# Traces

## Simple observation

Let $\Omega \subset \mathbb{R}^n$, $H \subset \mathbb{R}^n \leftarrow$ an $(n-1)$-dim hyperplane.

Assume $u \in C^1(\overline{\Omega})$ and let $\Omega' = \Omega \cap L$

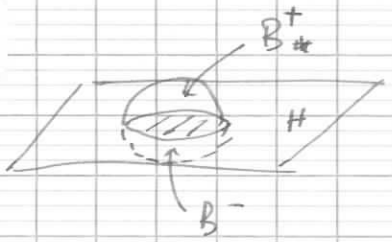
Take $x_0 \in \Omega'$, and $r > 0$ such that

$2B = B(x_0, 2r) \underset{\neq}{\subset} \Omega$. $\quad (B := B(x_0, r))$.

Let $\eta$ be a cut-off function: $\eta = 1$ on $B$

$\eta = 0$ off $2B$

$\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta \geq 0$.




Assume $H = \{x_n = 0\}$; $\mathbb{R}^n \ni (x', x_n)$.

$$\int_{\Omega' \cap B} |u|^p \, dx' \leq \int_H \eta |u|^p \, dx' = - \int_{(2B)^+_{\pm}} \frac{\partial}{\partial x_n} (\eta \cdot |u|^p) \, dx$$

$$= - \int_{(2B)^+} \left[ \frac{\partial \eta}{\partial x_n} |u|^p + p |u|^{p-1} \operatorname{sgn} u \, \frac{\partial u}{\partial x_n} \cdot \eta \right] dx$$

Young's ineq.

$$\leq \frac{C}{r} \int_{(2B)^+} |u|^p + C(p) \int_{(2B)^+} \left( |u|^p + \left| \frac{\partial u}{\partial x_n} \right|^p \right) dx$$

$$\leq C(r, p) \, \|u\|^p_{W^{1,p}((2B)^+)} .$$

By density argument $W^{1,p}_{loc}(\Omega) \hookrightarrow L^p_{loc}(\Omega)$.

~~and~~ ~~$t_H$~~

## Sobolev trace theorem

Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the cone condition; $H$ - a $k$-dimensional hyperplane in $\mathbb{R}^n$.
Assume $H \cap \Omega =: \Omega_k$ is non-empty.
Suppose $u \in W^{1,p}(\Omega)$. What can we say of $u|_{\Omega_k}$?

~~Theorem A: If $p>n$, then~~

Clarification: What do we mean by $u|_{\Omega_k}$?
This makes sense for smooth (or at least $C^1$) functions: $T: \underset{W^{1,p}(\Omega)}{\underset{\cap}{u}} \longrightarrow u|_{\Omega_k} \in ?$

If we identify ? as some $L^q$ and prove that $T$ is continuous as $\left(W^{1,p} \cap C^\infty(\overline{\Omega}) , \|\cdot\|_{W^{1,p}}\right) \to \left(L^p \cap C^\infty(\overline{\Omega}), \|\cdot\|_{L^p}\right)$

then it extends to $T: W^{1,p}(\Omega) \to L^p(\Omega_k)$ by density.
$T$ is called the trace operator; I shall abuse the notation and write $u|_{\Omega_k}$ instead of $Tu$ also for non-smooth $u$.

Theorem: A. If $p > n$, then $W^{1,p}(\Omega) \overset{T}{\hookrightarrow} L^q(\Omega_k)$ for all $q \in [p, \infty]$

B. If $p = n$, ——— " ——— for all $q \in [p, \infty)$

C. If $\overset{n-k}{\cancel{k}} < p < n$, ——— " ——— for all $q \in [p, p^*]$ where $p^* = \frac{kp}{n-p}$.

we shall prove only $[p, p^*)$

Remark: If $\mu_k(\Omega_k) < \infty$, then all the ranges of $q$ above can be extended from $[p, ?]$ to $[1, ?]$.

We shall prove the theorem in the case $\Omega$ bounded (unbounded $\Omega$ – not much more challenging, but proof is far more technical).
Of course this implies $\mu_k(\Omega_k) < \infty$.

