X_0 \; \forall\, i \text{ and the } P_i\text{'s are independent}\}.$$

Therefore for all $(P_1, \dots, P_s) \in U_s(X)$ the span $< P_1, \dots, P_s >$ is a $\mathbb{P}^{s-1}$.

Consider the following incidence variety:

$$I^s(X) = \{(Q, \pi) \in \mathbb{P}^N \times U_s(X) \mid Q \in \pi\}$$

The dimension of that variety is

$$\dim(I^s(X)) = n(s-1) + n + s - 1.$$

With this definition we can consider the usual projection

$$p : I^s(X) \to \mathbb{P}^N;$$

the s-th secant variety of X is just the image of the map p:

$$\sigma_s(X) = \overline{Im(p : I^s(X) \to \mathbb{P}^N)}.$$

Now, if $\dim(X) = n$, it is clear that, while $\dim(I^s(X)) = ns+s-1$, the dimension of $\sigma_s(X)$ can be smaller: it suffices that the generic fiber of p_1 has positive dimension to impose $\dim(\sigma_s(X)) < n(s-1)+n+s-1$. So it is a general fact that if $X \subset \mathbb{P}^N$ and $\dim(X) = n$ then:

$$\dim(\sigma_s(X)) \leq \min\{N, sn + s - 1\}.$$

Definition 5. A projective variety $X \subset \mathbb{P}^N$ of dimension n is said to be s-defective if $\dim(\sigma_s(X)) < \min\{N, sn+s-1\}$ and $\delta_s(X) := \min\{N, sn+s-1\} - \dim(\sigma_s(X))$ is called the s-th defect of X.

Alexander Hirschowitz Theorem ([2]) tells that the dimension of the s-th secant variety to the Veronese variety is not always the expected one and they are able to list all of them:

Theorem 1 (**Alexander-Hirschowitz**). *If $X = \sigma_s(\nu_d(\mathbb{P}^n))$, for $d \geq 2$. Then:*

$$\dim(X) = \min\left\{\binom{n+d}{d} - 1, s(n+1) - 1\right\}$$

except for:

- $d = 2$, $n \geq 2$, $s \leq n$;
- $d = 3$, $n = 4$, $s = 7$, $(\delta = 1)$;
- $d = 4$, $n = 2$, $s = 5$, $(\delta = 1)$;
- $d = 4$, $n = 3$, $s = 9$, $(\delta = 2)$;
- $d = 4$, $n = 4$, $s = 14$, $(\delta = 1)$.

The mail ingredient is Terracini's lemma (see [31], or [1]).

Lemma 1. (**Terracini's Lemma**) *Let X be an irreducible variety in $\mathbb{P}^N$, and let $P_1, \dots, P_s$ be s generic points on X. Then, the projectivised tangent space to $\sigma_s(X)$ at a generic point $Q \in < P_1, \dots, P_s >$ is the linear span in $\mathbb{P}^N$ of the tangent spaces $T_{P_i}(X)$ to X at P_i, $i = 1, \dots, s$, i.e.*

$$T_Q(\sigma_s(X)) = < T_{P_1}(X), \dots, T_{P_s}(X) > .$$

This "Lemma" can be proved in many ways, we present here a proof "made by hands".

Proof. We have already used the notation X^s for $X\times\cdots\times X$ taken s times. Suppose that $\dim(X)=n$. Let us consider the following incidences variety:

$$I=\{(P;P_1,\ldots,P_s)\in\mathbb{P}^n\times X^s \mid P\in\langle P_1,\ldots,P_s\rangle, P_1,\ldots,P_s \text{ generic in } X\}\subset\mathbb{P}^n\times X^s,$$

and the two following projections:

$$\pi_1: I\to\sigma_s(X)$$

and

$$\pi_2: I\to X^s.$$

The dimension of X^s is clearly sn. If $(P_1,\ldots,P_s)\in X^s$ the fiber $\pi_2^{-1}((P_1,\ldots,P_s))$ is generically a $\mathbb{P}^{s-1}$, $s<N$. Then $\dim(I)=sn+s-1$. If π_1 has finite fibers the $(s-1)$-secant variety to X is regular, otherwise it is defective with defect equal to the dimension of the generic fiber.

Suppose that each $P_i\in X\subset\mathbb{P}^N$ has coordinates $P_i=[a_{i,0},\ldots,a_{i,N}]$ for $i=1,\ldots,s$; around each P_i the variety X can be locally parameterized with some functions $f_{i,j}: K^{n+1}\to K^{n+1}$ for $i=1,\ldots,s$ and $j=0,\ldots,N$ that are zero at the origin:

$$X:\begin{cases} x_0=a_{i,0}+f_{i,0}(u_{i,0},\ldots,u_{i,n})\\ \vdots\\ x_N=a_{i,N}+f_{i,N}(u_{i,0},\ldots,u_{i,n})\end{cases}.$$

Now we need a parameterization φ for $\sigma_s(X)$. Consider a point in the subspace spanned by s points of X (for simplicity of notation we omit the dependence of the $f_{i,j}$ from the variables $u_{i,j}$): $<(a_{1,0}+f_{1,0},\ldots,a_{1,N}+f_{1,N}),\ldots,(a_{s,0}+f_{s,0},\ldots,a_{s,N}+f_{s,N})>$; an element of this subspace is of the form: $\lambda_1(a_{1,0}+f_{1,0},\ldots,a_{1,N}+f_{1,N})+\lambda_2(a_{2,0}+f_{2,0},\ldots,a_{2,N}+f_{2,N})+\cdots+\lambda_s(a_{s,0}+f_{s,0},\ldots,a_{s,N}+f_{s,N})$ for some $\lambda_1,\ldots,\lambda_s\in K$ (we can assume that $\lambda_1=1$). Therefore a parameterization of the s-th secant variety to X can be obtained by $(a_{1,0}+f_{1,0},\ldots,a_{1,N}+f_{1,N})+(\lambda_2+t_2)(a_{2,1}-a_{1,0}+f_{2,1}-f_{1,0},\ldots,a_{2,N}-a_{1,N}+f_{2,N}-f_{1,N})+\cdots+(\lambda_s+t_s)(a_{s,1}-a_{1,0}+f_{s,1}-f_{1,0},\ldots,a_{s,N}-a_{1,N}+f_{s,N}-f_{1,N})$ for some parameters $t_2,\ldots,t_s$, i.e. in coordinates the parameterization φ that we are looking for is that one that sends an element $(u_{1,0},\ldots u_{1,n},u_{2,0},\ldots,u_{2,n},\ldots\ldots,u_{s,0},\ldots,u_{s,n},t_2,\ldots,t_s)\in K^{s(n+1)+s-1}$ into

$$(\ldots,a_{1,j}+f_{1,j}+(\lambda_2+t_2)(a_{2,j}-a_{1,j}+f_{2,j}-f_{1,j})+\cdots+(\lambda_s-t_s)(a_{s,j}-a_{1,j}+f_{s,j}-f_{1,j}),\ldots)\in K^{N+1}.$$

For simplicity we have written only the j-th element of the image. Therefore we are able to write the Jacbian of φ. We are writing it in three blocks: the first one is $(N+1)\times(n+1)$, the second one is $(N+1)\times(s-1)(n+1)$ and the third one is $(N+1)\times(s-1)$:

$$J_{\underline{0}}(\varphi)=\left(\begin{array}{c|c|c}(1-\lambda_2-\cdots-\lambda_s)\frac{\partial f_{1,j}}{\partial u_{1,k}} & \lambda_i\frac{\partial f_{i,j}}{\partial u_{i,k}} & a_{i,j}-a_{1,j}\end{array}\right),$$

with $i=2,\ldots,s$; $j=0,\ldots,N$ and $k=0,\ldots,n$. Now the first block is a base of the tangent space to X at P_1, and in the second block we can find the bases for the tangent spaces to X at $P_2, \dots, P_s$; the rows of

$$\begin{pmatrix} \frac{\partial f_{i,0}}{\partial u_{i,0}} & \cdots & \frac{\partial f_{i,0}}{\partial u_{i,N}} \\ \vdots & & \vdots \\ \frac{\partial f_{i,N}}{\partial u_{i,0}} & \cdots & \frac{\partial f_{i,N}}{\partial u_{i,N}} \end{pmatrix}$$

give a base for $T_{P_i}(X)$. □

Corollary 1. *Let $(X, \mathcal{L})$ be an integral, polarized scheme. If $\mathcal{L}$ embeds X as a closed scheme in $\mathbb{P}^N$, then*

$$\dim(\sigma_s(X)) = N - \dim(h^0(\mathcal{I}_{Z,X} \otimes \mathcal{L}))$$

where Z is the union of s generic 2-fat points in X.

Proof. By Terracini's Lemma, $\dim(\sigma_s(X)) = \dim(< T_{P_1}(X), \dots, T_{P_s}(X) >)$, with $P_1, \dots, P_s$ generic points on X. Since X is embedded in $\mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{L})^*)$, we can view the elements of $H^0(X, \mathcal{L})$ as hyperplanes in $\mathbb{P}^N$; the hyperplanes which contain a space $T_{P_i}(X)$ correspond to elements in $H^0(\mathcal{I}_{2P_i,X} \otimes \mathcal{L})$, since they intersect X in a subscheme containing the first infinitesimal neighborhood of P_i. Hence the hyperplanes of $\mathbb{P}^N$ containing the subspace $< T_{P_1}(X), \dots, T_{P_s}(X) >$ are the sections of $H^0(\mathcal{I}_{Z,X} \otimes \mathcal{L})$, where Z is the scheme union of the first infinitesimal neighborhoods in X of the points P_i's. □

Remark 1. A hyperplane H contains the tangent space to a projective variety X at a smooth point P if and only if the intersection $X \cap H$ has a singular point at P.

In fact the tangent space $T_P(X)$ to X at P has the same dimension of X and $T_P(X \cap H) = H \cap T_P(X)$. Moreover P is singular in $H \cap X$ if and only if $\dim(T_P(X \cap H)) \geq \dim(X \cap H) = \dim(X) - 1$ and this happens if and only if $H \supset T_P(X)$.

Example 3. Consider the Veronese surface of $\mathbb{P}^5$. Let P be a general point of $\sigma_2(\nu_2(\mathbb{P}^2))$ and suppose that $P \in < R, Q >$ where $R, Q \in \nu_2(\mathbb{P}^2)$. By Terracini's Lemma $T_P(\sigma_2(\nu_2(\mathbb{P}^2))) =< T_R(\nu_2(\mathbb{P}^2)), T_Q(\nu_2(\mathbb{P}^2)) >$. The expected dimension for $\sigma_2(\nu_2(\mathbb{P}^2))$ is 5, so $\dim(T_P(\sigma_2(\nu_2(\mathbb{P}^2)))) < 5$ if and only if there exists a hyperplane H containing $T_P(\sigma_2(\nu_2(\mathbb{P}^2)))$. The Remark above tells us that this happens if and only if there exists a hyperplane H such that $H \cap \nu_2(\mathbb{P}^2)$ is singular at R, Q.
Now $\nu_2(\mathbb{P}^2)$ is the image of $\mathbb{P}^2$ via the map defined by complete linear system of quadrics hence $\nu_2(\mathbb{P}^2) \cap H$ is the image of plane conics. Let R', Q' be the pre-images via ν_2 of R, Q respectively. Then $2 < R', Q' >$ is a plane conic singular at R' and Q'; it corresponds to the hyperplane section of $\nu_2(\mathbb{P}^2)$ which is singular at R, Q. Since $2 < R', Q' >$ is the only one plane conic singular at R', Q' we can say that $\dim(T_P(\sigma_1(\nu_2(\mathbb{P}^2)))) = 4 < 5$.
Since the 2-Veronese surface is defined by the complete linear system of quadrics, the Corollary 1 allows to rephrase the defectivity of $\sigma_1(\nu_2(\mathbb{P}^2))$ in terms of number of conditions imposed by 2-fat points to forms of degree 2; i.e. "two 2-fat points of $\mathbb{P}^2$ do not impose independent conditions to the degree 2 forms of $K[x_0, x_1, x_2]$".

Corollary 1 can be generalized to non complete linear systems on X.

Remark 2. Let D be any divisor of an irreducible projective variety X. With $|D|$ we indicate the complete linear system defined by D. Let $V \subset |D|$ be a linear system. We use the notation

$$V(m_1P_1,\ldots,m_sP_s)$$

for the subsystem of divisors of V passing through the fixed points $P_1,\ldots,P_s$ with multiplicities at least $m_1,\ldots,m_s$ respectively.

