# Fractional order Sobolev spaces

## Attempt 1

Gagliardo - Slobodeckij spaces

For any $s \in (0,1)$, $p \geq 1$ we set

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\underset{\substack{\Downarrow \\ x}}{\Omega} \times \underset{\substack{\Downarrow \\ y}}{\Omega}) \right\}$$

In other words, $W^{s,p}(\Omega)$ consists of these ~~f~~ $u \in L^p(\Omega)$ for which

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} \, dx\, dy < \infty .$$

$$\overset{..}{\|}$$

$[u]^p_{W^{s,p}(\Omega)}$ (p-th power of Gagliardo's semi-norm)

This is a Banach space, when equipped with the norm

$$\|u\|_{W^{s,p}} = \|u\|_p + [u]_{W^{s,p}} .$$

Problem: How does this fit in the ~~St~~ standard Sobolev spaces scheme? What happens when $s \to 1$?

Theorem (Brezis, 2002?) (probably known earlier in ~~sm~~ less general versions)

Assume $\Omega \subset \mathbb{R}^n$ is open and connected, $f: \Omega \to \mathbb{R}$ measurable and

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+p}} < \infty .$$

Then $f$ is constant.

Exercise 1: Prove the above under additional assumption that $f$ is smooth

Exercise 2: As in Ex. 1, but assume only that $f \in L^p(\Omega)$. (Hint: Beyond what is covered by our course, you might need Rademacher's theorem: A Lipschitz function is a.e. differentiable).

~~Theorem: For any $f \in W^{1,p}(B)$ and $s \in (0,1)$~~

~~$$(1-s)[f]^p_{W^{s,p}(B)} \le C(n,p) \|\nabla f\|_p^p$$~~

~~We take $\Omega = B$ for simplicity, similar result holds for any bounded $\Omega$, with $C = C(n,p,\Omega)$~~

For the next theorem we need the extension property, that I delegate to the exercise course:

<u>Theorem</u>: Let $\Omega$ be an open subset of $\mathbb{R}^n$ such that $\partial\Omega$ is piecewise a graph of a Lipschitz function and such that $\partial\Omega$ is bounded. ~~The~~ (in other words, $\Omega$ is a $^{0,1}$ Lipschitz domain with bounded boundary). Then, for any $u \in W^{1,p}(\Omega)$ there exists $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ such that $\tilde{u} = u$ on $\Omega$, $\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq \underset{C(n,p,\Omega)}{\overset{\shortparallel}{C}} \|u\|_{W^{1,p}(\mathbb{R}^n)}$.

We shall use it here only for $\Omega = B$, to simplify calculations, but the same result as below holds for any $\Omega$ with the above extension property (with constants dependent on $\Omega$ !).

<u>Theorem</u> For any $f \in W^{1,p}(B)$ and $s \in (0,1)$

$$(1-s)[f]^p_{W^{s,p}(B)} \leq C(n,p) \|\nabla f\|_p^p$$

<u>Corollary</u>: $W^{1,p}(B) \subset W^{s,p}(B)$ for all $s \in (0,1)$

<u>Proof</u>:

$$[f]_{W^{1,s}(B)} = \int_B\int_B \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \qquad t = x-y \in B-B = 2B$$

$$\le \int_{2B} dt \int_B \frac{|f(x+t)-f(x)|^p}{|t|^{n+sp}}\,dx$$

$$\le \int_{2B} dt \int_{\mathbb{R}^n} \frac{|\tilde f(x+t)-\tilde f(x)|^p}{|t|^{n+sp}}\,dx = \int_{2B} \|\tilde f(\cdot+t)-\tilde f(\cdot)\|_p^p \frac{dt}{|t|^{n+sp}}$$

$$\le \int_{2B} \frac{\|\nabla \tilde f\|^p_{L^p(\mathbb{R}^n)}\,dt}{|t|^{n+(s-1)p}} \le C\int_{2B} \frac{\|f\|^p_{W^{1,p}(B)}\,dt}{|t|^{n+(s-1)p}} =$$