<u>Proof</u>: As mentioned, we need to prove embeddings for $u \in C^\infty \cap W^{1,p}(\Omega)$.

A. $\Omega$ has the cone property; ~~for~~ thus
$\Omega = \Omega^1 \cup \Omega^2 \cup \dots \cup \Omega^m$, all $\Omega^j$ star-shaped w.r. to a ball $B^j$
$\Omega_k = \Omega^1_k \cup \Omega^2_k \cup \dots \cup \Omega^m_k \qquad \Omega^j_k = \Omega_k \cap \Omega^j$

For $x \in \Omega^j_k$ we have the representation formula:

$$|u(x) - u_{B^j}| \le C \int_{\Omega^j} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy \overset{\text{Hölder}}{\le} C \Big(\int_{\Omega^j} |\nabla u|^p\Big)^{1/p} \cdot \Big(\int_{\Omega^j} \frac{dy}{|x-y|^{(n-1)\frac{p}{p-1}}}\Big)^{\frac{p-1}{p}}$$

$$\le C \|\nabla u\|_{L^p(\Omega^j)} |\Omega^j|^{\frac{1}{n} - \frac{1}{p}}$$

~~which proves that~~

$$|u(x)| \le \frac{1}{|B^j|} \|u\|_{L^p(B^j)} + C(n,p,\Omega^j) \|\nabla u\|_{L^p(\Omega^j)}$$

$$\le C(\Omega, n, p) \cdot \|u\|_{W^{1,p}(\Omega)}$$

thus $u$ is bounded in $\Omega_k$ ~~with the bound~~
and the mapping $T: W^{1,p}(\Omega) \cap C^\infty(\bar\Omega) \to L^\infty \cap C^\infty(\Omega_k)$ is bounded; ~~in particular~~
Since $\Omega$ (and thus $\Omega_k$) is bounded, this is enough to prove A (for unbounded $\Omega_k$ we still need to prove an estimate $\|u\|_{L^p(\Omega_k)} \le C \cdot \|u\|_{W^{1,p}(\Omega)}$ – Exercise: do it for half-space).

Case C. We need additional lemmata and notions

Let $H \subset \mathbb{R}^n$ be a $k$-dimensional hyperplane in $\mathbb{R}^n$, $n>1$

$p>1$

Define $\displaystyle J_r f(x) = \int_{B_r(x)} \frac{f(y)\,dy}{|x-y|^{n-1}}$ $\qquad J_\infty = I_1$

<u>Lemma 1</u> If $f \in L^p(\mathbb{R}^n)$, $\overline{p>n-k}$, then $J_r|f|\cancel{\in}\big|_H \in L^p(H)$, ~~p > n-k~~

$$\|J_r|f|\|^p_{L^p(H)} \le C\, r^{p-(n-k)} \|f\|^p_{L^p(\mathbb{R}^n)}$$

<u>Proof</u>:

$$J_r|f|(x) = \int_{B_r(x)} \frac{|f(y)|}{|x-y|^{n-1}}\,dy = \int_{B_r(x)} \frac{|f(y)|}{|x-y|^s} \cdot \frac{dy}{|x-y|^{n-s-1}}$$

$$\underset{\text{Hölder}}{\le} \left(\int_{B_r(x)} \frac{|f(y)|^p}{|x-y|^{sp}}\,dy\right)^{1/p} \underbrace{\left(\int_{B_r(x)} \frac{dy}{|x-y|^{(n-s-1)p/p-1}}\right)^{\frac{p-1}{p}}}_{\text{this we can calculate in radial coordinates, as long as } s > \frac{n}{p}-1 \text{ (Cond 1 ons)}}$$

$$= \left(\int_{B_r(x)} \frac{|f(y)|^p}{|x-y|^{sp}}\right)^{1/p} \cdot C\, r^{s+1-\frac{n}{p}}$$

Thus $\displaystyle \int_H (J_r|f|(x))^p\,dx' = C r^{(s+1-\frac{n}{p})p} \int_H \int_{B_r(x)} \frac{|f(y)|^p}{|x-y|^{sp}}\,dy\,dx'$