When the multiplicities m_i are equal to 2 for $i=1,\ldots,s$, the problem of the knowledge of $\dim(V(2P_1,\ldots,2P_s))$ is equivalent to that of the dimension of the s-th secant variety to a variety obtained as the closure of the image of the map we are going to define.

Suppose that V is associated to a morphism $\varphi_V : X_0 \to \mathbb{P}^r$ (if $\dim(V)=r$) which is an embedding on a dense open set $X_0 \subset X$. We will consider the variety $\overline{\varphi_V(X_0)}$.

In general we expect that if $\dim(X)=n$ then

$$expdim(V(2P_1,\ldots,2P_s)) = \dim(V) - s(n+1).$$

Proposition 2. *Let X be an integral scheme and V be a linear system on X such that the rational function $\varphi_V : X \dashrightarrow \mathbb{P}^r$ associated to V, is an embedding on a dense open subset X_0 of X. Then $\sigma_s\left(\overline{\varphi_V(X_0)}\right)$ is defective if and only if for general points $P_1,\ldots,P_s \in X$*

$$\dim(V(2P_1,\ldots,2P_s)) > \min\{-1, r-s(n+1)\}.$$

This statement can be reformulated via Apolarity (next section).

1.4. **Apolarity.** This section is an exposition of inverse systems techniques, and it follows [19].

Definition 6. Let $S=K[x_1,\ldots,x_n]$ and $R=K[y_1,\ldots,y_n]$ be polynomial rings and consider the action of R on S (called *Apolarity* of R on S) defined as follows:

$$y_i \circ x_j = \left(\frac{\partial}{\partial x_i}\right)(x_j) = \begin{cases} 0, & \text{if } i\neq j \\ 1, & \text{if } i=j \end{cases} ;$$

i.e. we view the polynomials of R as "partial derivative operator" on S.

Now we can extend this action to the whole rings R, S by linearity and using properties of differentiation:

$$R_i \times S_j \longrightarrow S_{j-i}$$

$$r_i \times s_j := r_i \circ s_j$$

in particular

$$y^\alpha \circ x^\beta = \begin{cases} 0, & \text{if } \alpha \nleq \beta; \\ \prod_{i=1}^n \frac{(b_i)!}{(b_i-a_i)!} x^{\beta-\alpha}, & \text{if } \alpha \leq \beta. \end{cases}$$

where $x^\beta := x_0^{b_1}\cdots x_n^{b_n}$ when $\beta=(b_1,\ldots,b_n)$ and $b_i \geq 0$, and also $\alpha=(a_1,\ldots,a_n) \leq \beta$ iff $a_i \leq b_i$ for all $i=1,\ldots,n$, that is equivalent to x^α divides x^β in S.

Remarks:

- The action of R on S makes S a (non finitely generated) R-module (but the converse is not true);
- the action of R on S lowers the degree;

- the apolarity action induces a non-singular K-bilinear pairing:
$$R_j \times S_j \longrightarrow K \ \ \forall\, j = 0, 1, \dots$$
that induces two bilinear maps; [1]
- Notice that if $\{y^A\}$ and $\{x^B\}$ are bases of R_j and S_j respectively, they are not exactly dual bases. The dual bases of R_j and S_j are: $\{y^{A_1}, \dots, y^{A_t}\}$ and $\{\frac{1}{c_1}x^{A_1}, \dots, \frac{1}{c_t}x^{A_t}\}$ for an appropriate choice of coefficients c_i. So $\{y_1, \dots, y_n\}$ in R_1 is a dual base of $\{x_1, \dots, x_n\}$, base of S_1, with respect to the apolarity action, but for $j > 1$ this is no longer true.

Definition 7. Let I be a homogeneous ideal of R. The Inverse System I^{-1} of I is the R-submodule of S containing all the elements of S annihilated by I.

Remarks:

- If $I = (F_1, \dots, F_t) \subset R$ and $G \in R$ then $G \in I^{-1} \Leftrightarrow F_1 \circ G = \cdots = F_t \circ G = 0$. Finding all such G's means finding all the polynomial solutions for the differential equations defined by the F_i's, so one can notice that determining I^{-1} is equivalent to solve (with polynomial solutions) a finite set of differential equations;
- I^{-1} is a graduated submodule of S but it is not necessarily multiplicatively closed and in general I^{-1} is not an ideal of S.

We need now a digression on the Hilbert function.

Let $X \subset \mathbb{P}^n(K)$ be a closed subscheme whose representative homogeneous ideal is $I := I(X) \subset S$. Let $A = S/I$ be the homogeneous coordinate ring of X; A_d will be its degree d component.

Definition 8. The Hilbert Function of the scheme X is:
$$H(X, \cdot) : \mathbb{N} \to \mathbb{N};$$
$$H(X, d) = \dim_K(A_d).$$

We can easy observe that
$$H(X, d) = \dim_K(A_d) = \dim_K(S_d) - \dim_K(I_d).$$

Let us introduce the following theorem known as "Hilbert Theorem":

In our work the importance of inverse systems will be given by the following theorem, for a particular choice of the ideal I:

Theorem 2. *The dimension of the part of degree d of the inverse system of an ideal $I \subset R$ is the Hilbert function of R/I in degree d:*

$$\boxed{\dim_K(I^{-1})_d = codim(I_d) = H(R/I, d)}\,. \tag{4}$$

---

[1] Let $V \times W \longrightarrow K$ be a K-bilinear parity given by $v \times w \longrightarrow v \circ w$. It induces two K-bilinear maps:
$\phi : V \longrightarrow Hom_K(W, K)$ such that $\phi(v) := \phi_v$ and $\phi_v(w) = v \circ w$ and $\chi : W \longrightarrow Hom_K(V, K)$ such that $\chi(w) := \chi_w$ and $\chi_w(v) = v \circ w$.
$V \times W \longrightarrow K$ is not singular iff for all the bases $\{w_1, \dots, w_n\}$ of W the matrix $(b_{ij} = v_i \circ w_j)$ is invertible.

Remark 3.
- $(I^{-1})_d \cong I_d^{\perp}$. [2]
- if I is a monomial ideal then $I_d^{\perp} =<$ monomials of R_d that are not in $I_d >$
- $(I \cap J)^{-1} = I^{-1} + J^{-1}$.

If $I = \wp_1^{\alpha_1+1} \cap \cdots \cap \wp_s^{\alpha_s+1} \subset S = K[x_0, \dots, x_n]$ with $\wp_i$ prime ideals of the points $P_1, \dots, P_s \in \mathbb{P}^n$ and $P_i = [p_{i_0}, p_{i_1}, \dots, p_{in}]$, $L_{P_i} = p_{i_0}y_0 + p_{i_1}y_1 + \cdots + p_{i_n}y_n \in R = K[y_0, \dots, y_n]$ then

$$\boxed{(I^{-1})_d = \begin{cases} R_d, & \text{for } d \leq \max\{\alpha_i\} \\ L_{P_1}^{d-\alpha_1}R_{\alpha_1} + \cdots + L_{P_s}^{d-\alpha_s}R_{\alpha_s}, & \text{for } d \geq \max\{\alpha_i + 1\} \end{cases}}$$

and also
(5)

$$\boxed{H(S/I, d) = \dim_K (I^{-1})_d = \begin{cases} \dim_K(R_d), & \text{for } d \leq \max\{\alpha_i\} \\ \dim_K(< L_{P_1}^{d-\alpha_1}R_{\alpha_1}, \dots, L_{P_s}^{d-\alpha_s}R_{\alpha_s} >), & \text{for } d \geq \max\{\alpha_i + 1\} \end{cases}}.$$

This last result gives a link between the Hilbert function of a set of fat points and ideals generated by sums of powers of linear forms. This implies that:

Proposition 3. *If $I = \wp_1^{\alpha_1+1} \cap \cdots \cap \wp_s^{\alpha_s+1} \subset S = K[x_0, \dots, x_n]$ then $(I^{-1})_d \subset R_d = K[y_0, \dots, y_n]_d$ is the d-th graded part of the ideal $(L_{P_1}^{d-\alpha_1}, \dots, L_{P_s}^{d-\alpha_s}) \subset R$ for $d \geq \max\{\alpha_i + 1, i = 1, \dots, s\}$.*

Finally the link between the big Waring problem and inverse systems is clear. If in (5) all the α_i are equal to 1, the dimension of the vector space $< L_{P_1}^{d-1}R_1, \dots, L_{P_s}^{d-1}R_1 >$ is at the same time the Hilbert function of the inverse system of a scheme of s double fat points, and the rank of the differential of the application ϕ defined in (1).

Thus we can say:

Theorem 3. *Let $L_1, \dots, L_s$ be linear forms of $R = K[y_0, \dots, y_n]$ such that:*

$$L_i = a_{i_0}y_0 + \cdots + a_{i_n}y_n$$

and let $P_1, \dots, P_s \in \mathbb{P}^n$ such that:

$$P_i = [a_{i_0}, \dots, a_{i_n}].$$

Let also $\wp_i \subset S = K[x_0, \dots, x_n]$ be the prime ideal associated to P_i for $i = 1, \dots, s$ and

$$\phi : \underbrace{R_1 \times \cdots \times R_1}_{s} \longrightarrow R_d$$

with

$$\phi(L_1, \dots, L_s) = L_1^d + \cdots + L_s^d;$$

then

$$rk(d\phi)|_{(L_1,\dots,L_s)} = \dim_K < L_1^{d-1}R_1, \dots, L_s^{d-1}R_1 > .$$

[2] If $V \times W \longrightarrow K$ is a non degenerate bilinear form and V_1 is a subspace of W, then $V_1^{\perp}$ is a subspace of W and precisely: $V_1^{\perp} = \{w \in W / v \circ w = 0 \ \forall \ v \in V_1\} = \{w \in W / \chi_w(V_1) = 0\}$. Let $V \times W \longrightarrow K$ be non singular simmetry with $\dim_K(V) = \dim_K(V_1) = t$, then $\dim_K(V_1^{\perp}) = n - t$.

And by (4), we have:

$$\dim(\langle L_1^{d-1}R_1,\dots,L_s^{d-1}R_1\rangle) = H\left(\frac{S}{\wp_1^2\cap\cdots\cap\wp_s^2},d\right).$$

Now it is quite easy to see that

$$\langle T_{P_1}\nu_d(\mathbb{P}^n),\dots,T_{P_s}\nu_d(\mathbb{P}^n)\rangle = \langle L_1^{d-1}R_1,\dots,L_s^{d-1}R_1\rangle.$$

Therefore, putting together Terracini's Lemma with this last Theorem 3 we get:

$$\dim(\sigma_s(\nu_d(\mathbb{P}^n)))+1 = \dim\langle T_{P_1}\nu_d(\mathbb{P}^n),\dots,T_{P_s}\nu_d(\mathbb{P}^n)\rangle = \dim\langle L_1^{d-1}R_1,\dots,L_s^{d-1}R_1\rangle =$$

$$= codim\langle L_1^{d-1}R_1,\dots,L_s^{d-1}R_1\rangle^{\perp} = \dim(\wp_1^2\cap\cdots\cap\wp_s^2)_d = H(S/(\wp_1^2\cap\cdots\cap\wp_s^2),d).$$

Example 4. Let $P\in\mathbb{P}^n$, $\wp\subset S$ be its representative prime ideal and $f\in S$. Then the order of all partial derivatives of f vanishing in P is almost t if and only if $f\in\wp^{t+1}$ i.e. iff P is a singular point of $V(f)$ of multiplicity grater or equal than $t+1$.

Therefore:

$$H(S/\wp^t,d)=\begin{cases}\begin{pmatrix}d+n\\n\end{pmatrix}, & \text{if } d<t\\ \begin{pmatrix}t-1+n\\n\end{pmatrix}, & \text{if } d\geq t\end{cases}. \tag{6}$$

It is easy to conclude that one t-fat point of $\mathbb{P}^n$ has the same Hilbert function of $\binom{t-1+n}{n}$ generic distinct points of $\mathbb{P}^n$. Therefore $\dim(\nu_d(\mathbb{P}^n)) = H(S/\wp^2,d)-1 = n+1-1$. In fact the Veronese varieties are never defective.

