Erratum for ‘Duality and integrability on contact Fano manifolds’

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Abstract

We fix a couple of bugs and mistakes in an earlier article, “Duality and integrability on contact Fano manifolds”. Along the way we collect several facts about $\mathbb{P}^1$-bundles admitting a contractible section, which might be of independent interest.

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The main issues we address in this erratum for [Bucz10] concern the claims about the results of [Kebe01] and [Kebe05]. There are two gaps in [Kebe05], one of which have been discovered shortly before the completion of [Bucz10], and another one was discovered more recently. Moreover, in [Bucz10, Thm 3.1] we claim that [Kebe01] and [Kebe05] prove more than it is actually written there, and given the gaps we clarify these issues in this note.

A consolidated version of [Bucz10] that takes into the account this erratum is available as arXiv:1002.0698. The statements of the main results [Bucz10, Thms 1.3–1.5] remain unchanged, the modifications only affect the proofs. In order to provide correct arguments we discuss facts about $\mathbb{P}^1$-bundles. This discussion should be inserted into the paper as Subsection 2.1 at the end of Section 2.

The mistakes in [Kebe05] and [Bucz10] have been discovered and confirmed by Jun-Muk Hwang, Stefan Kebekus, and the author, and in the last

years, between the three of us, we have thoroughly discussed if and how we can fix the gaps. The present erratum is based on some of these conversations, for which I sincerely thank Jun-Muk Hwang and Stefan Kebekus and also for their expressions of support on the initial version of this note. I am also grateful to Joachim Jelisiejew, Michał Kapustka, and Jarosław Wiśniewski for further suggestions. The author is supported by the NCN project "Complex contact manifolds and geometry of secants", 2017/26/E/ST1/00231.

2.1 Bundles of projective lines

The situation we are primarily interested in is the following:

Setting 2.3. Let Y, U, B be normal projective varieties, such that $\pi\colon U \to B$ is an analytically locally trivial $\mathbb{P}^1$-bundle with a section $s\colon B \to U$ and a map $\xi\colon U \to Y$ that is birational onto its image and the section $s(B)$ is contracted to a point $y \in Y$:

$$\begin{array}{ccc} U & \xrightarrow{\ \xi\ } & Y \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle \pi} & & \\ B. & & \end{array}$$

The lemmas in this subsection seem to be classical and known to experts, but hard to reference explicitly. Setting 2.3 (with more details spelled out as Setting 2.9) appears in [Kebe05, §6.1]. It seems that most of the statements in this subsection have been known to Kebekus, and have been implicitly used in his arguments. However, since there is a gap in his arguments (more on this below), we provide all necessary details in order to avoid analogous bugs at the cost of being somewhat tedious.

In the first two lemmas we focus on $\mathbb{P}^1$-bundles with one section.

Lemma 2.4. *Suppose U and B are normal projective varieties and $\pi\colon U \to B$ is an analytically locally trivial $\mathbb{P}^1$-bundle with a section $s\colon B \to U$. Then $U = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle of rank 2 over B. In particular, U is a Zariski locally trivial $\mathbb{P}^1$-bundle over B. Moreover,*

$$\pi_*(k \cdot s(B)) = \operatorname{Sym}^k(\mathcal{E} \otimes \mathcal{M}^*),$$

where $k \cdot s(B)$ is the divisor on U (the k-fold multiple of $s(B)$) and $\mathcal{M} \subset \mathcal{E}$ is the line subbundle corresponding to $s(B)$.

Proof. The image $s(B)$ is a divisor on U which is analytically locally given as vanishing of a single holomorphic function. Since U is projective, by

GAGA, $s(B)$ is a Cartier divisor. Let $\mathcal{O}_U(s(B))$ be the corresponding line bundle. Note that $\mathcal{O}_U(s(B))$ is π-ample by [Laza04, Thm 1.7.8], and its restriction to any fibre $\mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(1)$. Thus by [AD14, Prop. 4.10] the bundle is a projectivisation of a vector bundle $\mathcal{E}$ as claimed. □

Note that in the setting of Lemma 2.4 the vector bundle $\mathcal{E}$ such that $U \simeq \mathbb{P}(\mathcal{E})$ is not unique. Instead, $\mathcal{E}$ is well defined up to a twist by a line bundle on B. Nevertheless, the formula $\pi_*(k \cdot s(B)) = \operatorname{Sym}^k(\mathcal{E} \otimes \mathcal{M}^*)$ is independent of the choice of $\mathcal{E}$.