$x = (x', 0)$
$y = (y', y_n)$

$$= C r^{(s+1-\frac{n}{p})p} \int_{\cancel{B_r(x)}\,\mathbb{R}^n} |f(y)|^p \left(\int_H \frac{dx' \cdot \chi_{\{|x-y|\le r\}}}{|(x',0)-(y',y_n)|^{sp}}\right) dy$$

$$\le C r^{(s+1-\frac{n}{p})p} \int_{\cancel{B_r(x)}\,\mathbb{R}^n} |f(y)|^p \underbrace{\left(\int_{H \cap B_r(y',0)} \frac{dx'}{|(x',0)-(y',0)|^{sp}}\right)}_{\substack{\text{radial coordinates in } k \text{ dimensions} \\ \| \\ C \cdot r^{k-sp} \quad \text{if } sp<k \leftarrow \text{(Cond 2 ons)}}} dy =$$

$$= C r^{sp+p-n+k-sp} \cdot \|f\|^p_{L^p(\mathbb{R}^n)} = C \cdot r^{k+p-n} \|f\|^p_{L^p(\mathbb{R}^n)}$$

Check that $\exists s$ satisfying Cond 1 & Cond 2 iff ~~p~~ $p > n-k$.

~~Lemma 2~~

We say that $f \in M^p(\mathbb{R}^n)$ $L^p_w(\mathbb{R}^n)$ (Marcinkiewicz weak-$L^p$ space)

if $\exists_c$ for any $t>0$

$$|\{x \in \mathbb{R}^n : |f(x)| > t\}| < \frac{C}{t^p}$$ ; the best constant is the $L^p_w$-"norm" ($\Delta$ ineq. does not hold).

~~Lemma 2:~~

$$L^p(\mathbb{R}^n) \subsetneq M^p(\mathbb{R}^n)$$

$$f \in L^1(\mathbb{R}^n) \Rightarrow Mf \in M^1(\mathbb{R}^n)$$

Exercise: $f \in M^p(\mathbb{R}^n) \Rightarrow$ for any $\Omega$ with $|\Omega|<\infty$, $q<p$

$$f \in L^q(\Omega)$$

$$\|f\|_{L^q(\Omega)} \le C \cdot \|f\|_{M^p(\Omega)}$$

Lemma 2: If $f \in L^p(\mathbb{R}^n)$, then $I_1|f|\Big|_H \in M^{\frac{kp}{n-p}}(H)$;

~~Prop. and and $2\|I_1|f|\|_{L^{\frac{kp}{n-p}}(H)} \le C$~~

$$\mu_k\{x \in H : I_1|f|(x) > t\} \le C \cdot \left(\frac{\|f\|_{L^p(\mathbb{R}^n)}}{t}\right)^{\frac{kp}{n-p}}.$$

Proof: Denote $p^* = \frac{kp}{n-p}$.

Recall from the proof of H-L-S theorem that, for $r>0$,

$$\int_{\mathbb{R}^n \setminus B_r(x)} \frac{|f(y)|}{|x-y|^{n-1}}\,dy \le C_1 r^{1-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}$$ } Hölder with exp. $p$, $\frac{p}{p-1}$; we are away from singularity.

Take any $t>0$ and choose $r$ such that

$\frac{t}{2} = C_1 r^{1-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}$. Then $J_r(|f|)(x) = I_1(|f|)(x) - \int_{\mathbb{R}^n \setminus B_r(x)} \frac{|f(y)|}{|x-y|^{n-1}}\,dy$;

$$I_1|f|(x) > t \Rightarrow J_r|f|(x) > \frac{t}{2}$$

$$\mu_k\{x \in H : I_1|f|(x) > t\} \le \mu_k\{x \in H : J_r|f|(x) > t/2\} = (*)$$