Example 5. Let P_1,P_2 be two points of $\mathbb{P}^2$, $\wp_i\subset S=K[x_0,x_1,x_2]$ their associated prime ideals and let $\alpha_1=\alpha_2=2$ so that $I=\wp_1^2\cap\wp_2^2$. Is the Hilbert function of I equal to the Hilbert function of 6 points of $\mathbb{P}^2$ in general position? No, because the Hilbert function of 6 general points of $\mathbb{P}^2$ is 1 3 6 6 ... and this means that I should not contain conics, but this is clearly false because the double line through P_1 and P_2 is contained in I. This implies that $\sigma_2(\nu_2(\mathbb{P}^2))$ is defective.

The general problem is not yet solved: there is only a conjecture due first to Beniamino Segre (rephrased also by B. Harbourne, A. Gimigliano, A. Hirschowitz and others) which describes how the element of $S_d(P_1^{\alpha_1},\dots,P_s^{\alpha_s})$ should be done when it has not the expected dimension.

Definition 9. Let $P_1,\dots,P_s$ be s points of $\mathbb{P}^n$ in general position. If $S_d(P_1^{\alpha_1},\dots,P_s^{\alpha_s})$ is a linear system whose dimension is not the expected one, it is said to be a Special Linear System.

Conjecture 1 (**Segre**, 1961)**.** If $S_d(P_1^{\alpha_1},\dots,P_s^{\alpha_s})\subset K[x_0,x_1,x_2]$ is a special linear system, then there is a fixed double component for all curves through the scheme defined by $\wp_1^{\alpha_1}\cap\cdots\cap\wp_s^{\alpha_s}$.

Conjecture 2 (**Gimigliano**, 1987)**.** Consider the linear system $S_d(P_1^{\alpha_1},\dots,P_s^{\alpha_s})\subset K[x_0,x_1,x_2]$, then one has the following possibilities:

(1) the system is non-special and its general member is irreducible;

(2) the system is non-special, its general member is non-reduced, reducible, its fixed components are all rational curves, except for at most one (this may occur only if the system has dimension 0), and the general member of its movable part is either irreducible or composed of rational curves in a pencil;
(3) the system is non-special of dimension 0 and consists of a unique multiple elliptic curve;
(4) the system is special and it has some multiple rational curve as a fixed component.

Conjecture 3 (**Harbourne-Hirschowitz**, 1989)**.** A linear system of plane curves $S_d(P_1^{\alpha_1},\dots,P_s^{\alpha_s}) \subset K[x_0,x_1,x_2]$ with general multiple base points is special if and only if it is (-1)-special, i.e. it contains some multiple rational curve of self-intersection -1 in the base locus.

Conjecture 4 (**Nagata**, 1960)**.** $S_d(P_1^{\alpha},\dots,P_s^{\alpha}) \subset K[x_0,x_1,x_2]$ is empty as soon as $n \geq 10$ and $d \leq \sqrt{n}\dot{m}$.

1.5. **La méthode d'Horace.**

Definition 10. We say that a collection Z_r of r double points *imposes independent conditions on* $\mathcal{O}_{\mathbb{P}^n}(d)$ (hypersurfaces of degree d in $n+1$ variables if $codim(I_{Z_r}(d))$ in S^dV is $\min\{\binom{n+d}{d,r(n+1)}\}$.

Corollary 2. *$\sigma_s(\nu_d(\mathbb{P}^n))$ has the expected dimension if and only if Z_s, scheme of s generic double points in $\mathbb{P}^n$, imposes independent conditions on $\mathcal{O}_{\mathbb{P}^n}(d)$.*

Here the description of the Horace method ([2]).

Definition 11. Let $Z_r \subset \mathbb{P}^n$ be a scheme of r double points. It corresponds to the ideal sheaf $\mathcal{I}_{Z_r}$. Let $H \subset \mathbb{P}^n$ hyperplane. The trace of Z_r with respect to H is the schematic intersection $Tr_H(Z_r) = Z_r \cap H$. The Residual $Res_H(Z_r)$ of Z_r with respect to H is defined by the ideal sheaf $\mathcal{I}_{Z_r} : \mathcal{O}_{\mathbb{P}^n}(-H)$.

Example 6. If $Z_1 \subset \mathbb{P}^n$ is the scheme defined by $\wp^2$ with support on H, then $Res_H\wp^2 \subset \mathbb{P}^n$ is defined by $\wp$, while $Tr_H(\wp^2) \subset \mathbb{P}^{n-1}$ is still defined by the square of the ideal of a point $\tilde{\wp}^2$.

Taking the global sections of the restriction exact sequence

$$0 \to \mathcal{I}_{Res_H Z_r}(d-1) \to \mathcal{I}_{Z_r}(d) \to \mathcal{I}_{Tr_H(Z_r)}(d) \to 0,$$

we obtain the so called Castelnuovo exact sequence

$$0 \to I_{Res_H Z_r}(d-1) \to I_{Z_r}(d) \to I_{Tr_H(Z_r)}(d) \tag{7}$$

from which we get the following inequality

$$\dim I_{Z_r}(d) \leq \dim I_{Res_H Z_r}(d-1) + \dim I_{Tr_H(Z_r)}(d).$$

Consider the following:

(1) $Res_H Z_r$ imposes independent conditions to $\mathcal{O}_{\mathbb{P}^n}(d-1)$
(2) $Tr_H(Z_r)$ imposes independent conditions to $\mathcal{O}_{\mathbb{P}^{n-1}}(d)$.

Now items (1) and (2) above are equivalent to the following respectively:

(1) $\dim(I_{Res_H(Z_r)}(d-1)) = \max\{\binom{d-1+n}{d-1} - (r-t)(n+1) - t, 0\}$;
(2) $\dim(I_{Tr_H(Z_r)}(d)) = \max\{\binom{d+n-1}{d} - tn, 0\}$

So if $\max\{\binom{d+n-1}{d} - tn, 0\} = \binom{d+n-1}{d} - tn$ then $\dim I_{Z_r}(d) \leq \binom{d+n}{n} - r(n+1)$. While if $\max\{\binom{d+n-1}{d} - tn, 0\} = 0$ then $\dim I_{Z_r}(d) \leq 0$. But since $\dim I_{Z_r}(d)$ is always greater or equal than the expected dimension, we have that in both cases Z_r imposes independent conditions on the system $\mathcal{O}_{\mathbb{P}^n}(d)$. This proves the following:

Theorem 4 (Brambilla, Ottaviani, [13])**.** *Let Z_r be a union of r double points of $\mathbb{P}^n$ and let $H \subset \mathbb{P}^n$ be a hyperplane such that t of the r points of Z have support on H. Assume that $Tr_H(Z_r)$ imposes independent conditions on $\mathcal{O}_H(d)$ and that $Res_H Z_r$ imposes independent conditions on $\mathcal{O}_{\mathbb{P}^n}(d-1)$. Now, if*

(1) $tn \leq \binom{d+n-1}{n-1}$ $\qquad r(n+1) - tn \leq \binom{d+n-1}{n}$,
(2) $tn \geq \binom{d+n-1}{n-1}$ $\qquad r(n+1) - tn \geq \binom{d+n-1}{n}$.

Then Z_r imposes independent conditions on the system $\mathcal{O}_{\mathbb{P}^n}(d)$.

The technique used by Alexander and Hirschowitz to compute the dimension of secant varieties of Veronese varieties is mainly the Horace method via induction.

1.6. **Example of induction.** The idea of induction works well to prove regularity of secant varieties but it doesn't work at all for the defective cases that have to be proven case by case. We have already seen that the case of Veronese surfaces and of quadrics are defective, so we cannot take them as first step of the induction. Let us start with $\sigma_s(\nu_3(\mathbb{P}^3)) \subset \mathbb{P}^{19}$. The expected dimension of $\sigma_s(\nu_3(\mathbb{P}^3))$ is $4s-1$. Therefore we expect that $\sigma_5(\nu_3(\mathbb{P}^3))$ fills up the ambient space. Now $H(S/\wp_1^2 \cap \cdots \cap \wp_5^2) = 19 - (\sharp$ cubics through 5 double points in $\mathbb{P}^3) = 19 - 0$ hence $\sigma_s(\nu_3(\mathbb{P}^3)) = \mathbb{P}^{19}$ as expected. This implies that

$$\dim(\sigma_s(\nu_3(\mathbb{P}^3))) \text{ is the expected one for all } s \leq 5. \tag{8}$$

In fact:

Proposition 4. *Assume that X is k-defective, with k-defect δ_k. Assume $\sigma_{k+1}(X) \neq \mathbb{P}^N$. Then X is also $(k+1)$-defective.*

Proof. By assumptions and by Terracini's lemma, if $P_1, \ldots, P_k \in X$ are general points, then the span $T_{P_1,\ldots,P_k}$, which is the tangent space at a general point of $\sigma_k(X)$, has dimension $\min(N, kn+k-1) - \delta_k$. Hence adding one general point P_{k+1}, the space $T_{P_1,\ldots,P_k,P_{k+1}}$, which is the span of $T_{P_1,\ldots,P_k}$ and $T_{P_{k+1}}$, has dimension at most $\min(N, kn+k-1) - \delta_k + n + 1$. This last number, by assumptions, is smaller than N, while it is clearly smaller than $(k+1)n+k$. So X is $(k+1)$-defective. □

1.7. **Exercises.**

Exercise 1. Prove that all the equations $F_0 = 0, F_1 = 0, F_2 = 0$ are needed to define $\nu_3(\mathbb{P}^1)$. Moreover show that through a given point $P \in \mathbb{P}^3 \setminus \nu_3(\mathbb{P}^1)$ there exists at most one secant line to $\nu_3(\mathbb{P}^1)$.

Exercise 2. Let $Q \subset \mathbb{P}^3$ be a smooth quadric, and let $\nu_2(Q)$ be the image of Q into $\mathbb{P}^9$ via Veronese embedding $\mathcal{O}(2)$. Show that $\nu_2(Q)$ is 3-defective. How can one generalize this to $Q \subset \mathbb{P}^n$, $n > 2$?

Exercise 3. Prove that $\sigma_5(\nu_4(\mathbb{P}^2))$ is defective.

2. Lecture 2

Consider $d=4$, $s=8$ in $\mathbb{P}^3$, i.e. $\sigma_8(\nu_4(\mathbb{P}^3))$.
We need to compute $H(K[x_0,\dots,x_3]/(\wp_1^2\cap\cdots\cap\wp_8^2),4)$. In order to use Horace Lemma we need to know how many points among $\wp_1^2\cap\cdots\cap\wp_8^2$ have support on a given hyperplane. The good news is *upper semicontinuity* that allows us to specialize points on a hyperplane. In fact if the specialized scheme has the expected Hilbert function, then also the general scheme has the expected Hilbert function (again this argument cannot be used if the specialized one doesn't have the expected Hilbert function). We choose to specialize 4 points on H: $supp(V(\wp_1^2\cap\cdots\wp_4^2))\in H$. Therefore $Res_H(Z_8)=P_1+\cdots+P_4+2P_5+\cdots+2P_8$ and $Tr_H(Z_8)=2\tilde{P}_1+\cdots 2\tilde{P}_4$ where $2\tilde{P}_1+\cdots 2\tilde{P}_4$ are four double points of $\mathbb{P}^2$. Consider now the Castelunovo exact sequence (7). Now 4 double points in $\mathbb{P}^3$ imposes independent conditions to $\mathcal{O}_{\mathbb{P}^3}(3)$ by (8), then adding 4 simple general points imposes independent conditions, therefore $Res_H Z_r$ imposes independent condition to $\mathcal{O}_{\mathbb{P}^3}(3)$. Also $Tr_H(Z_r)$ imposes independent condition to $\mathcal{O}_{\mathbb{P}^2}(4)$. Therefore we have proved that

$$\sigma_s(\nu_4(\mathbb{P}^3)) \text{ has the expected dimension for any } s\leq 8.$$

This argument cannot be use to study $\sigma_9(\nu_4(\mathbb{P}^3))$ because it is defective (we cannot use induction on s in this case). Anyway we can use induction on d.

Exercise 4. Prove that $\sigma_{14}(\nu_5(\mathbb{P}^3))$ is regular.

To use induction we need to prove the first case $d=3$. We have done already $\mathbb{P}^3$. Now, $d=3$, $n=4$, $k=7$ is a defective case. So we need to start with $d=3$ and $n=5$. We expect that $\sigma_{10}(\nu_3(\mathbb{P}^5))$ fills up the ambient space. Let's try to apply Horace method. The hyperplane H is a $\mathbb{P}^4$, one double point in $\mathbb{P}^4$ has degree 5, so we can specialize up to 7 points on H (in $\mathbb{P}^4$ there are exactly $35=7\times 5$ cubics), BUT 7 double points in $\mathbb{P}^4$ are defective in degree 3, in fact $H(K[x_0,\dots,x_4]/\wp_1^2\cap\cdots\cap\wp_7^2,3)=\binom{4+3}{3}-1$. Therefore, if we specialize 7 points on H we are "not using all the room that we can use" if we want to get a 0 in the trace term of Castelunovo exact sequence we have to add one more condition on H. How can we get one single condition among 10 double points in $\mathbb{P}^5$ where each double point imposes 6 conditions?