Lemma 2.5. *Suppose U and B are reduced schemes and $\pi\colon U \to B$ is a projective morphism that is a Zariski locally trivial $\mathbb{P}^1$-bundle with a section $s\colon B \to U$. Suppose further that $S \subset U$ is a subscheme S supported on $s(B)$. Then $S = s(B)$ (as schemes) if and only if for all $b \in B$ the intersection of the fibre $\pi^{-1}(b)$ and S is the single reduced point $\{s(B)\}$.*

Proof. If $S = s(B)$, then the claim on intersection is clear. So suppose that S intersects a fibre $\pi^{-1}(b)$ in the single reduced point. We will show, that S agrees with $s(B)$ in some neighbourhood of $s(b)$.

Let $\mathcal{O}_{S,s(b)}$ be the local ring of S near $s(b)$ and let $I \subset \mathcal{O}_{S,s(b)}$ be the ideal of $s(B)$. Note that $\mathcal{O}_{S,s(b)}/I \simeq \mathcal{O}_{B,b}$ and by local triviality also $\mathcal{O}_{S,s(b)} = \mathcal{O}_{B,b} \oplus I$ as $\mathcal{O}_{B,b}$-modules. Let $\mathfrak{m}_b \subset \mathcal{O}_{B,b}$ be the maximal ideal. The assumption on the intersection is translated into algebra as

$$\mathcal{O}_{S,s(b)} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{B,b}/\mathfrak{m}_b = \mathcal{O}_{B,b}/\mathfrak{m}_b.$$

Thus $I \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{B,b}/\mathfrak{m}_b = 0$ and by Nakayama's Lemma $I = 0$ as claimed. □

In the following lemma we focus on $\mathbb{P}^1$-bundles that admit two disjoint sections.

Lemma 2.6. *Suppose U and B are normal projective varieties and $\pi\colon U \to B$ is an analytically locally trivial $\mathbb{P}^1$-bundle with a section $s\colon B \to U$. Suppose moreover that there exists another section $s'\colon B \to U$ whose image is disjoint from $s(B)$. Then*

(i) $\operatorname{Pic} U = \pi^* \operatorname{Pic}(B) \oplus \mathbb{Z} \cdot s'(B)$ *and the pullback* $s^*\colon \operatorname{Pic}(U) \to \operatorname{Pic}(B)$ *is the projection on the first component* $\pi^* \operatorname{Pic}(B) \simeq \operatorname{Pic}(B)$*, and*

(ii) $U = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{M})$*, where $\mathcal{M}$ is a line bundle on B such that the pullback of the corresponding Cartier divisor $\pi^*\mathcal{M}$ is linearly equivalent to the divisor $s(B) - s'(B)$.*

Proof. $U = \mathbb{P}(\mathcal{E})$ by Lemma 2.4, and by the assumptions $\mathcal{E}$ admits two line subbundles which do not intersect away from the zero section. Thus $\mathcal{E}$ is decomposable, and twisting by one of the components, without loss of generality we can assume $\mathcal{E} = \mathcal{O}_B \oplus \mathcal{M}$ for some line bundle $\mathcal{M}$, where $\mathbb{P}(\mathcal{O}_B) \subset \mathcal{U}$ is the image of the section s, while $\mathbb{P}(\mathcal{M})$ is the image of s'.

In this setting (i) is straightforward. To finish the proof of (ii) we must show $\pi^*\mathcal{M} = s(B) - s'(B)$ in $\operatorname{Pic}(U)$. Since $(s(B) - s'(B)).\mathbb{P}^1 = 0$ (where $\mathbb{P}^1$ is the general fibre of π) by (i) we must have $s(B) - s'(B) \in \pi^* \operatorname{Pic}(B)$, thus:

$$s^*(s(B) - s'(B)) = s^*(s(B)) = N_{s(B)\subset U} = N_{\mathbb{P}(\mathcal{O}_B)\subset\mathbb{P}(\mathcal{O}_B\oplus\mathcal{M})} = \mathcal{M},$$

where $N_{Z\subset V}$ denotes the normal sheaf of Z inside V. By (i) again we must have $\pi^*\mathcal{M} = s(B) - s'(B)$. □

Lemma 2.7. *In Setting 2.3*

(i) there exists a section $s' \colon B \to U$ disjoint from $s(B)$, and

(ii) the image of $\xi^* \colon \operatorname{Pic} Y \to \operatorname{Pic} U$ is contained in $\mathbb{Z} \cdot s'(B)$.