Recall: for $\tilde f \in W^{1,p}(\mathbb{R}^n)$: $\|\tilde f(\cdot+h)-\tilde f(\cdot)\|_p \le \|\nabla \tilde f\|_p \cdot |h|$

$$= \|f\|^p_{W^{1,p}(B)} \cdot C \int_0^2 \frac{n\omega_n r^{n-1}}{r^{n+(s-1)p}}\,dr = C\,\frac{n\omega_n 2^p}{p}\,\frac{1}{1-s}\,\|f\|_{W^{1,p}(B)}$$

$C = C(n,p)$ from the extension theorem.

$\square$.

In fact,

(1) for any open, connected $\Omega \subset \mathbb{R}^n$, $u \in W^{1,p}(\Omega)$,

$$\lim_{s\to 1^-} (1-s)[u]^p_{W^{s,p}(\Omega)} = C_1 \|\nabla u\|_p^p$$

Bourgain, Brezis, Mironescu, 2001

(2) $$\lim_{s\to 0+} s\,[u]^p_{W^{s,p}(\mathbb{R}^n)} = C_2 \|u\|^p_{L^p(\mathbb{R}^n)}$$

Maz'ya, Shaposhnikova, 2002

for $u \in \bigcap_{s\in(0,1)} W^{s,p}(\mathbb{R}^n)$

Thus, if we define an equivalent norm on $W^{s,p}(\mathbb{R}^n)$:

$$|||u|||_{W^{s,p}} = \left( \|u\|_p^p + s(1-s)\,[u]_{W^{s,p}}^p \right)^{1/p},$$

we get $\lim_{s \to 1^-} |||u|||_{W^{s,p}} \approx \|u\|_{W^{1,p}}$

$\lim_{s \to 0^+} |||u|||_{W^{s,p}} \approx \|u\|_{L^p}$.

For non-integer $s > 1$, we define

$$W^{s,p}(\Omega) = \left\{ u \in W^{[s],p}(\Omega) : D^\alpha u \in W^{s-[s],p}(\Omega) \text{ for all } \alpha \text{ such that } |\alpha| = [s] \right\}$$

with a norm

$$\|u\|_{W^{s,p}} = \left( \|u\|^p_{W^{[s],p}} + \sum_{|\alpha| = [s]} [D^\alpha u]^p_{W^{s-[s],p}(\Omega)} \right)^{1/p}$$

(this way, for $s \in \mathbb{N}$, $W^{s,p}$ coincides with standard Sobolev space $W^{s,p}$, because the second summand is non-existent by assumption.

The following result is also an expected one:

Theorem: For any $s, s'$ s.t. $0 < s \leq s' < 1$ and any measurable $u : \Omega \to \mathbb{R}^n$ (Ω open)

$$\|u\|_{W^{s,p}(\Omega)} \leq C(n,s,p) \|u\|_{W^{s',p}(\Omega)}.$$

In particular, $W^{s',p}(\Omega) \subset W^{s,p}(\Omega)$

<u>Proof:</u>

$$[u]^p_{W^{s,p}} = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy =$$

$$= \int_\Omega \int_{\underbrace{\Omega\cap\{|x-y|<1\}}_{\Omega_1(y)}} (\quad) + \int_\Omega \int_{\underbrace{\Omega\cap\{|x-y|\geq 1\}}_{\Omega_2(y)}} (\quad) = I_1 + I_2$$

$$I_1 = \int_\Omega \int_{\Omega_1(y)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \leq \int_\Omega \int_{\Omega_1(y)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+s'p}}\,dx\,dy$$

$$\leq \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+s'p}}\,dx\,dy = [u]^p_{W^{s',p}}$$

$$I_2 = \int_\Omega \int_{\Omega_2(y)} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \leq \int_\Omega \int_{\Omega_2(y)} 2^{p-1} \frac{|u(x)|^p + |u(y)|^p}{|x-y|^{n+sp}} =$$

both the integrand and the set of integration are symmetric w.r.t. $x \leftrightarrow y$