2.1. La méthode d'Horace differentielle.

This description follows the guide lines of [10].

Definition 12. In the algebra of formal functions $K[[\mathbf{x},y]]$, where $\mathbf{x}=(x_1,\dots,x_{n-1})$, a *vertically graded* (with respect to y) ideal is an ideal of the form:

$$I=I_0\oplus I_1y\oplus\cdots\oplus I_{m-1}y^{m-1}\oplus(y^m)$$

where for $i=0,\dots,m-1$, $I_i\subset\kappa[[\mathbf{x}]]$ is an ideal.

Let Q be a smooth n-dimensional integral scheme, let K be a smooth irreducible divisor on Q. We say that $Z\subset Q$ is a *vertically graded subscheme* of Q with base K and support $z\in K$, if Z is a 0-dimensional scheme with support at the point z such that there is a regular system of parameters $(\mathbf{x},y)$ at z such that $y=0$ is a local equation for K and the ideal of Z in $\widehat{\mathcal{O}}_{Q,z}\cong K[[\mathbf{x},y]]$ is vertically graded.

Let $Z\in Q$ be a vertically graded subscheme with base K, and $p\geq 0$ be a fixed integer; we denote by $Res_K^p(Z)\in Q$ and $Tr_K^p(Z)\in K$ the closed subschemes defined, respectively, by the ideals:

$$\mathcal{I}_{Res^p_K(Z)} := \mathcal{I}_Z + (\mathcal{I}_Z : \mathcal{I}_K^{p+1})\mathcal{I}_K^p, \qquad \mathcal{I}_{Tr^p_K(Z),K} := (\mathcal{I}_Z : \mathcal{I}_K^p) \otimes \mathcal{O}_K.$$

In $Res^p_K(Z)$ we take away from Z the $(p+1)^{th}$ "slice"; in $Tr^p_K(Z)$ we consider only the $(p+1)^{th}$ "slice". Notice that for $p = 0$ we get the usual trace and residual schemes: $Tr_K(Z)$ and $Res_K(Z)$.

Finally, let $Z_1, \dots, Z_r \in Q$ be vertically graded subschemes with base K and support z_i, $Z = Z_1 \cup \ldots \cup Z_r$, and $\mathbf{p} = (p_1, \dots, p_r) \in \mathbb{N}^r$.

We set:

$$Tr^{\mathbf{p}}_K(Z) := Tr^{p_1}_K(Z_1) \cup \cdots \cup Tr^{p_r}_K(Z_r), \quad Res^{\mathbf{p}}_K(Z) := Res^{p_1}_K(Z_1) \cup \cdots \cup Res^{p_r}_K(Z_r).$$

Proposition 5 (**Horace differential Lemma**, [3] Proposition 9.1)**.** *Let H be a hyperplane in $\mathbb{P}^n$ and let $W \wr \mathbb{P}^n$ be a 0-dimensional closed subscheme.*

Let $S_1, \dots, S_r$, $Z_1, \dots, Z_r$ be $0-$dimensional irreducible subschemes of $\mathbb{P}^n$ such that $S_i \cong Z_i$, $i = 1, \dots, r$, Z_i has support on H and is vertically graded with base H, and the supports of $S = S_1 \cup \cdots \cup S_r$ and $Z = Z_1 \cup \cdots \cup Z_r$ are generic in their respective Hilbert schemes. Let $\mathbf{p} = (p_1, \dots, p_r) \in \mathbb{N}^r$. Assume:

1. $H^0(\mathcal{I}_{Tr_H W \cup Tr^{\mathbf{p}}_H(Z),H}(n)) = 0$ *and*
2. $H^0(\mathcal{I}_{Res_H W \cup Res^{\mathbf{p}}_H(Z)}(n-1)) = 0$,

then

$$H^0(\mathcal{I}_{W \cup S}(n)) = 0.$$

For double fat points, this can be rephrased as follows.

Proposition 6 (**Horace differential lemma for double fat points**)**.** *Let $H \subset \mathbb{P}^n$ be a hyperplane, $P_1, \dots, P_r \in \mathbb{P}^n$ generic points, and $\tilde{Z}$ be a 0-dimensional scheme. Let $Z = \tilde{Z} + 2P_1 + \cdots + 2P_r \subset \mathbb{P}^n$, $\tilde{Z}' = Res_H(\tilde{Z})$, $\tilde{T} = Tr_H(\tilde{Z})$.*
Let $P_1', \dots, P_r'$ be generic points on H. Let $D_{2,H}(P_i') = 2P_i' \cap H$ and $Z' = \tilde{Z}' + D_{2,H}(P_1') + \cdots + D_{2,H}(P_r')$, $T = \tilde{T} + P_1' + \cdots + P_r'$. Then $\dim(I_Z)_t = 0$ if the following two conditions are satisfied.

- *DEGUE:* $\dim(I_{Z'})_{t-1} = \dim(I_{\tilde{Z}+D_{2,H}(P_1')+\cdots+D_{2,H}(P_r')})_{t-1} = 0$*;*
- *DIME:* $\dim(I_T)_t = \dim(I_{\tilde{T}+P_1'+\cdots+P_r'})_t = 0$.

Now with this proposition we can conclude the case of $\sigma_{10}(\nu_3(\mathbb{P}^5))$. Specialize $P_1, \dots, P_8$ on H and take $\tilde{Z} = 2P_1 + \cdots + 2P_7 + 2P_9 + 2P_{10} \subset \mathbb{P}^5$ with *supp*(P_i) on H for $i = 1, \dots, 7$ and out of H for $i = 9, 10$. Then $\tilde{Z}' = Res_H(\tilde{Z}) = P_1 + \cdots + P_7 + 2P_9 + 2P_{10} \subset \mathbb{P}^5$ and $\tilde{T} = Tr_H(\tilde{Z}) = 2\tilde{P}_1 + \cdots 2\tilde{P}_7 \subset H$ where $2\tilde{P}_i = 2P_i \cap H$. Then take $D_{2,H}(P_8) = 2P_8 \cap H$. Now $Z' = \tilde{Z}' + D_{2,H}(P_8) = P_1 + \cdots + P_7 + 2P_9 + 2P_{10} + D_{2,H}(P_8) \subset \mathbb{P}^5$, and $T = \tilde{T} + P_8 = 2\tilde{P}_1 + \cdots 2\tilde{P}_7 + P_8 \subset H$. Then DIME is now ok because we have added on the trace exactly the one condition that we were missing. Also DIME is ok because 7 simple points imposes independent conditions to quadrics, 2 double points does not impose independent conditions to quadrics but we know how many conditions they impose 6+6-1 and $D_{2,H}(P_8)$ can be proved to impose independent conditions to quadrics. All together the should impose 7+11+5=23 conditions that is sufficient to get DEGUE since there are only 21 quadrics in $\mathbb{P}^5$.

2.2. Exercises.

Exercise 5. Compute the dimensions of $\sigma_s(\nu_6(\mathbb{P}^3))$.

Exercise 6. Let $\tau(\nu_d(\mathbb{P}^n)) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ be the tangential variety to the Veronese variety. After having computed the tangent space to $\tau(\nu_d(\mathbb{P}^n))$ at one smooth point deduce the structure of the scheme Z such that $I(Z)_d$ is the inverse system of $T_P(\tau(\nu_d(\mathbb{P}^n)))$. Then prove that $\sigma_3(\tau(\nu_4(\mathbb{P}^2)))$ has the expected dimension.

Algorithms for the rank of a given polynomial

3. Lecture 3

On Sylvester's algorithm

This description can be found in [11].

If V is a two dimensional vector space, there is a well known isomorphism between $\bigwedge^{d-r+1}(S^dV)$ and $S^{d-r+1}(S^rV)$ (see [26]). Such isomorphism can be interpreted in terms of projective algebraic varieties; it allows to view the $(d-r+1)$-uple Veronese embedding of $\mathbb{P}^r$, as the set of $(r-1)$-dimensional projective subspaces of $\mathbb{P}^d$ that are r-secant to the rational normal curve. The description of this result, via coordinates, was originally given by A. Iarrobino, V. Kanev (see [22]). We give here the description appeared in [4] (Lemma 2.1).

Lemma 2. *Consider the map $\phi_{r,d-r+1} : \mathbb{P}(K[t_0,t_1]_r) \to \mathbb{G}(d-r+1, K[t_0,t_1]_d)$ that maps the class of $p_0 \in K[t_0,t_1]_r$ to the $(d-r+1)$-dimensional subspace of $K[t_0,t_1]_d$ of forms of the type p_0q, with $q \in K[t_0,t_1]_{d-r}$. Then the following hold:*
(i) The image of $\phi_{r,d-r+1}$, after the Plücker embedding of $\mathbb{G}(d-r+1, K[t_0,t_1]_d)$, is the r-dimensional $(d-r+1)$-th Veronese variety.
(ii) Identifying $\mathbb{G}(d-r+1, K[t_0,t_1]_d)$ with the Grassmann variety of subspaces of dimension $r-1$ in $\mathbb{P}(K[t_0,t_1]_d^*)$, the above Veronese variety is the set of r-secant spaces to a rational normal curve $C_d \subset \mathbb{P}(K[t_0,t_1]_d^*)$.

Proof. Write $p_0 = u_0t_0^r + u_1t_0^{r-1}t_1 + \cdots + u_rt_1^r$. Then a basis of the subspace of $K[t_0,t_1]_d$ of forms of the type p_0q is given by:

$$\left\{\begin{array}{l} u_0t_0^d + \cdots + u_rt_0^{d-r}t_1^r \\ \quad u_0t_0^{d-1}t_1 + \cdots + u_rt_0^{d-r-1}t_1^{r+1} \\ \qquad \ddots \\ \qquad\quad u_0t_0^rt_1^{d-r} + \cdots + u_rt_1^d. \end{array}\right. \tag{9}$$

The coordinates of these elements with respect to the basis $\{t_0^d, t_0^{d-1}t_1, \ldots, t_1^d\}$ of $K[t_0,t_1]_d$ are thus given by the rows of the matrix

$$\begin{pmatrix} u_0 & u_1 & \ldots & u_r & 0 & \ldots & 0 & 0 \\ 0 & u_0 & u_1 & \ldots & u_r & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & u_0 & u_1 & \ldots & u_r & 0 \\ 0 & \ldots & 0 & 0 & u_0 & \ldots & u_{r-1} & u_r \end{pmatrix}.$$

The standard Plücker coordinates of the subspace $\phi_{r,d-r+1}([p_0])$ are the maximal minors of this matrix. It is known (see for example [5]), that these minors form a basis of $K[u_0, \dots, u_r]_{d-r+1}$, so that the image of ϕ is indeed a Veronese variety, which proves (i).