By Lemma 1 and Markov's inequality

$$(\ast) \leq \frac{\| J_r |f| \|^p_{L^p(H)}}{(t/2)^p} \leq \frac{r^{p-(n-k)} \|f\|^p_{L^p(\mathbb{R}^n)}}{(t/2)^p}$$

$$= C \cdot \frac{r^{p-(n-k)} \|f\|^p_{L^p(\mathbb{R}^n)}}{r^{p-n} \|f\|^p_{L^p(\mathbb{R}^n)}} = C\, r^k$$

$$\left.\begin{array}{l} \cancel{\frac{t}{2} = C \cdot r^{1-\frac{n}{p}}} \\ \frac{t}{2} = C \cdot r^{1-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)} \end{array}\right\} = C \cdot \left( \frac{t}{\|f\|_{L^p(\mathbb{R}^n)}} \right)^{p^*}$$

□.

Proof of case C.

As before, $\Omega = \bigcup_{j=1}^{m} \Omega^j$; for any $x \in \Omega^j$ we have

$$\cancel{|u(x)| \leq C}\ |u(x) - u_{B_j}| \leq I_1(|\nabla u| \cdot \chi_{\Omega^j})(x)$$

$$\Rightarrow |u(x)| \leq |u_{B_j}| + I_1(|\nabla u| \cdot \chi_{\Omega^j})(x)$$

~~since $u \in C^\infty(\bar{\Omega})$, we~~

~~Fix $q \in (p,n)$. Since $u \in C^\infty$, we know that $|\nabla u| \cdot \chi_{\Omega^j} \in L^q(\mathbb{R}^n)$. By Lemma 2, with $q$ in place of $p$, $I_1(|\nabla u| \cdot \chi_{\Omega^j})|_H \in M^{q^*}(H)$, with $q^* = \frac{kq}{n-q}$ which (by Exercise) implies I~~

By Lemma 2, $I_1(|\nabla u| \cdot \chi_{\Omega^j})|_H \in M^{p^*}(H)$, thus

$$u|_{\Omega^j_k} \in M^{p^*}(\Omega^j_k); \quad \|u\|_{M^{p^*}(\Omega^j_k)} \leq C \cdot \|u\|_{W^{1,p}(\Omega^j)}$$

$$\Rightarrow \|u\|_{M^{p^*}(\Omega_k)} \leq C \|u\|_{W^{1,p}(\Omega)}.$$

Exercise gives $u|_{\Omega_k} \in L^q(\Omega_k)$ for any $q < p^*$

In fact, $u|_{\Omega_k} \in L^{p^*}(\Omega_k)$, but this is yet more involved.

Case B ~~Take any s<p; ...~~

$p = n$. Take any $s < n$; if $u \in W^{1,n}(\Omega)$, then $u \in W^{1,s}(\Omega)$ ($\Omega$ bounded)

$$\Downarrow$$

$$u|_{\Omega_k} \in \cancel{L^{\frac{ns}{n-s}}}\ L^{s^*}(\Omega_k)$$

$s$ arbitrarily close to $n$

$\Rightarrow s^* \xrightarrow{\quad n \quad} \infty$

$\Rightarrow u|_{\Omega_k} \in L^q(\Omega_k)$ for any $q < \infty$.

## 4. Traces & fractional Sobolev spaces

~~For simplicity, assum~~

Theorem 1. For simplicity, assume $\Omega = [-1,1]^n = Q$

$H = \{(x', x_n) \in \mathbb{R}^n : x_n = 0\}$, $p > 1$

$Q' = H \cap Q$. Then $T : W^{1,p}(Q) \longrightarrow W^{1-\frac{1}{p},p}(Q')$.

Sketch of the proof:

By our previous results, if $u \in W^{1,p}(Q)$, then $u \in L^q(Q')$ for any $q < p^*$ (or for any $q$ if $p \geq n$).

$p^* = \frac{(n-1)p}{n-p} > p$, thus in particular $u \in L^p(Q')$.