In particular, for a line bundle $\mathcal{L}$ on Y the linear equivalence of $\xi^*\mathcal{L}$ is uniquely determined by its intersection number with a general fibre of π.

Proof. Suppose $Z \subset Y$ is a very ample Cartier divisor in Y which does not contain $y = \xi(s(B))$. Then $\xi^* Z$ is an effective non-zero Cartier divisor on U which avoids $s(B)$. By [Wiśn89, Lem. A8.5] there exists a section $s' \colon B \to U$ disjoint from $s(B)$, as claimed in (i).

Let $\mathcal{L}$ be a line bundle on Y. In (ii) we claim that $\xi^*\mathcal{L} = k \cdot s'(B)$ for some integer k. By Lemma 2.6(i) this claim is equivalent to $s^*\xi^*\mathcal{L} \simeq \mathcal{O}_B$. The latter is true for very ample line bundles $\mathcal{L}$ by the argument above applied to Z in the linear system of $\mathcal{L}$. Any ample line bundle $\mathcal{L}$ is a difference $(m+1)\mathcal{L} - m\mathcal{L}$ with both $m\mathcal{L}$ and $(m+1)\mathcal{L}$ very ample for any sufficiently large m [Laza04, Thm 1.2.6(i),(iv)]. Therefore the claim also holds for ample line bundles. Finally, since Y is projective, any line bundle can be written as a difference of two ample line bundles, which concludes the proof. □

Lemma 2.8. *In Setting 2.3 suppose in addition that for every fibre $\mathbb{P}^1 \subset U$ the restriction $\xi|_{\mathbb{P}^1}$ is immersive and injective. Then the scheme-theoretic preimage $\xi^{-1}(y)$ is $s(B)$. In particular, the preimage is reduced and a Cartier divisor.*

Proof. The inclusion $s(B) \subset \xi^{-1}(y)$ is clear, and so is the reverse inclusion for the reduced structures $(\xi^{-1}(y))_{\mathrm{red}} \subset s(B)$. By Lemma 2.5 it is enough to prove that the intersection $\xi^{-1}(y) \cap \pi^{-1}(b)$ is reduced for any $b \in B$. We have:

$$\xi^{-1}(y) \cap \pi^{-1}(b) = \left(\xi|_{\pi^{-1}(b)}\right)^{-1}(y),$$

where the restriction $\xi|_{\pi^{-1}(b)}$ is immersive and injective. Thus the intersection $\xi^{-1}(y) \cap \pi^{-1}(b)$ is indeed reduced since $\{y\}$ is reduced. □

From now on we will assume in addition to Setting 2.3 that y is a smooth point of Y and the map ξ lifts to a closed embedding $\tilde{\xi}\colon U \to \tilde{Y}$, where $\tilde{Y} \to Y$ be the blow-up of Y at y. We also need the case of reducible base for the applications to the proof of Theorem 3.1.

Setting 2.9. Suppose $B = B_1 \sqcup \cdots \sqcup B_k$ is a projective normal reduced scheme with irreducible components B_i, Y is a normal projective variety, $\pi\colon U \to B$ is an analytically locally trivial $\mathbb{P}^1$-bundle with a section $s\colon B \to U$ and a map $\xi\colon U \to Y$ such that $s(B)$ is contracted to a smooth point $y \in Y$. In particular, restricting to each component B_i we obtain the situation of Setting 2.3. Let $\tilde{Y} \to Y$ be the blow-up of Y at y. We assume in addition that:

- ξ lifts to a regular immersive and injective map $\tilde{\xi}\colon U \to \tilde{Y}$, and
- each fibre $\mathbb{P}^1 \subset U$ is immersively mapped to Y via ξ.