To prove (ii), we recall some standard facts from [5]. Take homogeneous coordinates $z_0, \dots, z_d$ in $\mathbb{P}(K[t_0,t_1]_d^*)$ corresponding to the dual basis of $\{t_0^d, t_0^{d-1}t_1, \dots, t_1^d\}$. Consider $C_d \subset \mathbb{P}(K[t_0,t_1]_d^*)$ the standard rational normal curve with respect to these coordinates. Then, the image of $[p_0]$ by $\phi_{r,d-r+1}$ is precisely the r-secant space to C_d spanned by the divisor on C_d induced by the zeros of p_0. This completes the proof of (ii). $\square$

Since $\dim(V) = 2$, the Veronese variety of $\mathbb{P}(S^dV)$ is the rational normal curve $C_d \subset \mathbb{P}^d$. Hence, a symmetric tensor $t \in S^dV$ has symmetric rank r if and only if r is the minimum integer for which there exist a $\mathbb{P}^{r-1} = \mathbb{P}(W) \subset \mathbb{P}(S^dV)$ such that $T \in \mathbb{P}(W)$ and $\mathbb{P}(W)$ is r-secant to the rational normal curve $C_d \subset \mathbb{P}(S^dV)$ in r distinct points.
Consider the maps:
(10)
$$\mathbb{P}(K[t_0,t_1]_r) \stackrel{\phi_{r,d-r+1}}{\to} \mathbb{G}G(d-r, \mathbb{P}(K[t_0,t_1]_d)) \stackrel{\alpha_{r,d-r+1}}{\simeq} \mathbb{G}G(r-1, \mathbb{P}(K[t_0,t_1]_d)^*).$$
Clearly, since $\dim(V) = 2$, we can identify $\mathbb{P}(K[t_0,t_1]_d)^*)$ with $\mathbb{P}(S^dV)$, hence the Grassmannian $\mathbb{G}G(r-1, \mathbb{P}(K[t_0,t_1]_d)^*)$ can be identified with $\mathbb{G}G(r-1, \mathbb{P}(S^dV))$. Now, by Lemma 2, a projective subspace $\mathbb{P}(W)$ of $\mathbb{P}(K[t_0,t_1]_d)^* \simeq \mathbb{P}(S^dV) \simeq \mathbb{P}^d$ is r-secant to $C_d \subset \mathbb{P}(S^dV)$ in r distinct points if and only if it belongs to $\mathrm{Im}(\alpha_{r,d-r+1} \circ \phi_{r,d-r+1})$ and the preimage of $\mathbb{P}(W)$ via $\alpha_{r,d-r+1} \circ \phi_{r,d-r+1}$ is a polynomial with r distinct roots.
Therefore, a symmetric tensor $t \in S^dV$ has symmetric rank r if and only if r is the minimum integer for which:

(1) T belongs to an element $\mathbb{P}(W) \in \mathrm{Im}(\alpha_{r,d-r+1} \circ \phi_{r,d-r+1}) \subset \mathbb{G}G(r-1, \mathbb{P}(S^dV))$,
(2) there exist a polynomial $p_0 \in K[t_0t_1]_r$ such that $\alpha_{r,d-r+1}(\phi_{r,d-r+1}([p_0])) = \mathbb{P}(W)$ and p_0 has r distinct roots,

Fix the natural basis $\Sigma = \{t_0^d, t_0^{d-1}t_1, \dots, t_1^d\}$ in $K[t_0,t_1]_d$. Let $\mathbb{P}(U)$ be a $(d-r)$-dimensional projective subspace of $\mathbb{P}(K[t_0,t_1]_d)$. The proof of Lemma 2 shows that $\mathbb{P}(U)$ belongs to the image of $\phi_{r,d-r+1}$ if and only if there exist $u_0, \dots, u_r \in K$ such that $U =< p_1, \dots, p_{d-r+1} >$ with $p_1 = (u_0, u_1, \dots, u_r, 0, \dots, 0)_\Sigma$, $p_2 = (0, u_0, u_1, \dots, u_r, 0, \dots, 0)_\Sigma$, . . . ,$p_{d-r+1} = (0, \dots, 0, u_0, u_1, \dots, u_r)_\Sigma$.
Now let $\Sigma^* = \{z_0, \dots, z_d\}$ be the dual basis of Σ. Therefore there exist a $W \subset S^dV$ such that $\mathbb{P}(W) = \alpha_{r,d-r+1}(\mathbb{P}(U))$ if and only if $W = H_1 \cap \cdots \cap H_{d-r+1}$ and the H_i's are as follows:

$$\begin{array}{rl} H_1: & u_0z_0 + \cdots + u_rz_r = 0 \\ H_2: & \quad u_0z_1 + \cdots + u_rz_{r+1} = 0 \\ & \qquad \ddots \\ H_{d-r+1}: & \qquad u_0z_{d-r} + \cdots + u_rz_d = 0. \end{array}$$

This is sufficient to conclude that $T \in \mathbb{P}(S^dV)$ belongs to an $(r-1)$-dimensional projective subspace of $\mathbb{P}(S^dV)$ that is in the image of $\alpha_{r,d-r+1} \circ \phi_{r,d-r+1}$ defined in (10) if and only if there exist $H_1, \dots, H_{d-r+1}$ hyperplanes in S^dV as above such that $T \in H_1 \cap \ldots \cap H_{d-r+1}$.
Given $t = (a_0, \dots, a_d)_{\Sigma^*} \in S^dV$, $T \in H_1 \cap \ldots \cap H_{d-r+1}$ if and only if the following

linear system admits a non trivial solution:

$$\begin{cases} u_0a_0 + \cdots + u_ra_r = 0 \\ u_0a_1 + \cdots + u_ra_{r+1} = 0 \\ \vdots \\ u_0a_{d-r} + \cdots + u_ra_d = 0. \end{cases}$$

If $d-r+1 < r+1$ this system admits an infinite number of solutions.
If $r \leq d/2$, it admits a non trivial solution if and only if all the maximal $(r+1)$-minors of the following $(d-r+1)\times(r+1)$ catalecticant matrix vanish:

$$M_{d-r,r} = \begin{pmatrix} a_0 & \cdots & a_r \\ a_1 & \cdots & a_{r+1} \\ \vdots & & \vdots \\ a_{d-r} & \cdots & a_d \end{pmatrix}.$$

Remark 4. The dimension of $\sigma_r(C_d)$ is the minimum between $2r-1$ and d. Actually $\sigma_r(C_d) \subsetneq \mathbb{P}^d$ if and only if $1 \leq r < \lceil \frac{d+1}{2} \rceil$.

Remark 5. An element $T \in \mathbb{P}^d$ belongs to $\sigma_r(C_d)$ for $1 \leq r < \lceil \frac{d+1}{2} \rceil$ if and only if the catalecticant matrix $M_{r,d-r}$ defined in Definition 13 does not have maximal rank.

Sylvester Algorithm works as follows.

Let $p \in K[x_0,x_1]_d$ be a homogeneous polynomial of degree d in two variables: $p(x_0,x_1) = \sum_{k=0}^d a_k x_0^k x_1^{d-k}$; then we can associate to the form p a symmetric tensor $t \in S^dV \simeq K[x_0,x_1]_d$ where $t = (b_{i_1,\ldots,i_d})_{i_j\in\{0,1\};j=1,\ldots,d}$, and $b_{i_1,\ldots,i_d} = \binom{d}{k}^{-1} \cdot a_k$ for any d-uple $(i_1,\ldots,i_d)$ containing exactly k zeros. This correspondence is clearly one to one:

$$\begin{array}{ccc} K[x_0,x_1]_d & \leftrightarrow & S^dV \\ \sum_{k=0}^d a_k x_0^k x_1^{d-k} & \leftrightarrow & (b_{i_1,\ldots,i_d})_{i_j=0,1;\ j=1,\ldots,d} \end{array} \tag{11}$$

with $(b_{i_1,\ldots,i_d})$ as above.

Moreover, we can associate to a polynomial $p(x_0,x_1) = \sum_{k=0}^d a_k x_0^k x_1^{d-k}$, or to the symmetric tensor t associated to it, the so called Catalecticant matrix $M_{d-r,r}(t)$, defined as follows (for a definition of Catalecticant matrix see also [23]; $M_{d-r,r}(t)$ it is also called Hankel matrix in [12]):

Definition 13. Let $p(x_0,x_1) = \sum_{k=0}^d a_k x_0^k x_1^{d-k}$, and $t = (b_{i_1},\ldots,b_{i_d})_{i_j=0,1;\ j=1,\ldots,d} \in S^dV$ be the symmetric tensor associated to p, as above. Then the Catalecticant matrix $M_{d-r,r}(t)$ associated to t (or to p) is the $(d-r+1)\times(r+1)$ matrix with entries: $c_{i,j} = \binom{d}{i}^{-1} a_{i+j-2}$ with $i = 1,\ldots,d-r$ and $j = 1,\ldots,r$.

We describe here a version of the Sylvester algorithm ([30], [18], or [12]):

Algorithm 1. Input: A binary form $p(x_0,x_1)$ of degree d or, equivalently, its associated symmetric tensor t.
Output: A decomposition of p as $p(x_0,x_1) = \sum_{j=1}^k \lambda_j l_j(x_0,x_1)^d$ with $\lambda_j \in K$ and $l_j \in K[x_0,x_1]_1$ for $j = 1,\ldots,r$ with r minimal.

(1) Initialize $r = 0$;
(2) Increment $r \leftarrow r+1$;

(3) If the rank of the matrix $M_{d-r,r}$ is maximum, then go to step 2;
(4) Else compute a basis $\{l_1, \dots, l_h\}$ of the right kernel of $M_{d-r,r}$;
(5) Specialization:
   - Take a vector q in the kernel, e.g. $q = \sum_i \mu_i l_i$;
   - Compute the roots of the associated polynomial $q(x_0, x_1) = \sum_{h=0}^{r} q_h x_0^h x_1^{d-h}$. Denote them by (α_j, β_j), where $|\alpha_j|^2 + |\beta_j|^2 = 1$;
   - If the roots are not distinct in $\mathbb{P}^1$, go to step 2;
   - Else if $q(x_0, x_1)$ admits r distinct roots then compute coefficients λ_j, $1 \leq j \leq r$, by solving the linear system below:

$$\begin{pmatrix} \alpha_1^d & \cdots & \alpha_r^d \\ \alpha_1^{d-1}\beta_1 & \cdots & \alpha_r^{d-1}\beta_r \\ \alpha_1^{d-2}\beta_1^2 & \cdots & \alpha_r^{d-2}\beta_r^2 \\ \vdots & \vdots & \vdots \\ \beta_1^d & \cdots & \beta_r^d \end{pmatrix} \lambda = \begin{pmatrix} a_0 \\ 1/d a_1 \\ \binom{d}{2}^{-1} a_2 \\ \vdots \\ a_d \end{pmatrix};$$

(6) The decomposition is $p(x_0, x_1) = \sum_{j=1}^{r} \lambda_j l_j(x_0, x_1)^d$, where $l_j(x_0, x_1) = (\alpha_j x_1 + \beta_j x_2)$.

The following result has been proved by G. Comas and M. Seiguer in [18], and it describes the structure of the stratification by symmetric rank of symmetric tensors in $S^d V$ with $\dim(V) = 2$. This result allows to improve the classical Sylvester algorithm.

Theorem 5. *Let $X_{1,d} = C_d \subset \mathbb{P}(S_d f V)$, $\dim(V) = 2$, be the rational normal curve, parameterizing decomposable symmetric tensors ($C_d = \{T \in \mathbb{P}(S^d V) \mid rk(T) = 1\}$), i.e. homogeneous polynomials in $K[t_0, t_1]_d$ which are d-th powers of linear forms. Then:*

$$\forall\, r,\ 2 \leq r \leq \left\lceil \frac{d+1}{2} \right\rceil : \qquad \sigma_r(C_d) \setminus \sigma_{r-1}(C_d) = \sigma_{r,r}(C_d) \cup \sigma_{r,d-r+2}(C_d)$$

where $\sigma_{r,r}(C_d)$ and $\sigma_{r,d-r+2}(C_d)$ are subsets of $\sigma_r(C_d)$ containing only elements of ranks r and $d - r + 2$ respectively.

Proof. Of course, for all $t \in S^d V$, if $rk(t) = r$, with $r \leq \lceil \frac{d+1}{2} \rceil$, we have $T \in \sigma_r(C_d) \setminus \sigma_{r-1}(C_d)$. Thus we have to consider the case $rk(t) > \lceil \frac{d+1}{2} \rceil$.

If a point in $K[t_0, t_1]_d^*$ represents a tensor t with $rk(t) > \lceil \frac{d+1}{2} \rceil$, then we want to show that $rk(t) = d - r + 2$, where r is the minimum such that $T \in \sigma_r(C_d)$, $r \leq \lceil \frac{d+1}{2} \rceil$.

Let us consider the case $r = 2$ first: Let $T \in \sigma_2(C_d) \setminus C_d$. If $rk(t) > 2$, it means that T lies on a line t_P, tangent to C_d at a point P (since T has to lie on a $\mathbb{P}^1$ which is the image of a non-reduced form of degree 2: $p_0 = l^2$ with $l \in K[x_0, x_1]_1$, otherwise $rk(t) = 2$). We want to show that $rk(t) = d$; in fact, if $rk(t) = r < d$, there would exist distinct points $P_1, \dots, P_{d-1} \in C_d$, such that $T \in < P_1, \dots, P_{d-1} >$; in this case the hyperplane $H =< P_1, \dots, P_{d-1}, P >$ would be such that $t_P \subset H$, a contradiction, since $H \cap C_d = 2P + P_1 + \cdots + P_{d-1}$, which has degree $d + 1$.

Notice that $rk(t) = d$ is possible, since obviously there is a $(d-1)$-space (i.e. a hyperplane) through T cutting d distinct points on C_d (any generic hyperplane through T will do). This also shows that d is the maximum possible rank.