Recall that $f \in W^{1-\frac{1}{p},p}$ iff $f \in L^p$ and

$$\iint_{Q'Q'} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p-2}}\,dx\,dy < \infty .$$

$$\int_{Q'}\int_{Q'} \frac{|u(x)-u(y)|^p}{|x-y|^{n+p-2}}\,dx\,dy \leq \cancel{\sum_{i=1}\iint u(x_1,x_2,\dots,x_i,y_{i+1},\dots,y_n)}\cdot$$

$$\leq c\sum_{i=1}^{n-1}\int_{-1}^{1}\!\!\dots\!\int_{-1}^{1} \frac{|u(x_1,\dots,x_i,y_{i+1},\dots,y_{n-1}) - u(x_1,\dots,x_{i-1},y_i,y_{i+1},\dots,y_n)|^p}{|x-y|^{n+p-2}}$$

$$= c\sum_{i=1}^{n-1}\int_{-1}^{1}\dots\int_{-1}^{1} dx_1\dots dx_i\,dy_i\,dy_{i+1}\dots dy_{n-1}\,|u(\dots,x_i,\dots) - u(\dots,y_i,\dots)|^p \cdot$$

$$\cdot\underbrace{\left(\int_{-1}^{1}\dots\int_{-1}^{1}\frac{dx_{i+1}\cdot\ldots\cdot dx_{n-1}\cdot dy_1\cdot\ldots\cdot dy_{i-1}}{|x-y|^{n+p-2}}\right)}_{=\mathcal{F}_i(x_1,\dots,x_i,y_i,\dots,y_{n-1})} = (*)$$

Exercise: prove that $\exists_C\ \mathcal{F}_i \leq C\frac{1}{|x_i-y_i|^p}$ for all $i = 1,\dots,n$

$$(*) \lesssim \sum_{i=1}^{n-1} \int_{-1}^{1} \cdots \int_{-1}^{1} \underbrace{\left| \frac{u(\cdots, x_i, \cdots) - u(\cdots, y_i, \cdots)}{x_i - y_i} \right|^p}_{\text{difference quotient!}} d \cdots$$

$$\lesssim \|u\|^p_{W^{1,p}(Q)}.$$

D.

<u>Theorem 2''</u> The image of the trace operator is exactly $W^{1-1/p,p}(Q')$. More precisely, for any $u \in W^{1-1/p,p}(Q')$, there exists $v \in W^{1,p}(Q)$ such that $v|_{Q'} = u$ (in the sense of traces).

<u>Proof</u>:

Let $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ be a standard mollifier:

$\varphi \in C_0^\infty(B(0,1))$, $\varphi \geq 0$, $\int_{\mathbb{R}^{n-1}} \varphi = 1$.

$\varphi_\varepsilon(x') = \varepsilon^{1-n} \varphi(x'/\varepsilon)$.

Set $v(x', x_n) = \varphi_{x_n} * u(x')$ for $x_n > 0$ and extend $v$ on $Q \cap \{x_n < 0\}$ as an odd function.

- $v$ is smooth in $Q \cap \{x_n > 0\} =: Q^+$
- $\lim_{x_n \to 0^+} \|v(x', x_n) - u(x')\|_{L^p(Q')} = 0$ (DCT)

- $\|v\|^p_{L^p(Q^+)} = \int_0^1 \|\varphi_{x_n} * u(x')\|^p_{L^p(Q')} dx_n$

Young's conv. ineq. $\leq \int_0^1 \|\varphi_{x_n}\|_{L^1(Q')} \cdot \|u\|^p_{L^p(Q')} dx_n = \|u\|^p_{L^p(Q')}$

(and this implies $\|v\|^p_{L^p(Q)} \leq 2\|u\|^p_{W^{1-\frac{1}{p},p}(Q')}$) $\lesssim \|u\|^p_{W^{1-\frac{1}{p},p}(Q')}$

• for $i = 1, \dots, n-1$

$$D^i v = D^i \varphi_{x_n} * u$$

we can add this; $\int D^i\varphi = 0$.