Lemma 2.10. *In Setting 2.9 the projectivised tangent cone $\mathbb{P}\tau_y(\xi(U))$ is isomorphic to B via the map $\tilde{\xi} \circ s$. In particular, the projectivised tangent cone is reduced.*

Proof. The projectivised tangent cone $\mathbb{P}\tau_y(\xi(U))$ is equal (as a scheme) to the intersection of $\tilde{\xi}(U)$ and the exceptional divisor of the blow-up [Harr95, p. 254]. Since $\tilde{\xi}\colon U \to \tilde{Y}$ is a closed embedding, this intersection is isomorphic via $\tilde{\xi}$ to the preimage under $\tilde{\xi}$ of the exceptional divisor. The latter is in turn equal to the (scheme-theoretic) preimage $\xi^{-1}(y)$. Furthermore, $\xi^{-1}(y) = s(B)$ by Lemma 2.8 and clearly B and $s(B)$ are isomorphic via s, which concludes the proof of the lemma. □

For subvarieties $Z_1, Z_2 \subset \mathbb{P}^N$ recall that their *join* $Z_1 * Z_2$ is the closure of the locus of lines between points $z_1 \in Z_1$ and $z_2 \in Z_2$.[1] A special case of join that is relevant to this subsection is the projective cone: Suppose $Z \subset \mathbb{P}^{N-1}$ is a projective variety, then by $\operatorname{cone}(Z) \subset \mathbb{P}^N$ we denote the projective cone $Z * \{v\}$ over Z, where $v \in \mathbb{P}^N \setminus \mathbb{P}^{N-1}$.

[1] Defining the join here makes the same definition redundant in §3.2 of the orginal article.

Proposition 2.11. *In Setting 2.9 the normalisations of* $\operatorname{cone}(\mathbb{P}\tau_y(\xi(U)))$ *and of* $\xi(U)$ *are isomorphic in such a way that for each* $b \in B$ *the underlying birational map* $\operatorname{cone}(\mathbb{P}\tau_y(\xi(U))) \dashrightarrow \xi(U)$ *takes the line in* $\operatorname{cone}(\mathbb{P}\tau_y(\xi(U)))$ *joining the vertex* v *and* $\tilde{\xi} \circ s(b) \in \mathbb{P}\tau_y(\xi(U)$ *onto* $\xi(\pi^{-1}(b))$.

Proof. Since B is normal, it is a disjoint union of its irreducible components. Thus it is enough to argue for each component separately, and assume that B is irreducible. We use Lemma 2.10 to identify $\mathbb{P}\tau_y(\xi(U))$ and B. The embedding $\mathbb{P}\tau_y(\xi(U)) \subset \mathbb{P}(T_y Y)$ determines a natural very ample line bundle $\mathcal{O}_{\mathbb{P}\tau_y(\xi(U))}(1)$. We begin the proof by recognising this line bundle as a line bundle on B. Denote by $E \subset \tilde{Y}$ the exceptional divisor (preimage of y under the blow up map). Note that

$$\mathcal{O}_{\mathbb{P}\tau_y(\xi(U))}(-1) = E|_{\mathbb{P}\tau_y(\xi(U))} \text{ and } s^* \circ \tilde{\xi}^* E = s^*(s(B)) = \mathcal{M}$$

in the notation of Lemma 2.6. Thus $\mathcal{O}_{\mathbb{P}\tau_y(\xi(U))}(1) = \mathcal{M}^*$ via the isomorphism $\tilde{\xi} \circ s$ and in particular, $\mathcal{M}^*$ is very ample.

Therefore the normalisation of $\operatorname{cone}(\mathbb{P}\tau_y(\xi(U)))$ is equal to $\mathfrak{C}(B, \mathcal{M}^*)$, the normal cone on $(B, \mathcal{M}^*)$ in the language of [BS95, §1.1.8]. By construction,

$$\begin{aligned} \mathfrak{C}(B, \mathcal{M}^*) &= \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} H^0\left(\operatorname{Sym}^i\left(\mathcal{O}_B \oplus \mathcal{M}^*\right)\right)\right) \\ &= \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} H^0\left(\operatorname{Sym}^{ik}\left(\mathcal{O}_B \oplus \mathcal{M}^*\right)\right)\right) \end{aligned}$$

(the latter equality holds for any integer $k > 0$ by taking the Veronese subalgebra).