Now let us generalize the procedure above; let $T \in \sigma_r(C_d) \setminus \sigma_{r-1}(C_d)$, $r \leq \lceil \frac{d+1}{2} \rceil$; we want to prove that if $rk(t) \neq r$, then $rk(t) = d - r + 2$. Since $rk(t) > r$, we know that T must lie on a $\mathbb{P}^{r-1}$ which cuts a non-reduced divisor $Z \in C_d$ with $\deg(Z) = r$; therefore there is a point $P \in C_d$ such that $2P \in Z$. If we had $rk(t) \leq d - r + 1$, then T would be on a $\mathbb{P}^{d-r}$ which cuts C_d in distinct points $P_1, \dots, P_{d-r+1}$; if that were true the space $< P_1, \dots, P_{d-r+1}, Z - P >$ would be $(d - 1 - \deg(Z - 2P) \cap \{P_1, \dots, P_{d-r+1}\})$-dimensional and cut $P_1 + \dots + P_{d-r+1} + Z - (Z - 2P) \cap \{P_1, \dots, P_{d-r+1}\}$ on C_d, which is impossible.

So we got $rk(t) \geq d - r + 2$; now we have to show that the rank is actually $d - r + 2$. Let's consider the divisor $Z - 2P$ on C_d; we have $\deg(Z - 2P) = r - 2$, and the space $\Gamma =< Z - 2P, T >$ which is $(r - 2)$-dimensional since $< Z - 2P >$ does not contain T (otherwise $T \in \sigma_{r-3}(C_d)$). Consider the linear series cut on C_d by the hyperplanes containing Γ: we will be finished if we show that its generic divisor is reduced.

If it is not, there should be a fixed non-reduced part of the series, i.e. at least a divisor of type $2Q$. If this is the case, each hyperplane through Γ would contain $2Q$, hence $2Q \subset \Gamma$, which is impossible, since we would have $\deg(\Gamma \cap C_d) = r$, while $\dim \Gamma = r - 2$.

Thus $rk(t) = d - r + 2$, as required. □

This theorem allows to get a simplified version of the Sylvester algorithm (see also [18]), which computes only the symmetric rank of a symmetric tensor, without computing the actual decomposition.

Algorithm 2. The (Sylvester) Symmetric Rank Algorithm:

Input: The projective class T of a symmetric tensor $t \in S^dV$ with $\dim(V) = 2$
Output: $rk(t)$.

(1) Initialize $r = 0$;
(2) Increment $r \leftarrow r + 1$;
(3) Compute $M_{d-r,r}(t)$'s $(r + 1) \times (r + 1)$-minors; if they are not all equal to zero then go to step 2; else, $T \in \sigma_r(C_d)$ (notice that this happens for $r \leq \lceil \frac{d+1}{2} \rceil$); go to step 4.
(4) Choose a solution $(\overline{u}_0, \dots, \overline{u}_d)$ of the system $M_{d-r,r}(t) \cdot (u_0, \dots, u_r)^t = 0$. If the polynomial $\overline{u}_0 t_0^d + \overline{u}_1 t_0^{d-1} t_1 + \dots + \overline{u}_r t_1^r$ has distinct roots, then $rk(t) = r$, i.e. $T \in \sigma_{r,r}(C_d)$, otherwise $rk(t) = d - r + 2$, i.e. $T \in \sigma_{r,d-r+2}(C_d)$.

Remark 6. When a form $f \in K[x_0, \dots, x_n]$ can be written using less variables (i.e. $f \in K[l_0, \dots, l_m]$, for $l_j \in K[x_0, \dots, x_n]_1$, $m < n$) then the symmetric rank of the symmetric tensor associated to f (with respect to $X_{n,d}$) is the same one as the one with respect to $\nu_d(\mathbb{P}^m)$, (e.g. see [24], [25]). In particular, when a tensor is such that $T \in \sigma_r(X_{n,d}) \subset \mathbb{P}(S^dV)$, $\dim(V) = n + 1$, then, if $r < n + 1$, there is a subspace $W \subset V$ with $\dim(W) = r$ such that $T \in \mathbb{P}(S^dW)$; i.e. the form corresponding to T can be written with respect to r variables.

Consider the following construction

(12)
$$Hilb_r(\mathbb{P}^n) \overset{\phi}{\dashrightarrow} \mathbb{G}\left(\binom{d+n}{n} - r, K[x_0, \dots, x_n]_d\right) \cong \dots$$
$$\dots \cong \mathbb{G}\left(\binom{d+n}{n} - r - 1, \mathbb{P}(K[x_0, \dots, x_n]_d)\right) \to \mathbb{G}(r - 1, \mathbb{P}(K[x_0, \dots, x_n]_d)^*).$$

The map ϕ in (12) sends a scheme Z (0-dimensional with $\deg(Z) = r$) to the vector space $(I_Z)_d$; it is defined in the open set formed by the schemes Z which impose independent conditions to forms of degree d.

As in the case $n = 1$, the final image in the above sequence gives the $(r-1)$-spaces which are r-secant to the Veronese variety in $\mathbb{P}^N \cong \mathbb{P}(K[x_0,\dots,x_n]_d)^*$; moreover each such space cuts the image of Z on the Veronese.

Notation 1. From now on we will always use the notation Π_Z to indicate the projective linear subspace of dimension $r-1$ in $\mathbb{P}(S^dV)$, with $\dim(V) = n+1$, generated by the image of a 0-dimensional scheme $Z \subset \mathbb{P}^n$ of degree r via Veronese embedding.

Theorem 6. *Any* $T \in \sigma_2(X_{n,d}) \subset \mathbb{P}(V)$*, with* $\dim(V) = n+1$*, can only have symmetric rank equal to* 1*,* 2 *or* d*. More precisely:*

$$\sigma_2(X_{n,d}) \setminus X_{n,d} = \sigma_{2,2}(X_{n,d}) \cup \sigma_{2,d}(X_{n,d}),$$

moreover $\sigma_{2,d}(X_{n,d}) = \tau(X_{n,d}) \setminus X_{n,d}$.

Proof. The Theorem is actually a quite direct consequence of Remark 6 and of Theorem 5, but let us describe the geometry in some detail. Since $r = 2$, every $Z \in Hilb_2(\mathbb{P}^n)$ is the complete intersection of a line and a quadric, so the structure of I_Z is well known: $I_Z = (l_1,\dots,l_{n-1},q)$, where $l_i \in R_1$, linearly independent, and $q \in R_2 - (l_1,\dots,l_{n-1})_2$.

If $T \in \sigma_2(\nu_d(\mathbb{P}^n))$ we have two possibilities; either $rk(T) = 2$ (i.e. $T \in \sigma_2^0(\nu_2(\mathbb{P}^n))$), or $rk(T) > 2$ i.e. T lies on a tangent line Π_Z to the Veronese, which is given by the image of a scheme Z of degree 2, via the maps (12). We can view T in the projective linear space $H \cong \mathbb{P}^d$ in $\mathbb{P}(S_dV)$ generated by the rational normal curve $C_d \subset X_{n,d}$, which is the image of the line L defined by the ideal $(l_1,\dots,l_{n-1})$ in $\mathbb{P}^n$ with $l_1,\dots,l_{n-1} \in V^*$ (i.e. $L \subset |PPn$ is the unique line containing z); hence we can apply Theorem 5 in order to get that $rk(T) \leq d$.

Moreover, by Remark 6, we have $rk(T) = d$. □

Remark 7. Let us check that it is the annihilation of the (3×3)-minors of the first two catalecticant matrices, $M_{d-1,1}$ and $M_{d-2,2}$ which determines $\sigma_2(\nu_d(\mathbb{P}^n))$ (actually such minors are the generators of $I_{\sigma_2(\nu_d(\mathbb{P}^n))}$, see [23]).

Following the construction above (12), we can notice that the linear spaces defined by the forms $l_i \in V^*$ in the ideal I_Z, are such that their coefficients are the solutions of a linear system whose matrix is given by the catalecticant matrix $M_{d-1,1}$ defined in Definition 13 (where the a_i's are the coefficients of the polynomial defined by t); since the space of solutions has dimension $n-1$, we get $rk(M_{d-1,1}) = 2$. When we consider the quadric q in I_Z, instead, the analogous construction gives that its coefficients are the solutions of a linear systems defined by the catalecticant matrix $M_{d-2,2}$, and the space of solutions has to give q and all the quadrics in $(l_1,\dots,l_{n-1})_2$, which are $\binom{n}{2}+2n-1$, hence $rk(M_{d-2,2}) = \binom{n+2}{2} - (\binom{n}{2}+2n) = 2$.

Therefore we can write down an algorithm to test if an element $T \in \sigma_2(X_{n,d})$ has symmetric rank 2 or d.

Algorithm 3. Algorithm for the symmetric rank of an element of $\sigma_2(X_{n,d})$

Input: The projective class T of a symmetric tensor $t \in S^dV$, with $\dim(V) = n+1$;
Output: $T \notin \sigma_2(X_{n,d})$, or $T \in \sigma_{2,2}(X_{n,d})$, or $T \in \sigma_{2,d}(X_{n,d})$, or $T \in X_{n,d}$.

(1) Consider the homogeneous polynomial associated to t as in (11) and rewrite it with the minimum possible number of variables (methods are described in [15] or [27]), if this is 1 then $T \in X_{n,d}$; if it is > 2 then $T \notin \sigma_2(X_{n,d})$, otherwise T can be viewed as a point in $\mathbb{P}(S^dW) \cong \mathbb{P}^d \subset \mathbb{P}(S^dV)$, and $\dim(W) = 2$, so go to step 2.
(2) Apply the Algorithm 2 to conclude.

Everything that we have done in this section doesn't use anything more than Sylvester's Algorithm for the 2 variables case. What can be done if we have to deal with more variables?

3.1. **Beyond Sylvester's Algorithm using 0-dimensional schemes.** We keep following [11].

If $f \in \sigma_3(\nu_d(\mathbb{P}^n)) \setminus \sigma_2(\nu_d(\mathbb{P}^n))$ then we will need more than 2 variables.

Theorem 7. *Let $d \geq 3$, $X_{n,d} \subset \mathbb{P}^(V)$. Then:*
$\sigma_3(X_{n,3}) \setminus \sigma_2(X_{n,3}) = \sigma_{3,3}(X_{n,3}) \cup \sigma_{3,4}(X_{n,3}) \cup \sigma_{3,5}(X_{n,3})$, *while, for $d \geq 4$:*
$\sigma_3(X_{n,d}) \setminus \sigma_2(X_{n,d}) = \sigma_{3,3}(X_{n,d}) \cup \sigma_{3,d-1}(X_{n,d}) \cup \sigma_{3,d+1}(X_{n,d}) \cup \sigma_{3,2d-1}(X_{n,d})$.

Proof. For any scheme $Z \in Hilb_3(\mathbb{P}(V))$ there exist a subspace $U \subset V$ of dimension 3 such that $Z \subset \mathbb{P}(U)$. Hence, when we make the construction in (12) we get that Π_Z is always a $\mathbb{P}^2$ contained in $\mathbb{P}(S^dU)$ and $\nu_d(\mathbb{P}(U))$ is a Veronese surface $X_{2,d} \subset \mathbb{P}(S^dU) \subset \mathbb{P}(S^dV)$. Therefore, by Remark 6, it is sufficient to prove the statement for $X_{2,d} \subset \mathbb{P}(S^dU)$.

We will consider first the case when there is a line L such that $Z \subset L$. In this case, let $C_d = \nu_d(L)$, we get that $T \in \sigma_3(C_d)$, hence either $T \in \sigma_{3,3}(C_d)$ (hence $T \in \sigma_{3,3}(X_{2,d})$), or (only when $d \geq 4$) $T \in \sigma_{3,d-1}(C_d)$, hence $rk(T) \leq d-1$. It is actually $d-1$ by Remark 6.

Now we let Z not to be on a line; the scheme $Z \in Hilb_3(\mathbb{P}^n)$ can have support on 3 , 2 distinct points or on one point.

If $Supp(Z)$ is the union of 3 distinct points then clearly Π_Z, that is the image of Z via (12), intersects $X_{2,d}$ in 3 different points and hence any $T \in \Pi_Z$ has symmetric rank precisely 3, so $T \in \sigma_{3,3}(X_{2,d})$.