$$= 2^p \int_\Omega \int_{\Omega_2(y)} \frac{|u(x)|^p}{|x-y|^{n+sp}} \lesssim 2^p \int_\Omega |u(x)|^p \cdot \int_{\{|z|\geq 1\}} \frac{dz}{|z|^{n+sp}}$$

$$= 2^p n\omega_n \cdot \frac{1}{n+sp-1}\,\|u\|_p^p = C(n,s,p)\,\|u\|_p^p$$

Combining the estimates,

$$\|u\|_{W^{s,p}} \approx \left(\|u\|_p^p + [u]^p_{W^{s,p}}\right)^{1/p} \lesssim \left(\|u\|_p^p + [u]^p_{W^{s',p}}\right)^{1/p} \approx \|u\|_{W^{s,p}}$$

$\square$.

# Sobolev embedding theorem

(1) If $sp < n$, $p \geq 1$, $s \in (0,1)$, then

$$\|f\|^p_{L^{\frac{np}{n-sp}}(\mathbb{R}^n)} \leq C(n,p,s)\,[f]_{W^{s,p}(\mathbb{R}^n)};$$

in particular $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$

for any $q \in [p, \frac{np}{n-sp}]$

($W^{s,p} \subset L^p$ by definition, for $L^q$, $q \in [p, \frac{np}{n-sp}]$, use Hölder)

The same holds for any $\Omega \subset \mathbb{R}^n$ for which the extension theorem holds ($\Omega$ an extension domain).

(2) If $sp = n$, $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$

for any $q \geq p$.

~~(3)~~ (2′) Also, $W^{s,p}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ and $VMO(\mathbb{R}^n)$

I haven't found a simple proof of (2′);
a possible reference: J. van Schaftingen, J. Func. Anal. 2006

Proofs of (1) and (2) usually go through much more sofisticated embeddings of Besov spaces, ~~that~~ of which fractional ~~and~~. S. spaces are particular cases. However, „Hitchhikers guide..." contains a rather simple, if tricky, proof based on estimates of level sets of $u$.

Fourier transform approach (Bessel potential spaces)

Recall: Schwartz space $\mathcal{S}(\mathbb{R}^n)$

$$\mathcal{S}(\mathbb{R}^n) = \left\{ u \in C^\infty(\mathbb{R}^n) : \sup_{\mathbb{R}^n} x^\alpha D^\beta u < \infty \text{ for all multiindices } \alpha, \beta \right\}.$$

Since $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R})$, we have $\mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$

$\left\{ \right.$ the inclusion is a cont. embedding, when $\mathcal{S}$ carries its usual Fréchet space topology defined by $\{[u]_{\alpha,\beta} = \sup_{\mathbb{R}^n} x^\alpha D^\beta u\}_{\alpha,\beta}$.

The distributions in $\mathcal{S}'(\mathbb{R}^n)$ are called tempered distributions, their advantage is that Fourier transform

$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, thus we can define
$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$; $\langle \mathcal{F}T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle$

Or we can prove, just like with $\mathcal{D}$ and $\mathcal{D}'$, that $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$ $\Rightarrow$ $\mathcal{F}$ extends.

All standard formulae for behavior of $\mathcal{F}$ versus translation, convolution or differentiation carry over to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Define $\Lambda^s : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by the formula

$$\Lambda^s u = \mathcal{F}^{-1}\left((1+|\xi|^2)^{s/2} \mathcal{F}u\right)$$

Obviously $\Lambda^{-s}\Lambda^s = id$.

Define

$$H^{s,p}(\mathbb{R}^n) = \Lambda^{-s} L^p(\mathbb{R}^n) = \{ u \in \mathcal{S}'(\mathbb{R}^n) : \Lambda^s u \in L^p(\mathbb{R}^n) \}$$

These spaces are particularly useful for $p=2$ (Fourier transform on $L^p$ spaces, for $p \neq 2$, is difficult, the image of $L^p$ in $\mathcal{F}$ is not well described, there are $L^p$ functions $u$, $p>2$, such that $\mathcal{F}u$ is not a regular distribution. (see Stein, Weiss, Intro to Fourier Analysis on Eucl. Spaces).