Let $\mathcal{L}$ be a very ample line bundle on Y and suppose that for a general fibre $\mathbb{P}^1 \subset U$ the image $\xi(\mathbb{P}^1)$ has degree $k > 0$ with respect to $\mathcal{L}$. Recall from Lemma 2.7 that $\xi^*(i \cdot \mathcal{L}) = ik \cdot s'(B)$ and from Lemma 2.4 that $\pi_*(ik \cdot s'(B)) = \operatorname{Sym}^{ik}(\mathcal{O}_B \otimes \mathcal{M}^*)$. Therefore we obtain a map of algebras:

$$\bigoplus_{i=0}^{\infty} H^0(Y, i \cdot \mathcal{L}) \xrightarrow{\xi^*} \bigoplus_{i=0}^{\infty} H^0(U, ik \cdot s'(B)) \overset{\pi_*}{\cong} \bigoplus_{i=0}^{\infty} H^0(B, \operatorname{Sym}^{ik}(\mathcal{O}_B \otimes \mathcal{M}^*)).$$

The map induces a morphism of projective spectra:

$$\zeta \colon \mathfrak{C}(B, \mathcal{M}^*) \to Y$$

that factorises ξ through $U \to \mathfrak{C}(B, \mathcal{M}^*) \xrightarrow{\zeta} Y$. This is because

$$U = \mathcal{P}roj_B\left(\bigoplus_{i=0}^{\infty} \operatorname{Sym}^{ik}(\mathcal{O}_B \otimes \mathcal{M}^*)\right),$$

that is, U is the relative $\mathcal{P}roj$ of a sheaf of algebras, while $\mathfrak{C}(B, \mathcal{M}^*)$ is the non-relative Proj of the sections of the same sheaf (moreover, $\mathcal{O}_B \otimes \mathcal{M}^*$ and its symmetric powers are base point free, since $\mathcal{M}^*$ is very ample). Furthermore, the map $U \to \mathfrak{C}(B, \mathcal{M}^*)$ contracts $s(B) = \mathbb{P}(\mathcal{O}_B) \subset \mathbb{P}(\mathcal{O}_B \oplus \mathcal{M}) = U$ to a single point by construction.

Therefore ζ is bijective onto $\xi(U)$, in particular, finite and birational, hence ζ must be the normalisation of $\xi(U)$. Thus the two varieties, $\xi(U)$ and $\operatorname{cone}(\mathbb{P}\tau_y(\xi(U)))$, indeed have isomorphic normalisations, and the desired correspondence between lines follows from the naturality of the construction. $\square$

Note that we do not know if the birational map $\operatorname{cone}(\mathbb{P}\tau_y(\xi(U))) \dashrightarrow \xi(U)$ or its inverse is regular. In [Kebe05, §6.1] Kebekus claims (without proof) that under additional assumptions the map is an isomorphism. Unfortunately, it is not clear how to prove the claim, and this gap invalidates the proof of the claim about cone in [Kebe05, Thm 1.1].

On Theorem 3.1

In Theorem 3.1(iii) of [Bucz10] we claim that each irreducible component of $\mathscr{C}_x$ is linearly non-degenerate in $\mathbb{P}(F_x)$, which is not sufficiently justified. That is, the last three sentences of the proof begining "Each irreducible component $\mathscr{C}_x$ is non-degenerate in $\mathbb{P}F_x$ by [Kebe01, Thm 4.4] (...)" and concluding at the end of the proof are incorrect.

Moreover, the same item also mentions that C_x is a projective cone over $\mathscr{C}_x$ and the claim is attributed to [Kebe05, Thm 1.1]. Jun-Muk Hwang pointed out that the proof in [Kebe05, §6.1] has a gap (see [Hwan19], or the concluding paragraph of §2.1).

The correct version of Theorem 3.1 and its proof should be:

Theorem 3.1. *With X as in Notation 2.1 let $x \in X$ be any point. Then:*

(i) There exist lines through x, in particular C_x and $\mathscr{C}_x$ are non-empty.

(ii) C_x is Legendrian in X and $\mathscr{C}_x \subset \mathbb{P}(F_x)$ and $\mathscr{C}_x$ is Legendrian in $\mathbb{P}(F_x)$.