If $Supp(Z) = \{P, Q\}$ with $P \neq Q$, then the scheme Z is the union of a simple point, Q, and of a 2-jet J at P. The structure of 2-jet on P implies that there exist a line $L \subset \mathbb{P}^n$ whose intersection with Z is a 0-dimensional scheme of degree 2. Hence $\Pi_Z =< T_{\nu_d(P)}(C_d), \nu_d(Q) >$ where $T_{\nu_d(P)}(C_d)$ is the projective tangent line at $\nu_d(P)$ on $C_d = \nu_d(L)$. Since $T \in \Pi_Z$, the line $< T, \nu_d(Q) >$ intersects $T_{\nu_d(P)}(C_d)$ in a point $Q' \in \sigma_2(C_d)$. From Theorem 5 we know that $rk(Q') = d$. We may assume that $T \neq Q'$ because otherwise T should belong to $\sigma_2(X_{2,d})$.

We have $Q \notin L$ because Z is not in a line, so T can be written as a combination of a tensor of symmetric rank d and a tensor of symmetric rank 1, hence $rk(t) \leq d+1$. Now suppose that $rk(t) = d$, hence there should exist $Q_1, \ldots, Q_d \in X_{2,d}$ such that $T \in < Q_1, \ldots, Q_d >$; notice that $Q_1, \ldots, Q_d$ are not all on C_d, otherwise $T \in \sigma_2(X_{2,d})$. Let $P_1, \ldots, P_d$ be the pre-image via ν_d of $Q_1, \ldots, Q_d$; then $P_1, \ldots, P_d$ together with J and Q should not impose independent conditions to curves of degree d, so, by Lemma 3, either $P_1, \ldots, P_d, J$ are on L, or $P_1, \ldots, P_d, P, Q$ are on a line L'. The first case is not possible, since $Q_1, \ldots, Q_d$ are not on C_d. In the other case notice that, by Lemma 3 and the Remark 8, should have that $< Q_1, \ldots, Q_d, T_{\nu_d(P)}(C_d), \nu_d(Q) > \cong \mathbb{P}^d$, but since $< Q_1, \ldots, Q_d >$ and

$< T_{\nu_d(P)}(C_d), \nu_d(Q) >$ have $T, \nu_d(P)$ and $\nu_d(Q)$ in common, they generate a $(d-1)$-dimensional space, a contradiction. Hence $rk(t) = d+1$.

This construction shows also that $T \in \sigma_{3,d+1}(X_{2,d})$, and that there exist $W \subset V$ with $\dim(W) = 2$ and $l_1, \dots, l_d \in W^*$ and $l_{d+1} \in V^*$ such that $t = l_1^d + \cdots + l_d^d + l_{d+1}^d$ and $t = [T]$.

If $Supp(Z)$ is only one point $P \in \mathbb{P}^2$, then Z can only be one of the following: either Z is 2-fat point (i.e. I_Z is I_P^2), or there exists a smooth conic containing Z. If Z is a double fat point then Π_Z is the tangent space to $X_{2,d}$ at $\nu_d(P)$, hence if $T \in \Pi_Z$, then the line $< \nu_d(P), T >$ turns out to be a tangent line to some rational normal curve of degree d contained in $X_{2,d}$, hence in this case $T \in \sigma_2(X_{2,d})$.
If there exists a smooth conic $C \subset \mathbb{P}^2$ containing Z, write $Z = 3P$ and consider $C_{2d} = \nu_d(C)$, hence $T \in \sigma_3(C_{2d})$, therefore by Theorem 5 clearly $rk(t) \leq 2d-1$. Suppose that $rk(t) \leq 2d-2$, hence there exist $P_1, \dots, P_{2d-2} \in \mathbb{P}^2$ distinct points that are neither on a line nor on a conic containing $3P$, such that $T \in \Pi_{Z'}$ with $Z' = P_1 + \cdots + P_{2d-2}$ and $Z + Z' = 3P + P_1 + \cdots + P_{2d-2}$ doesn't impose independent conditions to the planes curves of degree d. Now, by Lemma 3 we get that $3P + P_1 + \cdots + P_{2d-2}$ doesn't impose independent conditions to the plane curves of degree d if and only if there exists a line $L \subset \mathbb{P}^2$ such that $\deg((Z+Z') \cap L) \geq d+2$. Observe that Z' cannot have support contained in a line because otherwise $T \in \sigma_2(X_{2,d})$. Moreover $Z + Z'$ cannot have support on a conic $C \subset \mathbb{P}^2$ because in that case T would have symmetric rank $2d-1$ with respect to $\nu_d(C) = C_{2d}$.
We have to check the following cases:

(1) There exist $P_1, \dots, P_{d+2} \in Z'$ on a line $L \subset \mathbb{P}^2$;
(2) There exist $P_1, \dots, P_{d+1} \in Z'$ such that together with $P = Supp(Z)$ they are on the same line $L \subset \mathbb{P}^2$;
(3) There exist $P_1, \dots, P_d \in Z'$ such that together with the 2-jet $2P$ they are on the same line $L \subset \mathbb{P}^2$.

Case 1. Let $P_1, \dots, P_{d+2} \in L \subset \mathbb{P}^2$, then $\nu_d(L) = C_d \subset \mathbb{P}^d \subset \mathbb{P}^N$ with $N = \binom{d+2}{2} - 1$. Clearly $T \in \Pi_Z \cap \Pi_{Z'}$, then $\dim(\Pi_Z + \Pi_{Z'}) \leq \dim(\Pi_Z) + \dim(\Pi_{Z'})$, moreover $\Pi_{Z'}$ doesn't have dimension $2d-3$ as expected because $\nu_d(P_1), \dots, \nu_d(P_{d+2}) \in C_d \subset \mathbb{P}^d$, hence $\dim(\Pi_{Z'}) \leq 2d-4$ and $\dim(\Pi_Z + \Pi_{Z'}) \leq 2d-2$. But this is not possible because $Z + Z'$ imposes to the plane curves of degree d only one condition less then the expected, hence $\dim(I_{Z+Z'}(d)) = \binom{d+1}{2} - d + 1$ and then $\dim(\Pi_Z + \Pi_{Z'}) = 2d-1$, that is a contradiction.

Case 2. Let $P_1, \dots, P_{d+1}, P \in L \subset \mathbb{P}^2$, then $\nu_d(P_1), \dots, \nu_d(P_{d+1}), \nu_d(P) \in \nu_d(L) = C_d$. Now $\Pi_Z \cap \Pi_{Z'} \supset \{\nu_d(P), T\}$, then again $\dim(\Pi_Z + \Pi_{Z'}) \leq 2d-2$.

Case 3. Let $P_1, \dots, P_d, 2P \in L \subset \mathbb{P}^2$, as previously $\nu_d(P_1), \dots, \nu_d(P_{d+1}), \nu_d(2P) \in \nu_d(L) = C_d$, then now $T_{\nu_d(P)}(C_d)$ is contained in $< C_d > \cap \Pi_Z$. Since $< \nu_d(P_1, \dots, \nu_d(P_d) >)$ is an hyperplane in $< C_d >= \mathbb{P}^d$, it will intersect $T_{\nu_d(P)}(C_d)$ in a point Q different form $\nu_d(P)$. Again $\dim(\Pi_Z \cap \Pi_{Z'}) \geq 1$ and then $\dim(\Pi_Z + \Pi_{Z'}) \leq 2d-2$.

□

It is worth to stress the importance of Lemma 3 used many times in the proof.

Lemma 3. *Let $Z \subset \mathbb{P}^n$, $n \geq 2$, be a 0-dimensional scheme, with* $\deg(Z) \leq 2d+1$. *A necessary and sufficient condition for Z to impose independent conditions to hypersurfaces of degree d is that no line $L \subset \mathbb{P}^n$ is such that* $\deg(Z \cap L) \geq d+2$.

Proof. The statement was probably classically known, we prove it here for lack of a precise reference. Let us work by induction on n and d; if $d = 1$ the statement is trivial; so let us suppose that $d \geq 2$ and now let's work by induction on n. Let us consider the case $n = 2$ first. If there is a line L which intersects Z with multiplicity $\geq d + 2$, then trivially Z cannot impose independent condition to curves of degree d, since the fixed line imposes $d + 1$ conditions, hence we have already missed one. So, suppose that no such line exist, and let L be a line such that $Z \cap L$ is as big as possible (but $Z \cap L \leq d + 1$). Let $Tr_L Z$, the *Trace* of Z on L, be the schematic intersection $Z \cap L$ and $Res_L Z$, the *Residue* of Z with respect to L, be the scheme defined by $(I_Z : I_L)$. We have the following exact sequence of ideal sheaves:

$$0 \to \mathcal{I}_{Res_L Z}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Tr_L Z}(d) \to 0.$$

Then no line can intersect $Res_L Z$ with multiplicity $\geq d+1$, because $\deg(Z) \leq 2d+1$ and L is a line with maximal intersection with Z; so if $\deg(L' \cap res_L Z) = d + 1$, we'd have that also $\deg(L \cap Z) = d + 1$, which is impossible because it would give $\deg(L \cap Z) + \deg(L' \cap res_L Z) = \deg(L' \cap res_L Z) = 2d + 2$, while $\deg Z \leq 2d + 1$. Hence we have $h^1(\mathcal{I}_{Res_L Z}(d-1)) = 0$, by induction on d; on the other hand, we have $h^1(\mathcal{I}_{Tr_L Z}(d)) = h^1(\mathcal{O}_{\mathbb{P}^1}(d - \deg(Tr_L Z))) = 0$, hence also $h^1(\mathcal{I}_Z(d)) = 0$, i.e. Z imposes independent conditions to curves of degree d (notice that the condition $\deg(Z) \leq 2d + 1$ yields $h^0(\mathcal{I}_Z(d)) > 0$).

With the case $n = 2$ done, let us finish by induction on n; let $n \geq 3$ now; again, if there is a line L which intersects Z with multiplicity $\geq d + 2$, we can conclude that Z does not impose independent conditions to forms of degree d, as in the case $n = 2$. Otherwise, consider a hyperplane H, with maximum multiplicity of intersection with Z, and consider the exact sequence:

$$0 \to \mathcal{I}_{Res_H Z}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Tr_H Z}(d) \to 0.$$

We have $h^1(\mathcal{I}_{Res_H Z}(d-1)) = 0$, by induction on d, and $h^1(\mathcal{I}_{Tr_H Z}(d)) = 0$, by induction on n, so we get that $h^1(\mathcal{I}_Z(d)) = 0$ again, and we are done. □

Remark 8. Notice that if $\deg L \cap Z$ is exactly $d + 1 + k$, then the dimension of the space of curves of degree d through them increases exactly by k with respect to the generic case.

It is quite easy to see that Lemma 3 can be improved as follows (see [6]).

Lemma 4. *Let $Z \subset \mathbb{P}^n$, $n \geq 2$, be a 0-dimensional scheme, with* $\deg(Z) \leq 2d + 1$. *If* $h^1(\mathbb{P}^n, \mathcal{I}_Z(d)) > 0$ *there there exists a unique line* $L \subset \mathbb{P}^n$ *such that* $\deg(Z \cap L) = d + 1 + h^1(\mathbb{P}^n, \mathcal{I}_Z(d)) > 0$.

We can go back to our problem of finding the rank of a given tensor. If now we want to treat the case of $\sigma_4(\nu_d(\mathbb{P}^n))$ analogously we did for $\sigma_3(\nu_d(\mathbb{P}^n))$, we can (see [8]) but it requires a very complicate analysis on the schemes of length 4. Despite the long procedure required for the classification of the rank with respect to the minimal length of 0-dimensional scheme whose span contains the given polynomial we are dealing with, there is a more intrinsic problem. We cannot use the same technique for classifying the ranks in the case of $\sigma_5(\nu_d(\mathbb{P}^n))$. In fact there is a famous contra-example due to W. Buczyńska, J. Buczyśki (see [14]) that shows that in $\sigma_5(\nu_3(\mathbb{P}^4))$ there is at least a polynomial for which it doesn't exist any 0-dimensional scheme contained in $\nu_3(\mathbb{P}^4)$ whose span contains it. The example is the following:

Example 7. The following polynomial has border rank ≤ 5 but smoothable rank ≥ 6:

$$f = x_0^2 x_2 + 6x_1^2 x_3 - 3\,(x_0 + x_1)^2 x_4.$$

One can easily check that the following polynomial

$$f_\epsilon = (x_0 + \epsilon x_2)^3 + 6(x_1 + \epsilon x_3)^3 - 3(x_0 + x_1 + \epsilon x_4)^3 + 3(x_0 + 2\,x_1)^3 - (x_0 + 3x_1)^3$$

has rank 5 for $\epsilon > 0$, and that $\lim_{\epsilon \to 0} \frac{1}{3\epsilon} f_\epsilon = f$.