The Sobolev embedding holds in the same way as for $W^{s,p}$; note that $H^{s,p}$ are well defined also for $s \geq 1$.

From now on we restrict our attention to the case $p=2$. Then $H^{s,2} =: H^s$ is a Hilbert space,

$$\langle u, v \rangle_{H^s} = \langle \Lambda^s u, \Lambda^s v \rangle_{L^2}$$

$$= \langle (1+|\xi|^2)^s u, v \rangle_{L^2}$$

Theorem: $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ for $s \in (0,1)$.

Proof:

$$[u]^2_{W^{s,2}} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(z+y)-u(y)|^2}{|z|^{n+2s}}\, dz\, dy$$

$$= \int_{\mathbb{R}^n} \left\| \frac{u(z+\cdot)-u(\cdot)}{|z|^{n/2+s}} \right\|^2_{L^2} dz$$

Plancherel's theorem

$$= \int_{\mathbb{R}^n} \left\| \mathcal{F}\left( \frac{u(z+\cdot)-u(\cdot)}{|z|^{n/2+s}} \right) \right\|^2_{L^2} dz$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|e^{i\langle \xi, z\rangle}-1|^2}{|z|^{n+2s}} |\mathcal{F}u(\xi)|^2\, d\xi\, dz$$

$$= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1-\cos\langle \xi, z\rangle}{|z|^{n+2s}} |\mathcal{F}u(\xi)|^2\, dz\, d\xi$$

$$= 2\int_{\mathbb{R}^n} |\mathcal{F}u(\xi)|^2 \underbrace{\left( \int_{\mathbb{R}^n} \frac{1-\cos\langle \xi, z\rangle}{|z|^{n+2s}}\, dz \right)}_{\text{Exercise: prove that this is equal to } C(n,s)\cdot|\xi|^{2s}.} d\xi$$

$$= C(n,s) \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2\, d\xi .$$

Thus

$$\|u\|^2_{W^{s,2}(\mathbb{R}^n)} \approx \|u\|^2_{L^2} + [u]^2_{W^{s,2}} \approx \int_{\mathbb{R}^n} (1+|\xi|^{2s}) |\mathcal{F}u(\xi)|^2$$

$$\approx \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\mathcal{F}u(\xi)|^2 = \langle \Lambda^s u, \Lambda^s u\rangle = \|u\|^2_{H^s(\mathbb{R}^n)}$$

Advantage of $H^s$: easy proofs of super-critical Sobolev embeddings

1. If $s > \frac{n}{2}$, then $H^s(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

Proof: $$\int_{\mathbb{R}^n} |\mathcal{F}u(\xi)|\, d\xi \le \underbrace{\left(\int_{\mathbb{R}^n} |\mathcal{F}u(\xi)|^2 (1+|\xi|^2)^s\right)^{1/2}}_{\text{this is } \|u\|^2_{H^s}} \underbrace{\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} d\xi\right)^{1/2}}_{\text{this is finite precisely when } s > \frac{n}{2}}$$

Exercise!

This shows that
if $u \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2}$,
then $\mathcal{F}u$ is in $L^1(\mathbb{R}^n)$,
thus $u$ is a continuous, bounded function,
~~decay~~ vanishing at infinity.

Trivial observation: If $s > 1$, $u \in H^s(\mathbb{R}^n)$, then
for any $\alpha : |\alpha| = [s]$ $\quad D^\alpha u \in H^{s-[s]}(\mathbb{R}^n)$
Thus if $s > \frac{n}{2} + k$, $k \in \mathbb{N}$, then $u \in H^s(\mathbb{R}^n)$ is in $C^k(\mathbb{R}^n)$

Exercise: Prove Morrey's theorem:
$u \in H^s(\mathbb{R}^n)$, $s = \frac{n}{2} + \alpha$, $\alpha \in (0,1)$, then $u \in C^{0,\alpha}(\mathbb{R}^n)$.