(iii) If in addition x is a general point of X, then $\mathscr{C}_x$ is smooth. Further, all lines through x are smooth and two different lines intersecting at x do not intersect anywhere else, nor they do share a tangent direction. Every point in $\mathscr{C}_x$ is a tangent direction to a contact line through x. Moreover the normalisations of $\operatorname{cone}(\mathscr{C}_x)$ and C_x are isomorphic in such a way that the natural birational map $\operatorname{cone}(\mathscr{C}_x) \dashrightarrow C_x$ maps lines through the vertex of the cone to contact lines through x.

Proof. Part (i) is proved in [Kebe01, §2.3].

The proof of (ii) is essentially contained in [KPSW00, Prop. 2.9]. Explicit statements are in [Kebe01, Prop. 4.1] for C_x and in [Wiśn00, Lemma 5] for $\mathscr{C}_x$. Also [HM99] may claim the authorship of this observation, since the proof in the homogeneous case is no different than in the general case.

Assume $x \in X$ is a general point. The statements of (iii) are basically [Kebe05, Thm 1.1], which however assumes (in the statement) that $\mathcal{H}$ is irreducible. This is never used in the proof, with the exception of the argument for the irreducibility of C_x — see however Remark 3.2.

More explicitly, let B be the scheme parametrising contact lines through x. Since every curve through x is free [Kebe01, Lem. 3.5], thus B is smooth by [Koll96, Thms II.1.7, II.2.16]. (See also [BKK20, Prop. 7.3] for a related discussion). We are going to verify that we are in the situation of Setting 2.9 with $Y = X$, $y = x$, B as above, U the universal space of lines through x, $\xi \colon U \to X$ is the evaluation map, and $C_x = \xi(U)$. Indeed, B is smooth, hence normal and reduced, the section $s \colon B \to U$ assigns to $b \in B$ the point x in the corresponding $\mathbb{P}^1 \subset X$ (in particular, ξ contracts the section to x). Every line through x is smooth [Kebe01, Prop 3.3], hence $\xi^{-1}(x) = s(B)$ by Lemma 2.8. In particular, $\xi^{-1}(x)$ is a Cartier divisor, and by [Hart77, Prop. II.7.14] (the universal property of blowing up) ξ factorises through $\tilde{\xi} \colon U \to \tilde{X}$, where $\beta \colon \tilde{X} \to X$ is the blow-up of X at x. The only thing left to check from Setting 2.9 is that $\tilde{\xi}$ is injective and immersive.

To check the injectivity we take two distinct points $u_1, u_2 \in U$ and suppose $\tilde{\xi}(u_1) = \tilde{\xi}(u_2)$. In particular, $\xi(u_1) = \xi(u_2)$. Since ξ is injective upon restricting to any fibre, we must have $\pi(u_1) \neq \pi(u_2)$, and either both $u_1, u_2 \notin s(B)$ or both $u_1, u_2 \in s(B)$. By [Kebe05, Thm 5.1] the former case is impossible, and by [Kebe05, Thm 4.1] the latter case is impossible, concluding the proof of injectivity of $\tilde{\xi}$.

To check that $\tilde{\xi}$ is immersive, we note that $\pi \colon U \to B$ is flat, and since $\tilde{\xi}|_{\pi^{-1}(b)}$ is an embedding for any fibre $\pi^{-1}(b)$, thus $\tilde{\xi} \times \pi \colon U \to \tilde{X} \times B$ is an embedding. Therefore, by the universal property of Hilbert scheme, there is a natural map $B \to \operatorname{Hilb}(\tilde{X})$. This map is injective, since each b represents a different curve in $\tilde{X}$. The tangent space to $\operatorname{Hilb}(\tilde{X})$ at the image of b is $H^0\left(N(\mathbb{P}^1 \subset \tilde{X})\right)$, where $\mathbb{P}^1$ is $\tilde{\xi}(\pi^{-1}(b))$ and N denotes the normal bundle [Koll96, Thm I.2.8.1]. We will now calculate this tangent space. Let $E \subset \tilde{X}$ denote the exceptional divisor of the blow-up β. Then we have the following short exact sequence [Fult98, Lem. 15.4(iv)]:

$$0 \to T\tilde{X} \to \beta^* TX \to TE(E) \to 0.$$

Then we can restrict this sequence to $\mathbb{P}^1 = \tilde{\xi}(\pi^{-1}(b))$ and obtain:

$$0 \to T\tilde{X}|_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(2, 1^{n-1}, 0^{n+1}) \to T_{\tilde{\xi}(s(b))}E \to 0$$

where the middle term is $\mathcal{O}_{\mathbb{P}^1}(2, 1^{n-1}, 0^{n+1}) = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}^{\oplus(n+1)}$ by [Kebe01, Lem. 3.5] and $T_{\tilde{\xi}(s(b))}E$ is the skyscraper sheaf supported at $\xi(s(b))$ (the intersection point of $\mathbb{P}^1$ and E) with the fibre $T_{\tilde{\xi}(s(b))}E$. Since $\mathbb{P}^1$ is transversal to E and the $\mathcal{O}(2)$ corresponds to the tangent space to $\mathbb{P}^1$, it follows that $T\tilde{X}|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2, 0^{n-1}, (-1)^{n+1})$. Therefore, $N(\mathbb{P}^1 \subset \tilde{X}) = \mathcal{O}_{\mathbb{P}^1}(0^{n-1}, (-1)^{n+1})$ and

$$\dim H^0\left(N(\mathbb{P}^1 \subset \tilde{X})\right) = n - 1 = \dim B.$$

We conclude that the connected component of the Hilbert scheme containing the image of B is smooth and isomorphic to B and U is the restriction of the universal subscheme to this component. By the decomposition of the normal bundle its sections do not vanish anywhere. Thus $\tilde{\xi}$ is immersive by [Koll96, Prop. II.3.4], taking into account the correspondence between the Hom-scheme and the Chow variety [Koll96, Thm II.2.16] and the fact that the Hilbert to Chow is an isomorphism on this component [Koll96, Thms I.3.17, I.3.21].

Once we have estalished that this is the situation of Setting 2.9, the claim of (iii) follows from Lemma 2.10 and Proposition 2.11. □

On the proof of Theorem 3.6

The modification of Theorem 3.1 has an impact on the proof of Theorem 3.6 (the statement remains unchanged). At the end of the proof an additional case needs to be excluded. Thus the three paragraphs after Equation (3.7) should be replaced by the following four paragraphs.

Now we claim that $F_x \subset \tau_x D_x$. For this purpose we analyse three cases.

In the first case $C' = (C_x)^\bullet$ and the corresponding component of $\mathscr{C}_x$ is nondegenerate in $\mathbb{P}F_x$. Then $\mathbb{P}\tau_x C'$ is non-degenerate in $\mathbb{P}F_x$ by Theorem 3.1 and thus

$$(\mathbb{P}\tau_x C') * (\mathbb{P}\tau_x (C_x)^\bullet) = \sigma_2(\mathbb{P}\tau_x C') = \mathbb{P}(F_x)$$

by Proposition 3.5. Combining with (3.7) we obtain the claim.

In the second case C' and $(C_x)^\bullet$ are different components of C_x. Then by generality of x and by Theorem 3.1, the two tangent cones $(\mathbb{P}\tau_x C')$ and $(\mathbb{P}\tau_x (C_x)^\bullet)$ are disjoint. Thus again

$$(\mathbb{P}\tau_x C') * (\mathbb{P}\tau_x (C_x)^\bullet) = \mathbb{P}(F_x)$$

by Proposition 3.5. Combining with (3.7) we obtain the claim.

Finally, we exclude the possibility that $C' = (C_x)^\bullet$ and the corresponding component of $\mathscr{C}_x$ is contained in a hyperplane $H \subset \mathbb{P}F_x$. Then, since the component of $\mathscr{C}_x$ is smooth by Theorem 3.1(iii), it must be a linear space by [Bucz06, Thm 3.4] or [LM07, Prop. 17, item 1]. By Theorem 3.1(iii) the normalisation of $(C_x)^\bullet$ is $\iota\colon \mathbb{P}^n \to (C_x)^\bullet$ and since contact lines through x in $(C_x)^\bullet$ are pulled back to ordinary lines in $\mathbb{P}^n$, we must have $\iota^* L|_{(C_x)^\bullet} = \mathcal{O}_{\mathbb{P}^n}(1)$. Therefore images of all lines in $\mathbb{P}^n$ are also contact lines in X and since $C' = (C_x)^\bullet$, we would have $(D_x)^\bullet = (C_x)^\bullet$. However, there are other lines through x by [Kebe01, Thm 4.4] and $(C_x)^\bullet$ is strictly contained in another component of D_x, which is a contradiction, as it shows that $(D_x)^\bullet$ is not an irreducible component of D_x.