$$D^i v(x', x_n) = -\int_{|x'-y'|<x_n} x_n^{-n} D^i\varphi\left(\frac{y'-x'}{x_n}\right)\big(u(x')-u(y')\big)\,dy'$$

$$= -\int_{|z'|<1} D^i\varphi(z')\,\frac{u(x')-u(x'+x_n z')}{x_n}\,dz'$$

$$\|D^i v\|^p_{L^p(Q^+)} \le C \int_{Q^+}\int_{|z'|<1} \frac{|u(x')-u(x'+x_n z')|^p}{x_n^p}\,dz'$$

$$= C\int_{Q^+}\int_{|x'-y'|<x_n} \frac{|u(x')-u(y')|^p}{|x'-y'|^{n+p-2}}\,\frac{|x'-y'|^{n+p-2}}{x_n^{n+p-1}}\,dy'$$

$$\le c\int_{Q^+}\int_{|x'-y'|<x_n} \frac{|u(x')-u(y')|^p}{|x'-y'|^{n+p-2}}\,\frac{|x'-y'|}{x_n^2}\,dy'$$

$$= c\int_0^1\left(\int_{Q'}\left(\int_{Q'} \frac{|u(x')-u(y')|^p}{|x'-y'|^{n+p-2}}\,\chi_{\{1\ge x_n>|x'-y'|\}}\cdot\frac{|x'-y'|}{x_n^2}\,dy'\right)dx'\right)dx_n$$

$$= \int_{Q'}\int_{Q'} \frac{|u(x')-u(y')|^p}{|x'-y'|^{n+p-2}}\underbrace{\left\{|x'-y'|\int_{|x'-y'|}^1 \frac{dx_n}{x_n^2}\right\}}_{|x'-y'|\cdot\left(-\frac{1}{x_n}\right)\Big|_{|x'-y'|}^{1} = 1-|x'-y'|}\cdot\chi_{\{|x'-y'|<1\}}\,dx'\,dy'$$

$$\le \int_{Q'}\int_{Q'} \frac{|u(x')-u(y')|^p}{|x'-y'|^{n+p-2}}\,dx'\,dy' \le \|u\|_{W^{1-\frac1p,p}(Q')}.$$

And the same holds on $Q^-$, thus

$$\|v\|_{W^{1,p}(Q)} \le C\cdot\|u\|_{W^{1-\frac1p,p}(Q')}.$$

~~The inverse trace~~ Wait! How about $D^n v$?

$$\frac{\partial}{\partial x_n} v(x', x_n) = \int_{|x'-y'|<x_n} \frac{\partial}{\partial x_n}\left(x_n^{1-n}\varphi\left(\frac{x'-y'}{x_n}\right)\right) u(y')\,dy'$$

$$= \int_{|x'-y'|<x_n} \left[(1-n)x_n^{-n}\varphi\left(\frac{x'-y'}{x_n}\right) + \left\langle \nabla\varphi\left(\frac{x'-y'}{x_n}\right), \frac{x'-y'}{x_n}\right\rangle x_n^{-n}\right](u(y') - u(x'))\,dy'$$

$$= \int_{|z'|<1} \underbrace{\left[(n-1)\varphi(z') + \langle\nabla\varphi(z'), z'\rangle\right]}_{\text{bounded by some } C^{1/p}} \frac{u(x'-x_n z') - u(x')}{x_n}\,dz'$$

$z' = \frac{x'-y'}{x_n}$

$dz' = -x_n^{1-n}\,dy'$

$$\left\|\frac{\partial}{\partial x_n} v(x', x_n)\right\|^p_{L^p(Q^+)} \leq C \int_{Q^+}\int_{|z'|<1} \frac{|u(x'-x_n z') - u(x')|^p}{x_n^p}\,dz'$$

$$= C \int_{Q^+}\int_{|x'-y'|<x_n} \frac{|u(y') - u(x')|^p}{|y'-x'|^{n+p-2}} \cdot \frac{|y'-x'|^{n+p-2}}{x_n^{n+p-1}}\,dy'$$

$< \frac{|y'-x'|}{x_n^2}$

And the rest of the calculations exactly as for the other derivatives.

The remark I made ~~for~~ during the last exercise course was unnecessary – or, in fact, needed only to obtain a function defined ~~for~~ on $Q' \times [0,\infty)$ and in $W^{1,p}$ instead on $Q^+ = Q' \times [0,1]$.