Therefore $r_\sigma(f) \leq 5$.

An explicit computation of $(f^\perp)$ yields to the following Hilbert function for $H_{R/(f^\perp)} = [1, 5, 5, 1, 0, \ldots]$. Let us prove, by contradiction, that there is no saturated ideal $I \subset (f^\perp)$ of degree ≤ 5. Suppose on the contrary that I is such an ideal. Then $H_{R/I}(n) \geq H_{R/(f^\perp)}(n)$ for all $n \in \mathbb{N}$. As $H_{R/I}(n)$ is an increasing function of $n \in \mathbb{N}$ with $H_{R/(f^\perp)}(n) \leq H_{R/I}(n) \leq 5$, we deduce that $H_{R/I} = [1, 5, 5, 5, \ldots]$. This shows that $I^1 = \{0\}$ and that $I^2 = (f^\perp)^2$. As I is saturated, $I^2 : (x_0, \ldots, x_4) = I^1 = \{0\}$ since $H_{R/(f^\perp)}(1) = 5$. But an explicit computation of $((f^\perp)^2 : (x_0, \ldots, x_4))$ gives $\langle x_2, x_3, x_4 \rangle$. We obtain a contradiction, so that there is no saturated ideal of degree ≤ 5 such that $I \subset (f^\perp)$. Consequently, $r_{\mathrm{smooth}^0}(f) \geq 6$ so that $r_\sigma(f) < r_{\mathrm{smooth}^0}(f)$.

On our knowledge the tow main results that are nowadays available to treat these "wild" cases are the following.

Proposition 7 ([11])**.** *Let $X \subset \mathbb{P}^N$ be a non degenerate smooth variety. Let H_r be the irreducible component of the Hilbert scheme of 0-dimensional schemes of degree r of X containing r distinct points, and assume that for each $y \in H_r$, the corresponding subscheme Y of X imposes independent conditions to linear forms. Then for each $P \in \sigma_r(X) \setminus \sigma_r^0(X)$ there exist a 0-dimensional scheme $Z \subset X$ of degree r such that $P \in < Z > \cong \mathbb{P}^{r-1}$.*

Conversely if there exists $Z \in H_r$ such that $P \in < Z >$, then $P \in \sigma_r(X)$.

Proof. Let us consider the map $\phi : H_r \to \mathbb{G}(r-1, \mathbb{P}^N)$, $\phi(y) =< Y >$; ϕ is well defined since $\dim < Y >= r - 1$ for all $y \in H_r$ by assumption. Hence $\phi(H_r)$ is closed in $\mathbb{G}(r-1, \mathbb{P}^N)$.

Now let $\mathcal{I} \subset \mathbb{P}^N \times \mathbb{G}(r-1, \mathbb{P}^N)$ be the incidence variety, and p, q its projections on $\mathbb{P}^N$, $\mathbb{G}(r-1, \mathbb{P}^N)$ respectively; then, $A := pq^{-1}(\phi(H_r))$ is closed in $\mathbb{P}^N$. Moreover, A is irreducible since H_r is irreducible, so $\sigma_r^0(X)$ is dense in A. Hence $\sigma_r(X) = \overline{\sigma_r^0(X)} = A$. □

Obviously, 5 points on a line don't impose independent conditions to cubics in any $\mathbb{P}^n$ for $n \geq 5$, therefore this could be one reason why such contra-exmple is possible.

Another reason is in the following.

Proposition 8 ([14])**.** *Suppose there exist points $x_1, \ldots x_r \in X$ that are linearly degenerate, that is $\dim\langle x_1, \ldots x_r \rangle < r - 1$. Then the join of the r tangent stars at these points is contained in $\sigma_r(X)$. In the case X is smooth at $x_1, \ldots x_r$ then $\langle \mathbb{P}\hat{T}_{x_1} X, \ldots, \mathbb{P}\hat{T}_{x_r} X \rangle \subset \sigma_r(X)$.*

Lecture 4

4. Beyond Sylvester's Algorithm via Apolarity

We have already defined apolarity and inverse system. One crucial Lemma that we have not introduced yet is the so called Apolarity Lemma.

Lemma 5 (**Apolarity Lemma**). *Let $Z = \{[L_1], \dots, [L_r]\} \subset \mathbb{P}(S^1 V)$, then $f = \sum_{i=1}^r \lambda_i L_i^d$ iff $I(Z) \subseteq f^\perp$.*

Proof. The implication $\Rightarrow$ is obvious. The other direction can be obtained with a dimension argument. □

With this Lemma we can rephrase Sylvester's Algorithm.

Let $f(x,y) = \sum_{i=0}^d c_i \binom{d}{i} x^{d-i} y^i$. Such an f can be decomposed as sum of r distinct powers of linear forms iff there exists $q(x,y) = q_0 x^r + q_1 x^{r-1} y + \cdots + q_r y^r = 0$ with

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_r \\ c_1 & & \cdots & c_{r+1} \\ \vdots & & & \vdots \\ c_{d-r} & & \cdots & c_d \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_r \end{pmatrix} = 0$$

and $q(x,y) = \mu \Pi_{k=1}^r (\beta_k x - \alpha_k y)$. In this case then $f(x,y) = \sum_{k=1}^r \lambda_k (\alpha_k x + \beta_k y)^d$.

This is possible because

$$I(Z) \subset (f^\perp) = ker(Cat) = (G_1, G_2)$$

where the scheme Z is the scheme defined by the zeros of q and G_1, G_2 are the two generators of the apolar ideal to f.

Since the Apolarity Lemma is true for any number of variables, what can we say about a possible relation between $I(Z)$, $(f^\perp)$ and $ker(Cat)$? Obviously, by definition, we have that $I(Z) \subset (f^\perp)$, but in general the $(f^\perp) = ker(Cat)$ is not true anymore. In particular $rk(f) \geq \max\{rkCat\}$.

A generalization of Sylvester's algorithm to any number of variables that uses this techniques is given by Iarrobino and Kanev (see [22]) only under the hypothesis $rk(f) = \max\{rkCat\}$.

Algorithm 4 (**Iarrobino and Kanev**, [22])**.**

Input: $f \in S^d V$, where $\dim V = n + 1$.

(1) Construct the most square possible catalecticant $C_f^m = C_f$ with $m = \lceil \frac{d}{2} \rceil$.
(2) Compute $\ker C_f$. If $rk(f) = rk(C_f)$ then continue, otherwise stop here.
(3) Find the zero-set $Z' = [L_1], \dots, [L_s]$ of the polynomials in $\ker C_f$.
(4) Solve the linear system defined by $f = \sum_{i=1}^s c_i L_i^d$ in the unknowns c_i.

Output: Waring decomposition of f.

This method works only if $rk(f) = \max\{rkCat\}$. The idea developed in [12] is to construct a general Henkel matrix as follows:

$$H_f : S^* \to S, \text{ such that } \partial \mapsto \partial(f).$$

Such an application is linear, where the entries of an associated matrix are known, they coincides with the entries of catalecticant matrices, but they are not all known.

Proposition 9. *$Ker(H_f)$ is an ideal.*

Proposition 10. *If $rk(H_f) = r < \infty$, then $\dim R^*/I_f = r$ and there exist $L_1, \dots L_k$, $g_i \in S^{d_i}V$ such that $f = \sum_{i=1}^k L_i^{d-d_i} g_i$ and the apolar of f contains schemes Z_i with support on $[L_i^*]$ and they have multiplicity equal to the dimension of the vector space spanned by the inverse system generated by $L_i^{d-d_i} g_i$.*

Theorem 8 (Brachat, Comon, Mourrain, Tsigaridas [12])**.** *$f = \sum_{i=1}^r L_i^d$ if and only if $rkH_f = r$ and $I_f := KerH_f$ is a radical ideal.*

How to do it in practice? We give here only an idea.

Given $f \in S^d$, find $f^* \in S$ which extends f with $H_{f^*}|_{\text{Domain of } H_f} = H_f$, $rkH_{f^*} = rkf$ and I_{f^*} is a radical ideal. Those f^* are elements of the following set

$$\mathcal{E}_r^{d,0} := \{[f] \in \mathbb{P}(S^d V) \,|\, \exists L \in S^1V \setminus \{0\}, \exists \tilde{f} \in Y_r^{m,m'} \text{ with}$$
$$m = \max\{r, \lceil d/2 \rceil\}, m' = \max\{r-1, \lfloor d/2 \rfloor\} \text{ s.t. } L^{m+m'-d}\tilde{f}^* = f^*\}$$

where $Y_r^{i,d-i} = \{[f] \in \mathbb{P}(S^dV) \,|\, rk(C^{i,d-i}) \leq r\}$. If $f \in \mathcal{E}_r^{d,0}$ we say that f has a generalized affine decomposition of size r.

Now, multiplying by a linear (power of) form means introduce multiplying operators in $S/I_f = A_f$:

$$M_a : A_f \to A_f$$
$$b \mapsto a \cdot b$$

and

$$M_a^t : A_f^* \to A_f^*$$
$$\gamma \mapsto a * \gamma.$$

Now define

$$M_{a*f} := M_a^t \circ H_f. \tag{13}$$

Theorem 9. *If $\dim A_f < \infty$ then $f = \sum_{i=1}^k L_i^{d-d_i} g_i$ and*

- *the eigenvalues of the operators M_a and M_a^t are given by $\{a(L_i^*), \dots, a(L_r^*)\}$,*
- *the common eigenvectors of the operators $(M_{x_i}^t)_{1 \leq i \leq n}$ are up ti scalar L_i.*

One can recover the points L_i by eigenvector computations: Take B a basis $|B| = rkH_f$, $H_{a*f}^B = M_a^t H_f^B = H_f^B M_a$ (M_a is the matrix multiplication by a in the basis B of A_f). The common solutions of the generalized eigenvalue problem

$$(H_{a*f} - \lambda H_f)v = 0$$

for all $a \in S$ yield the common eigenvectors $H_f^B v$ of M_a^t that is the evaluation of L_i at the roots. Therefore these common eigenvectors $H_f^B v$ are up to scalar the vectors $[b_i(L_i^*), \dots, b_r(L_i^*)]$.

If $f = \sum_{i=1}^r L_i^d$, then the roots are simple and one eigenvector computation is enough for $a \in S$, M_a is diagonalizable and the generalized eigenvectors $H_f^B v$ are up to scalar the evaluations 1_{L_i} of the roots.

Theorem 10. *If B and B' are connected to 1 (i.e. $m \in B \neq 1$ then $m = x_{i,j} m'$ with $m' \in B$) and $\tilde{f}$ known on $B'^+ \times B^+$ and $H_{\tilde{f}}^{B',B}$ is invertible, then $\tilde{f}$ extends uniquely to S if and only if $M_i^{B',B} \circ M_j^{B'B} = M_j^{B',B} \circ M_i^{B'B}$ where $M_i^{B',B} := H_{\tilde{f}}^{B',x_iB}(H_{\tilde{f}}^{B',B})^{-1}$, $1 \leq i,j \leq n$, $B \subset R$, $B^+ = B \cup x_1B \cup \dots \cup x_nB$.*

Algorithm 5 (**Brachat, Comon, Mourrain, Tsigaridas**)**.**

- Compute a set B of monomials of deg $\leq d$ connected to 1 with $|B| = r$.
- Find parameters h s.t. $\det(H_f^B) \neq 0$ and the operators $M_i = H_{x_i f}^B (H_f^B)^{-1}$ commute.
- If there is no solution re-start the loop with $r = r + 1$.
- Else compute the $n \times r$ eigenvalues $z_{i,j}$ and the eigenvectors v_j such that $M_j v_j = z_{i,j} v_j$, $i = 1, \ldots, n$, $j = 1, \ldots, r$. until the eigenvalues are simple.
- Solve the linear system in $(v_j)_{j=1,\ldots,k}$, $f = \sum_{j=1}^r v_j z_j^d$ where z_j are the eigenvectors found above.

4.1. Exercises.

Exercise 7. Compute the rank of $F = -x^5 + 3x^4y - 3x^3y^2 + x^2y^3 + 2x^4z - 6x^3yz + 6x^2y^2z - 2xy^3z - x^3z^2 + 3x^2yz^2 - 3xy^2z^2 + y^3z^2$.

Exercise 8. Let $F = x^2yz$. Prove that $5 \leq rk(F) \leq 6$. (Actually $rk(F) = 6$ **).

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