The concluding paragraph of the proof remains unchanged.

Quaternion-Kähler manifolds

The discussion of quaternion-Kähler manifolds in the introduction is not accurate. This concerns the paragraph between Conjectures 1.1 and 1.2, the statement of Conjecture 1.2, and the following paragraph. The assumption on positivity of metric of M and simple-connectedness of M is missing in this discussion. In particular, the claim that "if M is compact, then it has positive scalar curvature," is incorrect. For a correct and more detailed discussion we refer the reader to [BM19, Sect. 4].

On Theorem 6.3

In item (ii) of Theorem 6.3 the statement should be:

(ii) For general y in any of the irreducible components of C_x all lines through y are either D_x-integral or contained in the singular locus of D_x.

The second sentence of the proof should read:

To prove part (ii) let l be a line through y which is not contained in the singular locus of D_x.

References

[AD14] Carolina Araujo and Stéphane Druel. On codimension 1 del Pezzo foliations on varieties with mild singularities. *Math. Ann.*, 360(3-4):769–798, 2014.

[BKK20] Jarosław Buczyński, Grzegorz Kapustka, and Michał Kapustka. Special lines on contact manifolds. arXiv: 1405.7792; to appear in Annales de l'Institut Fourier, 2020.

[BM19] Jarosław Buczyński and Giovanni Moreno. Complex contact manifolds, varieties of minimal rational tangents, and exterior differential systems. In *Geometry of Lagrangian Grassmannians and nonlinear PDEs*, volume 117 of *Banach Center Publ.*, pages 145–176. Polish Acad. Sci. Inst. Math., Warsaw, 2019.

[BS95] Mauro C. Beltrametti and Andrew J. Sommese. *The adjunction theory of complex projective varieties*, volume 16 of *De Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1995.

[Bucz06] Jarosław Buczyński. Legendrian subvarieties of projective space. *Geom. Dedicata*, 118:87–103, 2006.

[Bucz10] Jarosław Buczyński. Duality and integrability on contact Fano manifolds. *Doc. Math.*, 15:821–841, 2010.

[Fult98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[Harr95] Joe Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. A first course, Corrected reprint of the 1992 original.

[Hart77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[HM99] Jun-Muk Hwang and Ngaiming Mok. Varieties of minimal rational tangents on uniruled projective manifolds. In *Several complex variables (Berkeley, CA, 1995–1996)*, volume 37 of *Math. Sci. Res. Inst. Publ.*, pages 351–389. Cambridge Univ. Press, Cambridge, 1999.

[Hwan19] Jun-Muk Hwang. Rigidity properties of holomorphic Legendrian singularities. *Épijournal Géom. Algébrique*, 3:Art. 18, 18, 2019.

[Kebe01] Stefan Kebekus. Lines on contact manifolds. *J. Reine Angew. Math.*, 539:167–177, 2001.

[Kebe05] Stefan Kebekus. Lines on complex contact manifolds. II. *Compos. Math.*, 141(1):227–252, 2005.

[Koll96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[KPSW00] Stefan Kebekus, Thomas Peternell, Andrew J. Sommese, and Jarosław A. Wiśniewski. Projective contact manifolds. *Invent. Math.*, 142(1):1–15, 2000.

[Laza04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[LM07] Joseph M. Landsberg and Laurent Manivel. Legendrian varieties. *Asian J. Math.*, 11(3):341–359, 2007.

[Wiśn89] Jarosław A. Wiśniewski. Length of extremal rays and generalized adjunction. *Math. Z.*, 200(3):409–427, 1989.

[Wiśn00] Jarosław A. Wiśniewski. Lines and conics on Fano contact manifolds. `http://www.mimuw.edu.pl/~jarekw`, 2000.