EXAMPLE TO "DETERMINANTAL EQUATIONS FOR SECANT VARIETIES AND THE EISENBUD-KOH-STILLMAN CONJECTURE"

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We sketch how to construct an example of a smoothable scheme $R \subset \mathbb{P}V$ and a smooth variety $X \subset \mathbb{P}V$, such that $R \cap X$ is locally Gorenstein, but not smoothable. Such an example illustrates that in the course of the proof of Theorem 1.1.1 in [BGL10] one really needs to treat this special case. We wrote down this example upon a request of an anonymous referee of this paper, and also motivated by questions of audiences during the author's presentation in Grenoble and Berlin.

To begin with note that unless $R \cap X = R$, or $R \cap X$ is "small enough" (so that all schemes of given degree and embedding dimension are smoothable), there is no obvious reason, why should $R \cap X$ be smoothable. In general, smoothability issues are very delicate and often rely on a case by case study, rather than general statements, see for instance proofs in [CEVV09] or [CN09]. Thus even if $X \cap R$ was always smoothable, for some weird reason, then the proof would be much more complicated than the proof of Theorem 1.1.1 in [BGL10]. Below we present a series of steps how we one can construct R and X for which $R \cap X$ is non-smoothable, but without giving all the details. By the very nature of the smoothability issue, both R and X will be quite large. We keep in mind that it is desirable to construct R and X such that $R \cap X$ is locally Gorenstein, see [BB10, Thm 1.6].

Firstly, observe there exist non-smoothable locally Gorenstein zero dimensional schemes. One of them is given in [IK99, Cor. 6.21], it is a degree 14 scheme in $\mathbb{C}^6$, supported at a single point. We set R' to be this scheme (abstractly, we will produce an appropriate embedding later). Equally well, R' could also be any other non-smoothable Gorenstein zero dimensional scheme. We aim to construct R and X such that $R \cap X \simeq R'$.

Secondly, observe that any scheme is contained in a smoothable scheme. To see this, first reduce to schemes supported at a single point. Then take the scheme defined by a sufficiently high power of the maximal ideal. This is a scheme defined by monomials, and this is famously known to be smoothable (it is possible to explicitly write out the smoothing).

Thirdly, set R to be a smoothable zero dimensional subscheme in $\mathbb{C}^n$ (with $n \geq 6$) containing $R' \subset \mathbb{C}^6 \subset \mathbb{C}^n$. Let $f_1, \ldots, f_m$ be polynomials in n variables, such that the ideal of R' is generated by the ideal of R together with $f_1, \ldots, f_m$.

Fourthly, in $\mathbb{C}^n \times \mathbb{C}^m \simeq \mathbb{C}^{n+m}$ consider Y to be a scheme given by $f_1 - t_1, \ldots, f_m - t_m$, where f_i are the polynomials as above on the $\mathbb{C}^n$ factor, whereas $t_1, \ldots, t_m$ are the coordinates on the $\mathbb{C}^m$ factor. That is, Y is a graph of the map $\mathbb{C}^n \to \mathbb{C}^m$ given by $(f_1, \ldots, f_m)$. In particular, Y is smooth of dimension n. Note also that $Y \cap (\mathbb{C}^n \times \{0\}) = R'$. Similarly, $Y \cap R = R'$, where $R \subset \mathbb{C}^n \times \{0\}$. Y is our local model for X.

Finally, we have to find a closed-open embedding of $\mathbb{C}^n \times \mathbb{C}^m$ into a large projective space $\mathbb{P}^N$ such that the closure of Y in $\mathbb{P}^N$ is smooth. Set X to be this closure. Then X, R and $R' = X \cap R$ have the required properties. In this case, the smoothable scheme $Q \subset X$, such that $\deg Q = \deg R$ and $R' \subset Q$ can be obtained by reembedding R into Y using the natural map $\mathbb{C}^n \to Y$ and setting Q to be the image of this reembedding.

In general Q needs not to be (abstractly) isomorphic to R. In the above example, modify R to be a larger scheme. For instance, set $S \subset \mathbb{C}^o$ to be any finite subscheme supported at $0 \in \mathbb{C}^o$, and replace $(Y \subset \mathbb{C}^{n+m}, R)$ with $(Y \times \{0\} \subset \mathbb{C}^{n+m} \times \mathbb{C}^{o}, R \times S)$. If our choices were sufficiently minimal (for instance R does not embed in a smaller affine space than $\mathbb{C}^n$, and similarly for S), then the embedding dimension of $R \times S$ is strictly larger than the dimension of $Y \times \{0\}$, so $Q \subset Y \times \{0\}$ cannot be isomorphic to $R \times S$.

References

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